On the impact of short laser pulses on cold low-density plasmas

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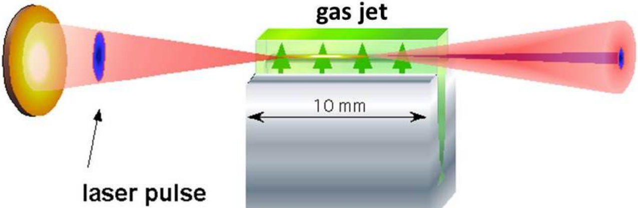
GF, J. Phys. A 47 (2014); Acta Appl. Math. 132 (2014); Ricerche Mat. 65 (2016); arXiv:1607.03482.
GF, U. de Angelis, R. Fedele, Phys. Plasmas 21 (2014);
GF, S. DeNicola, Phys Rev Acc Beams 19; NIMA 829 (2016)

Introduction

Today ultra-intense laser-plasma interactions allow extremely compact acceleration mechanisms of charged particles to relativistic regimes.

In the *Wake-Field Acceleration* [Tajima, Dawson 79] electrons accelerate "surfing" a plasma wake wave driven by a very short laser pulse or charged particle beam, e.g. in a supersonic diluted gas jet.

Singling out the relevant parts in parameter space, and then solving PDEs through PIC or other codes involves huge and costly computations. Any theoretical insight that can reduce or simplify the job should be welcome!



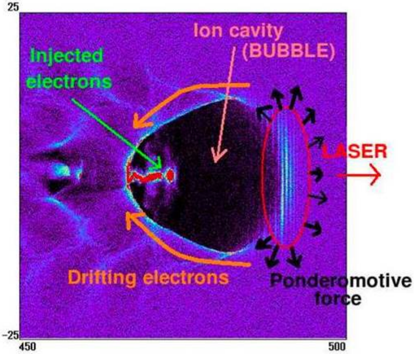
Here I'll argue: with very little computational power can get important information on the impact of a very short and intense laser pulse onto a cold diluted plasma:
i) generation of Plasma Wakes;
ii) conditions for the *bubble regime*, or else
iii) conditions for the *slingshot effect* [GF et al 2014-16].

We first determine the motion of the plasma electrons up to shortly after the impact in a *plane hydrodynamical model* [GF 2014-16] (the pulse is modeled as a plane traveling-wave the z-direction, i.e. spot size $R=\infty$): we reduce the system of Lorentz-Maxwell and continuity PDEs into a family of decoupled systems of non-autonomous Hamilton Equations in dim 1 (ODE!); achieved neglecting the ions' motion and the pump depletion, and adopting $\xi = ct - z$ instead of time t as an independent variable, electrons' $p^0 - cp^z$ instead of p^z as an unknown.

Solving these Hamilton Equations we derive:
i) how long the hydrodynamical picture holds, when & where it breaks;
ii) the main features of the induced *plane* plasma WF, with strict lower bounds for the electron density n_e well inside the plasma (in particular, $n_e > n_0/2$ if the initial one $\tilde{n}_0$ was uniform).

Then we use causality and geometric arguments to qualitatively correct predictions for the "real world" ($R < \infty$). We suggest that:
1. a *ion bubble* can arise only at the vacuum-plasma interface and only with sufficiently small $R, \tilde{n}_0$, while
2. with slightly larger $R, \tilde{n}_0$ the *slingshot effect* may occur (backward expulsion of energetic electrons from the plasma surface).

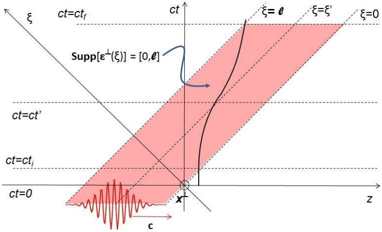
Point **1.** gives a solution to the problem of explaining how continuous PDEs with continuous initial & boundary conditions *inside the bulk* can allow the formation of a *singularity*, which develops into a cavity in n_e (the *bubble*).

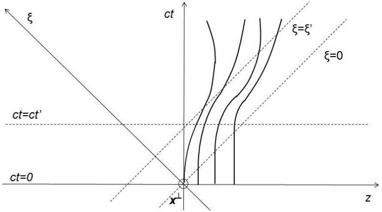


Plan

Plane model

As no particle can travel at speed c, $\tilde{\xi}(t) = ct - z(t)$ is strictly growing: we can adopt $\xi = ct - z$ as the independent parameter on the worldline λ & in EoM.





$$\text{initial position} \quad \mathbf{X}=(X,Y,Z) \;\underset{\substack{\mathbf{X}_e(t,\cdot)\\ \hat{\mathbf{X}}_e(\xi,\cdot)}}{\overset{\substack{\mathbf{x}_e(t,\cdot)\\ \hat{\mathbf{x}}_e(\xi,\cdot)}}{\rightleftarrows}}\; \mathbf{x}=(x,y,z) \quad \text{position at } t,\ \xi \quad \text{must be 1-to-1}$$

Eulerian $f(t,\mathbf{x}) = \tilde{f}(t,\mathbf{X}) = \hat{f}(\xi,\mathbf{X})$ Lagrangian observables. Use CGS units.

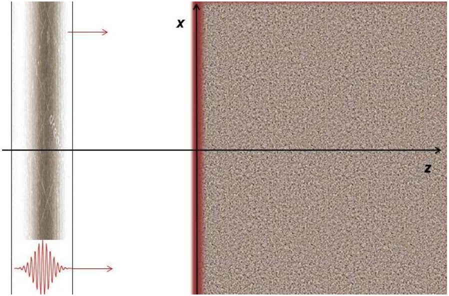
Dimensionless: $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} = \frac{\dot{\mathbf{x}}}{c}$, $\gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}$, 4-vel. $u = (u^0, \mathbf{u}) \equiv (\gamma, \gamma\boldsymbol{\beta}) = \left(\frac{p^0}{mc^2}, \frac{\mathbf{p}}{mc}\right)$

Eqs: **Maxwell, continuity and equation of motion of the electron fluid**.

Impact of EM **plane** wave on a plasma at equilibrium; $t=0$ initial conditions:

$$n_h(0,\mathbf{x}) = 0 \quad \text{if } z \leq 0, \qquad n_p(0,\mathbf{x}) = n_e(0,\mathbf{x}) \equiv \widetilde{n_0}(z), \qquad \mathbf{u}_h(0,\mathbf{x}) = \mathbf{0},$$
$$\mathbf{E}(0,\mathbf{x}) = \boldsymbol{\epsilon}^{\perp}(-z), \quad \mathbf{B}(0,\mathbf{x}) = \mathbf{k} \wedge \boldsymbol{\epsilon}^{\perp}(-z) + \mathbf{B}_s, \quad \boldsymbol{\epsilon}^{\perp}(\xi) = 0 \text{ if } \xi \notin]0, l[. \tag{1}$$

$h = e, p$ ($p \equiv$proton). No assumption on the Fourier analysis of the pump $\boldsymbol{\epsilon}^{\perp}$.



$A^\mu, \mathbf{u}_h, n_h$ **will depend only on** z,t; $\Delta\mathbf{x}_e \equiv \mathbf{x}_e - \mathbf{X}$ **on** Z,t.

Then $\mathbf{B} = \mathbf{k}\partial_z \wedge \mathbf{A}^\perp$, $c\mathbf{E}^\perp = -\partial_t \mathbf{A}^\perp$, $\mathbf{A}^\perp(t,z) = -c\int_{-\infty}^{t} dt'\, \mathbf{E}^\perp(t',z)$ (phys obs!)

Prop. 1 in [GF'14]: Maxwell eqs $\nabla\cdot\mathbf{E} = 4\pi\rho$, $\partial_0 E^z + 4\pi j^z = (\nabla\wedge\mathbf{B})^z = 0$ imply

$$E^z(t,z) = 4\pi e\left\{\widetilde{N}[Z_p(t,z)] - \widetilde{N}[Z_e(t,z)]\right\}, \qquad \widetilde{N}(Z) \equiv \int_0^Z dZ'\, \widetilde{n}_0(Z'). \quad (2)$$

We thus eliminate the unknown E^z in terms of the (still unknown) longitudinal motion. Neglecting the ions' motion we find $Z_p(t,z) = z$, $n_p(t,\mathbf{x}) = \widetilde{n}_0(z)$.

$$\frac{d\mathbf{p}_e}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{v}_e}{c}\wedge\mathbf{B}\right) \quad \& \quad \text{initial cond.} \quad \overset{\perp}{\Rightarrow} \quad \mathbf{u}_e^\perp = \frac{e}{mc^2}\mathbf{A}^\perp. \quad (3)$$

So $\mathbf{u}_e^\perp$ in terms of $\mathbf{A}^\perp$. Remaining unknowns $\mathbf{A}^\perp, n_e, u_e^z, \mathbf{x}_e$ are all observables.

$\mathbf{A}^\perp$ fulfills $\Box\mathbf{A}^\perp = 4\pi\mathbf{j}^\perp$ (Landau gauge). Including (1) this amounts to

$$\mathbf{A}^\perp - \boldsymbol{\alpha}^\perp = 2\pi c\int dt'dz'\,\theta\left(ct - ct' - |z - z'|\right)\theta(t')\,\mathbf{j}^\perp(t',z'), \quad \text{(integral eq.)} \quad (4)$$

$$\text{where} \quad \boldsymbol{\alpha}^\perp(\xi) \equiv -\int_{-\infty}^{\xi} d\xi'\,\boldsymbol{\epsilon}^\perp(\xi') \qquad (\boldsymbol{\alpha}^\perp(\xi)\to 0 \quad \text{as } \xi\to-\infty).$$

rhs=0 for $t\le 0$, because the laser-plasma interaction starts at $t=0$. Within *small times* (to be determined aposteriori) can approximate $\mathbf{A}^\perp(t,z) \simeq \boldsymbol{\alpha}^\perp(ct-z)$.

It is convenient to use the electron "s-factor" s instead of u^z as an unknown:

$$s \equiv u^0 - u^z = \gamma - u^z. \qquad \text{(positive definite!)} \qquad (5)$$

Insensitive to rapid oscillations of $\mathbf{u}^\perp \sim \boldsymbol{\alpha}^\perp$; $\gamma, \mathbf{u}, \boldsymbol{\beta}$ are rational functions of $\mathbf{u}^\perp$, s:

$$\gamma = \frac{1 + \mathbf{u}^{\perp 2} + s^2}{2s}, \qquad u^z = \frac{1 + \mathbf{u}^{\perp 2} - s^2}{2s}, \qquad \boldsymbol{\beta} = \frac{\mathbf{u}}{\gamma}. \qquad (6)$$

Then the left eqs of motion for the electron fluid amount to [GF, DeNicola '16]

$$\hat{\Delta}'(\xi, Z) = \frac{1+\hat{v}}{2\hat{s}^2} - \frac{1}{2}, \qquad \hat{s}_e'(\xi, Z) = \frac{4\pi e^2}{mc^2} \left\{ \widetilde{N}[\hat{z}_e] - \widetilde{N}(Z) \right\}, \qquad (7)$$

$$\hat{\mathbf{x}}_e(0, \mathbf{X}) - \mathbf{X} = \mathbf{0}, \qquad \hat{\mathbf{u}}_e(0, \mathbf{X}) = \mathbf{0} \quad \Rightarrow \quad \hat{s}_e(0, \mathbf{X}) = 1. \qquad (8)$$

Here $\hat{\Delta} \equiv \hat{z}_e - Z$, $\hat{v} \equiv \hat{\mathbf{u}}^{\perp 2}$. (7) is a family parametrized by Z of *decoupled ODEs*.
Eq (7) can be put in the form of *Hamilton equations* in 1 $\sharp$ of freedom:
ξ plays the role of "time", $(\Delta, -s)$ play the role of (q, p).

$$H(\Delta, s, \xi; Z) \equiv \gamma(s, \xi) + \mathcal{U}(\Delta; Z), \quad \mathcal{U}(\Delta; Z) \equiv \tfrac{4\pi e^2}{mc^2} \left[\widetilde{\mathcal{N}}(Z+\Delta) - \widetilde{\mathcal{N}}(Z) - \widetilde{N}(Z)\Delta \right],$$

$$\gamma(s, \xi) \equiv \frac{1}{2}\left[s + \frac{1 + v(\xi)}{s} \right], \qquad \widetilde{\mathcal{N}}(Z) \equiv \int_0^Z d\zeta\, \widetilde{N}(\zeta) = \int_0^Z d\zeta\, \widetilde{n_0}(\zeta)\, (Z - \zeta). \qquad ($$

(7) is solved numerically where $\boldsymbol{\epsilon}^\perp(\xi) \neq 0$, by quadrature elsewhere.

All other unknowns can be determined explicitly using $\hat{s}, \hat{z}$, in particular

$$\hat{\mathbf{x}}_e^{\perp}(\xi, \mathbf{X}) = \mathbf{X}^{\perp} + \int_0^{\xi} dy \, \frac{\hat{\mathbf{u}}^{\perp}(y)}{\hat{s}_e(y, Z)}, \tag{10}$$

$$c\hat{t}(\xi, Z) = \xi + \hat{z}_e(\xi, Z) \equiv Z + \hat{\Xi}(\xi, Z). \tag{11}$$

Clearly $\hat{\Xi}(\xi, Z)$ is strictly increasing for each Z. Inverting (11) we find $\tilde{\xi}(t, Z) = \hat{\Xi}^{-1}(ct - Z, Z)$ and e.g. the position of the **X**-electrons from

$$\mathbf{x}_e(t, \mathbf{X}) = \hat{\mathbf{x}}_e[\tilde{\xi}(t, Z), \mathbf{X}]. \tag{12}$$

By derivation we obtain several useful relations, e.g.

$$\frac{\partial \hat{z}_e}{\partial Z}(\xi, Z) = 1 + \partial_Z \hat{\Delta}(\xi, Z), \qquad \frac{\partial Z_e}{\partial z}(t, z) = \left.\frac{\hat{\gamma}}{\hat{s}\, \partial_Z \hat{z}_e}\right|_{(\xi,Z)=\left(ct-z, Z_e(t,z)\right)}. \tag{13}$$

By (13), $\partial_Z \hat{\Delta} > -1$ is thus a necessary and sufficient condition for the invertibilities of $\hat{z}_e : Z \mapsto z$, $\hat{\mathbf{x}}_e : \mathbf{X} \mapsto \mathbf{x}$ at fixed ξ, of $z_e : Z \mapsto z$, $\mathbf{x}_e : \mathbf{X} \mapsto \mathbf{x}$ at fixed t, what justifies the hydrodynamic description adopted so far and ensures the existence of the inverse function $Z_e(t,z)$. Then it is also

$$n_e(t, z) = \widetilde{n_0}[Z_e(t,z)] \left.\frac{\hat{\gamma}}{\hat{s}[1 + \partial_Z \hat{\Delta}]}\right|_{(\xi,Z)=\left(ct-z, Z_e(t,z)\right)}. \tag{14}$$

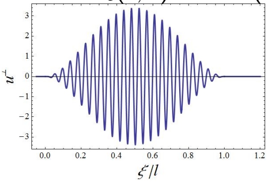
Approximation $\mathbf{A}^{\perp}(t,z) \simeq \boldsymbol{\alpha}^{\perp}(ct - z)$ is acceptable as long as the found motion makes $|\text{rhs}(4)| \ll |\boldsymbol{\alpha}^{\perp}|$; otherwise (4) determines the 1st correction to $\mathbf{A}^{\perp}$; etc.

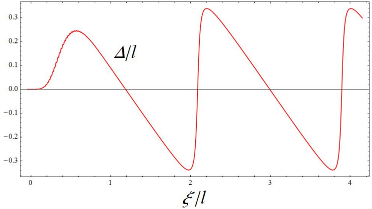
If $\widetilde{n}_0(Z) \equiv n_0 = \textbf{const}$ (7-8) and its solution is in fact Z-independent:

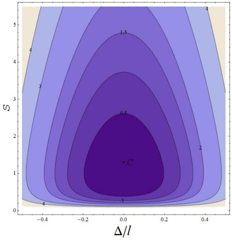
$$\Delta' = \frac{1+v}{2s^2} - \frac{1}{2}, \qquad s' = M\Delta, \qquad \Delta(0)=0, \quad s(0)=1, \quad (15)$$

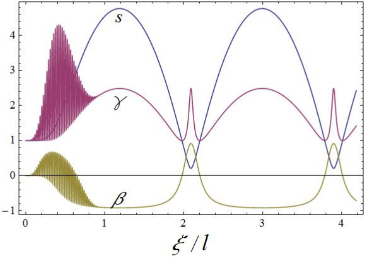
where $M \equiv 4\pi e^2 n_0/mc^2$, $v(\xi) \equiv \mathbf{u}^{\perp 2}(\xi)$, and $\mathcal{U}(\Delta, Z) \equiv M\Delta^2/2$: copy of the same relativistic harmonic oscillator. $\partial_Z \hat{z}_e \equiv 1$, invertibility Ok, and

$$Z_e(t,z) = z - \Delta(ct-z).$$







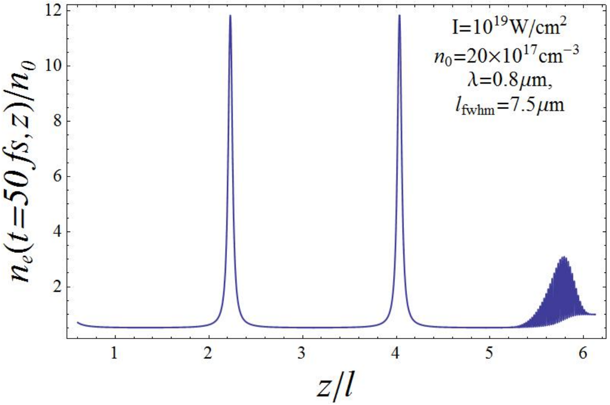


Solution of (15) for $\mathrm{I} = 10^{19} W/cm^2$, $n_0 = 2\times 10^{18} cm^{-3}$, $\lambda = .8\mu$m, $l_{fwhm} = 7.5\mu$m.

$$n_e(t,z) = \frac{n_0}{2}\left[1+\frac{1+v(ct-z)}{s^2(ct-z)}\right] = \frac{n_0}{1-\beta^z(ct-z)}, \tag{16}$$

$n(t,z), \mathbf{u}(t,z), \ldots$ evolve as forward travelling waves. Remarkable consequences:

$$n_e(t,z) > \frac{n_0}{2}, \qquad n_e(t,z) \simeq \frac{n_0}{2} \quad \text{if } s^2(ct-z) \gg 1+v(ct-z). \tag{17}$$



The wake-field by "rules of thumb"

Solved eq. (7), one can calculate the final energy variation h of the electrons after the interaction with the pulse, normalized to mc^2 (energy gain):

$$h(Z) = 1 - v(l) + \int_0^l d\xi \, \frac{\hat{v}'(\xi)}{2s(\xi)} \simeq \int_0^l d\xi \, \frac{\hat{v}'(\xi)}{2s(\xi)}. \tag{18}$$

If $h \gg 1$ then the main physical features can be expressed by powers of h:

$$\text{max displacement } \Delta_M = \sqrt{\frac{2h}{M}}, \qquad \text{oscill. period } T_H \simeq \frac{4}{c}\Delta_M = \frac{4}{c}\sqrt{\frac{2h}{M}}, \tag{19}$$

$$\text{max el. field } \hat{E}^z_M = 4\pi e n_0 \Delta_M = 4\pi e n_0 \sqrt{\frac{2h}{M}} = \sqrt{8\pi h n_0 mc^2}, \tag{20}$$

$$\text{max electron density } n_M \simeq 2\, n_0\, h^2 \tag{21}$$

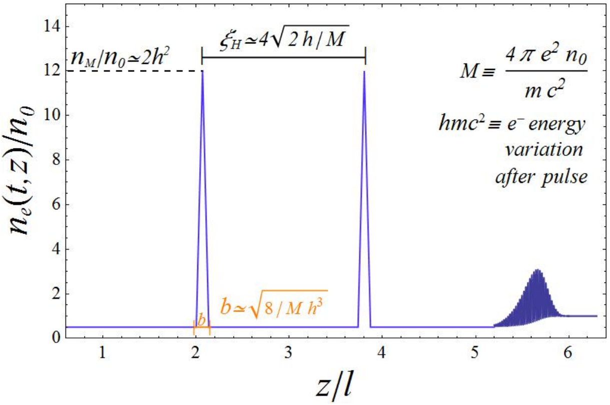
Can get a sequence of approximations of (Δ, s), h even *without solving* (15): $\left(\Delta^{(0)}(\xi), s^{(0)}(\xi)\right) = \left(\int_0^\xi dy \frac{v(y)}{2}, 1\right)$, $h^{(0)} =! \frac{v(l)}{2}$ is bad; the next one $\left(\Delta^{(1)}(\xi), s^{(1)}(\xi)\right)$

$$s^{(1)}(\xi) = 1 + \frac{M}{2}\int_0^\xi dy\, v(y)(\xi - y), \qquad h^{(1)} = \int_0^l d\xi \, \frac{v'(\xi)}{2s^{(1)}(\xi)} \tag{22}$$

is better; and so on. They are better for lower n_0.

We can schematize the graph of n_e as the polygonal line depicted below: it is made of isosceles triangles of heignt n_M and base b separated by intervals of length $\xi_H - b \simeq \xi_H$ where $n_e = n_0/2$ =const. b is easily determined from conservation of the total number of electrons, $(\xi_H - b)n_0/2 = bn_M/2$, leading to

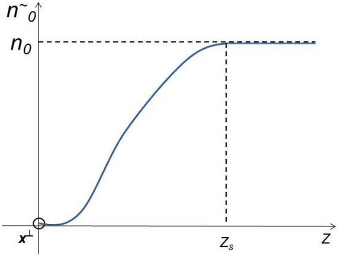
$$b \simeq \sqrt{8/Mh^3} = 8/M^2\Delta_M^3 \tag{23}$$



Impact of the pulse on an increasing $\widetilde{n}_0(Z)$

We assume:

1. $\widetilde{n}_0(Z)$ growing with Z, and $\exists Z_s > 0$ s.t. $Z \geq Z_s \;\Rightarrow\; \widetilde{n}_0(Z) \geq n_0 =$const.
2. Pulses of duration $\tau = l/c \leq T_H$
   $T_H \equiv$plasma oscillation period, depends on osc. amplitude. $T_H \geq T_H^{nr} = \sqrt{\frac{\pi m}{n_0 e^2}}$



For $Z > Z_S \equiv \max\{Z_s, \Delta_M\}$ eq. (7-8) reduce to (15), and have the same solution; no collisions between electrons with different $Z, Z' > Z_S$ intersect. 1st collision occurs at $t = t_c$ and involves electrons with $Z = Z_c < Z_S$ suitable. $t_c^m \equiv$lowest t_c arises if $\widetilde{n}_0(Z) = n_0\theta(Z)$ (*worst case*). We show $t_c^m > \frac{5}{4}T_H + \frac{Z_c}{c}$.

Summarizing, for $t \leq t_c$ there are collisions nowhere (see fig. 2), the maps $z_e(t,\cdot) : Z \mapsto z$ are invertible, and the hydrodynamic description is justified.

For $t > t_c$ the perturbations due to collisions can propagate only with a *velocity* $v_p < c$, hence do not affect causally disconnected regions D. Since the pulse speed is c, the part of the Wake travelling-wave behind the pulse not affected and looking as in the figures becomes longer and longer. The $Z \simeq 0$ electrons go very far backwards; also not affected for very long t.

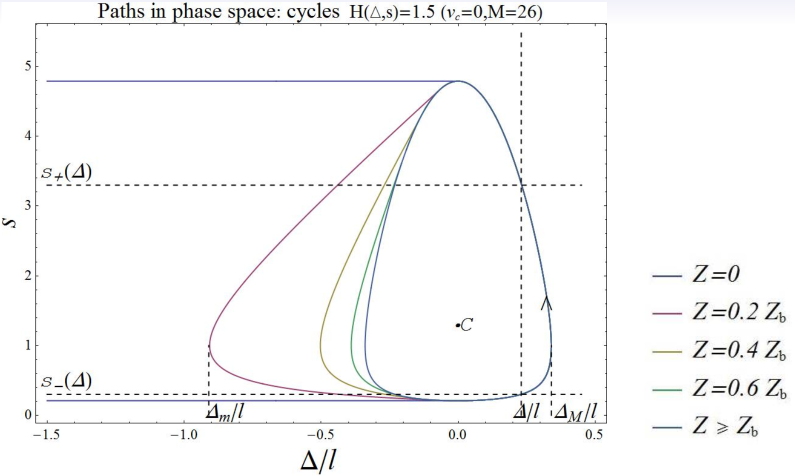


Figure 1 : Phase portraits for $\widetilde{n}_0(Z) = n_0\theta(Z)$ ($n_0 = 2\times10^{18}$cm^{-3}), $v(\xi) = 0$. The paths of all $Z > 0$ electrons are cycles around $C \simeq (0, 1)$. Those of the $Z > \Delta_M$ electrons do not cross the $\hat{\Delta} = -Z$ axis (no exit from the bulk). The path of the $Z = 0$ electrons is unbounded.

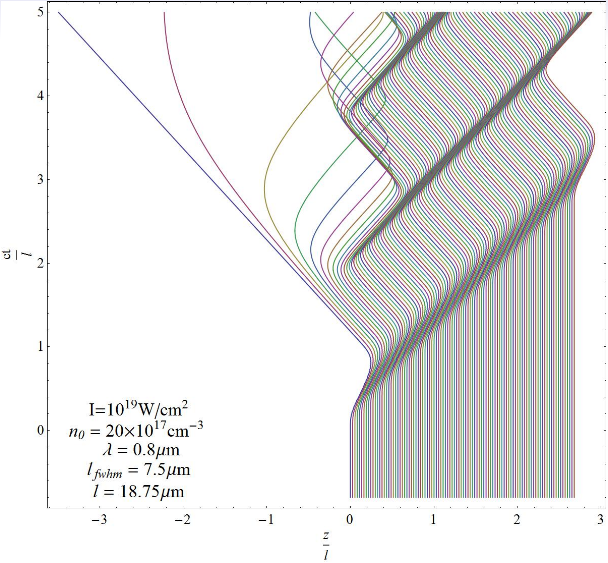


Figure 2 : Electrons' worldlines if $\widetilde{n}_0(Z)=n_0\theta(Z)$. They first intersect after 5/4 oscillations induced by the pulse, here at $t = t_c^m \simeq 3l/c \simeq 56$fs.

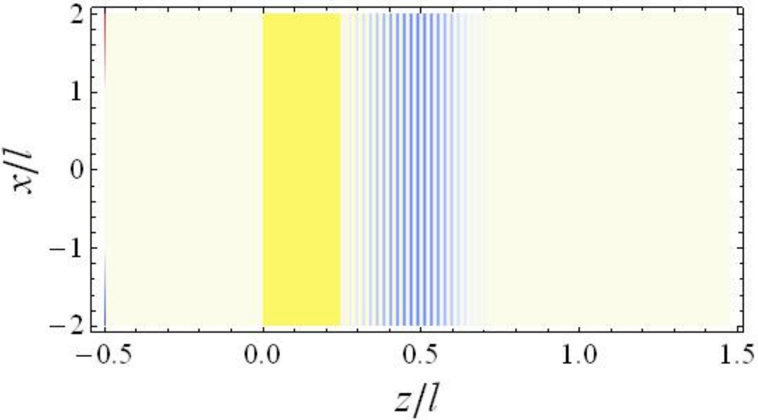


Figure 3 : Normalized charge density plot after 37 fs, for pulse intensity $I = 10^{19}$ W/cm^2 & step-shaped initial electron density $n_0 = 2 \times 10^{18}$cm^{-3}. The forward boost of the most external (i.e. small Z) electrons by the ponderomotive force has left a layer (here in yellow) containing only ions.

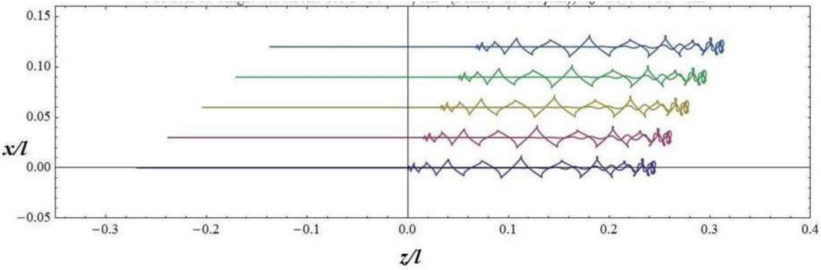
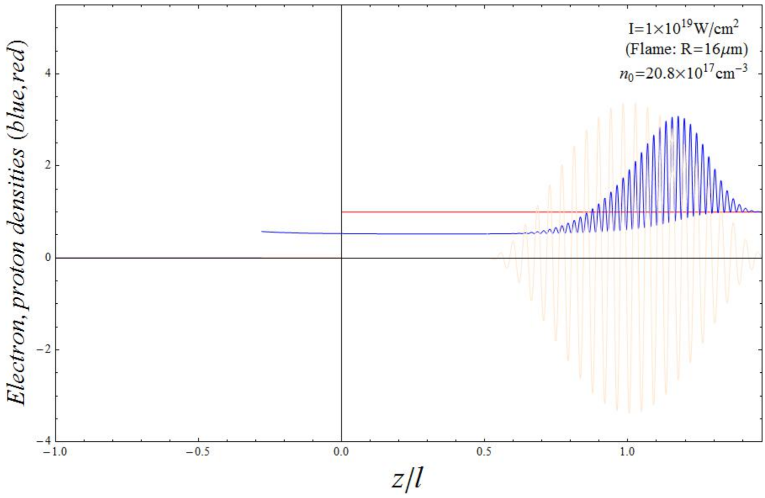


Figure 4 : The trajectories of the most external (i.e. small Z) electrons after a few tens of fs have exited the bulk ($z < 0$), completely filling the previously formed ion cavity (here the pulse intensity is $I = 10^{19}$ W/cm^2, the initial electron density is $\widetilde{n_0}(Z) = n_0\theta(Z)$, $n_0 = 2 \times 10^{18}$cm^{-3}).



Normalized charge density plots of the electrons (blue) and the ions (red) corresponding to the the figure 4.

Finite R corrections & discussion

Within a sufficiently small distance $R < \infty$ from the $\vec{z}$-axis the real laser pulse is indistinguishable from a plane wave ($R = \infty$) travelling in the $\vec{z}$-direction. By causality, the electrons in the cylinder C with axis $\vec{z}$ and radius R experience no change with respect to the $R = \infty$ until the information about the different charge distribution contained in the retarded potential reaches them, i.e. until they remain the causal cone depicted in fig. 5; those along the $\vec{z}$-axis are the latest to experience any change.
Let t_e be time of backward expulsion of the first electrons on the $\vec{z}$-axis hit by the pulse ($\mathbf{X} = \mathbf{0}$ electrons). If R is sufficiently large, $R \gtrsim t_e c$, a thin bunch of $\mathbf{X} \simeq \mathbf{0}$ electrons succeeds in going out of the plasma before their way out can be obstructed by the Lateral Electrons (LE) outside the surface of the ion cavity C_R created by the pulse (the LE are attracted towards the $\vec{z}$-axis).

Part of them will succeed in escaping to $z = -\infty$ (slingshot effect).
As a consequence, the ion cavity closes forever behind the pulse.

If R is sufficiently small, $R < t_e c$, the Lateral Electrons (LE) attracted towards the z-axis $\vec{z}$ can reach it, collide around the $\vec{z}$-axis and close the cavity before the backward espulsion of any electrons.
Actually, if R is small enough the $\mathbf{X} = 0$ electrons are still moving forward behind the pulse when the LE reach the $\vec{z}$-axis: a ion bubble can form.

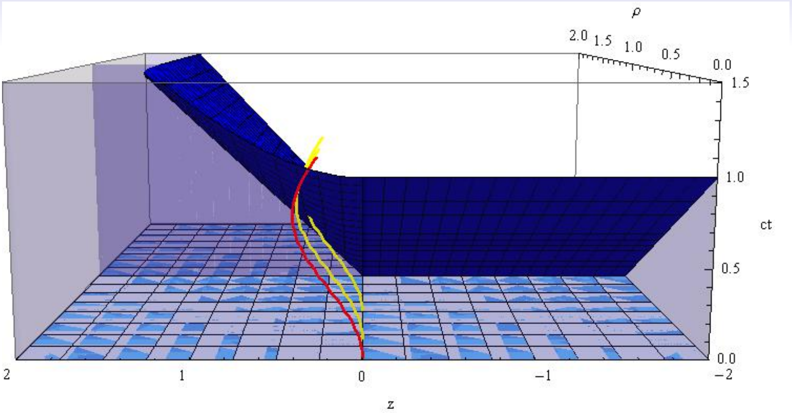
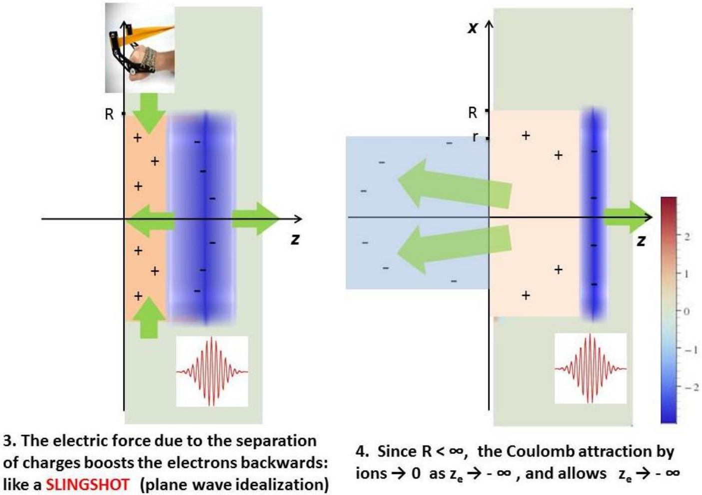


Figure 5 : The $t = 0$ initial data of the $R = \infty$ model coincide with the real ones on the light blue domain $\mathcal{D}_1^0$ at the base. Therefore also all the physical consequences coincide within the corresponding future Cauchy development $D^+(\mathcal{D}_1^0)$ (shaded region between the blue and light blue hypersurfaces), here represented in (ρ, z, ct) coordinates (we have dropped the inessential angle φ). The worldlines of the $\mathbf{X}=0$ electrons (red) remain in $D^+(\mathcal{D}_1^0)$ longer than those of off-$\vec{z}$-axis electrons (yellow)



Schematic picture of the *slingshot effect*. The effect is enhanced if the pulse duration τ fulfills $\tau \sim T_H/2$.

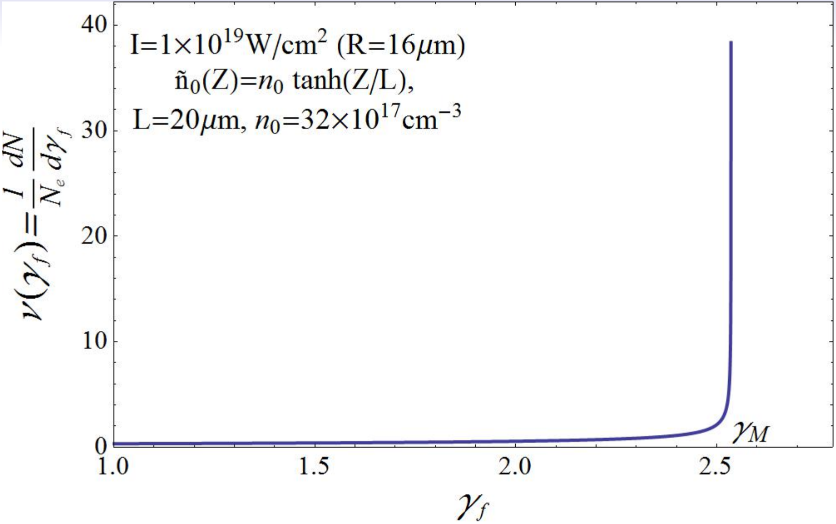


Figure 6 : Fraction ν of expelled electrons vs. the relativistic factor, for pulse intensity $I = 10^{19}$ W/cm^2 & smooth initial electron density with asymptotic value $n_0 = 32 \times 10^{17}$cm^{-3}

References

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