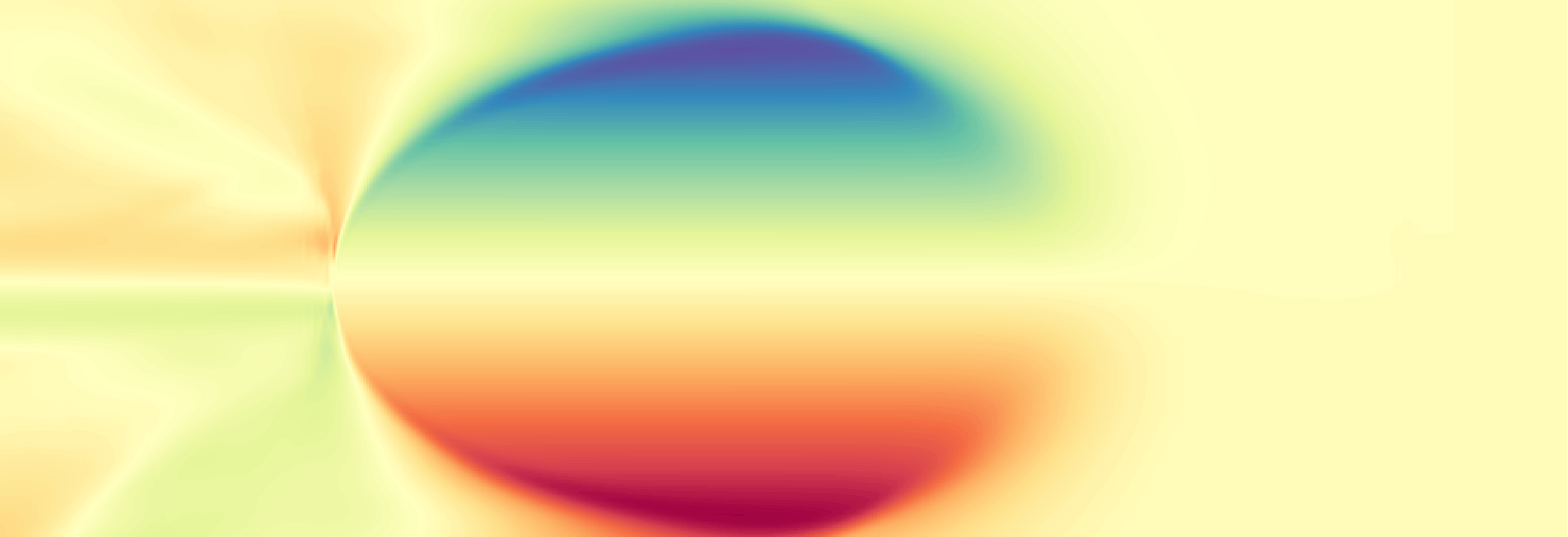
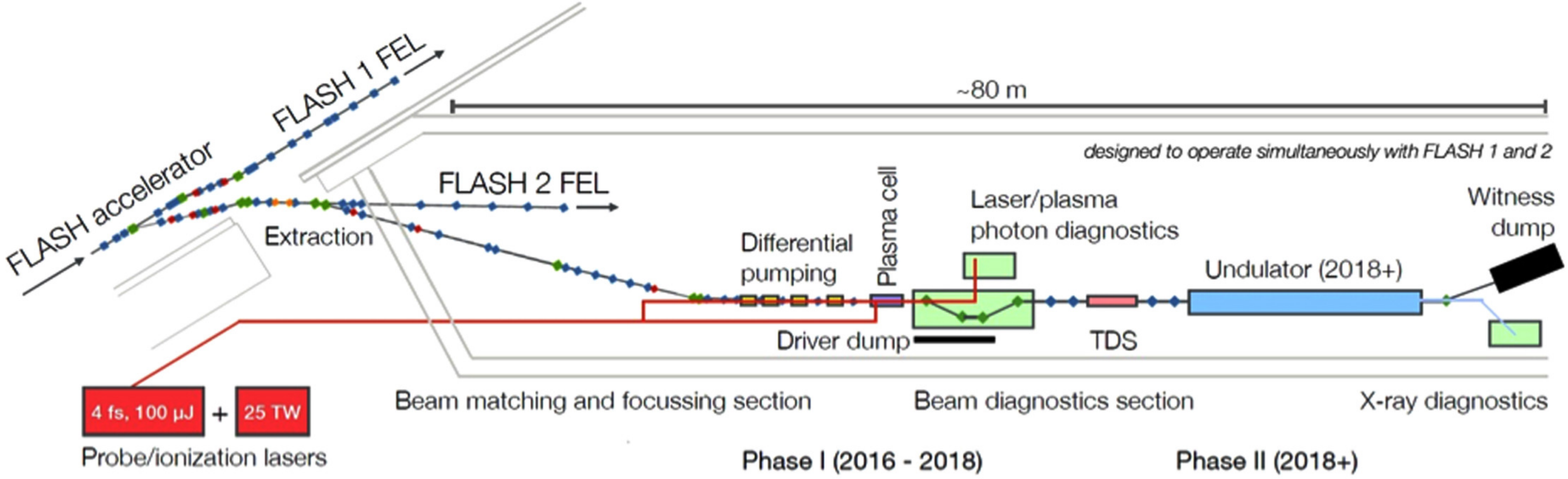


# Development of an analytical model for emittance calculation in external injection scenarios

A. Aschikhin, A. Martinez de la Ossa, T. Mehrling, J. Osterhoff

*Deutsches Elektronen-Synchrotron DESY*
*Universität Hamburg*

# The FLASHForward project



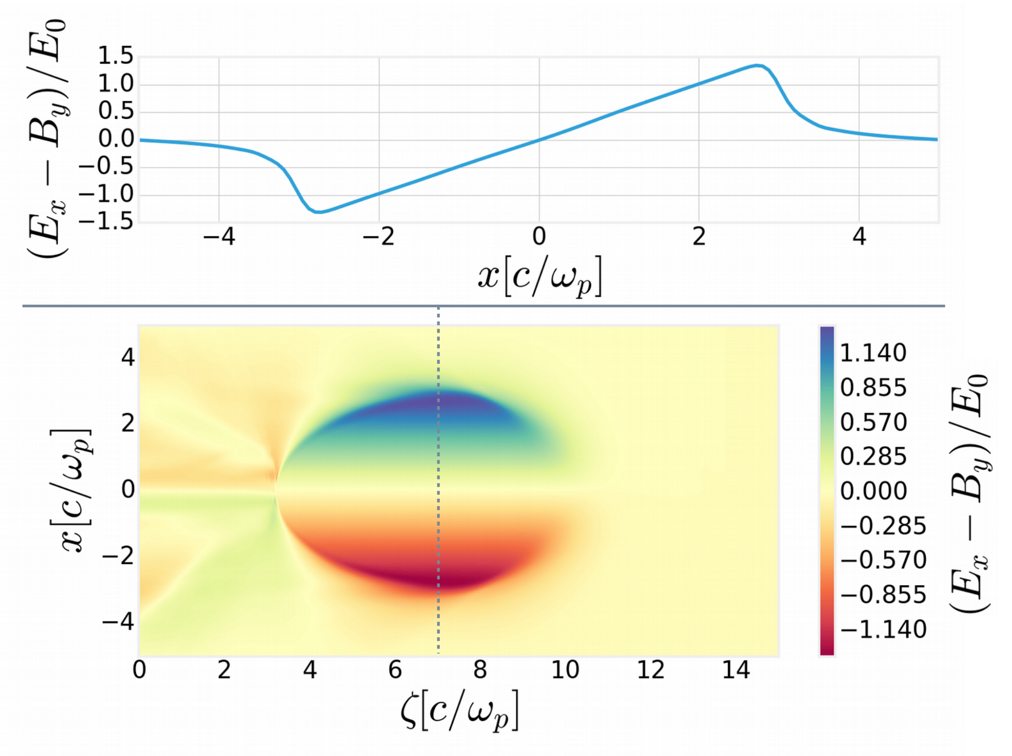

**FLASH**Forward▸▸

Future-oriented wakefield-accelerator research and development at FLASH

*FLASHForward - A Future-Oriented Wakefield-Accelerator Research and Development Facility at FLASH,* J. Osterhoff
Wednesday, 18:45, WG1-WG8 Joint Session

# Beam transport in the blowout regime

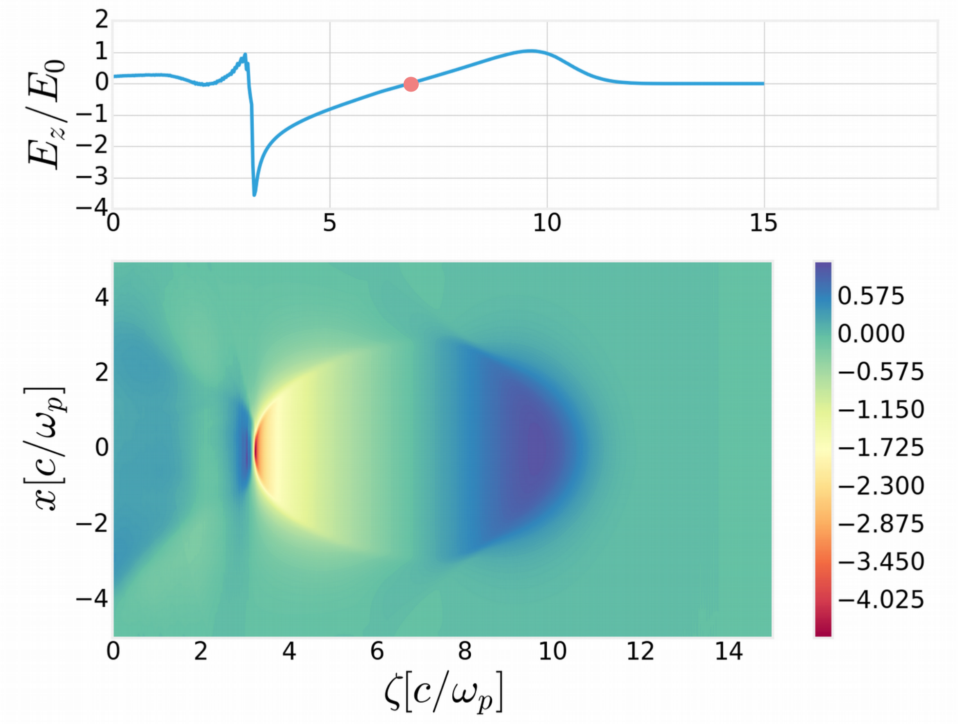
- The situation concerns typical external injection blowout scenarios
  - Linear focusing channel
  - A preexisting beam following a driver generating a wakefield
- Emittance preservation and analysis is a crucial aspect in these acceleration scenarios
- Driver and witness were assumed to have no offset in respect to each other




T. Mehrling et al., Plasma Phys. Control. Fusion 56 084012

# Scenario I – single slice without energy gain

- Single slice with predefined Twiss parameters moving through the ion channel
- No energy gain
- Use field information obtained from Particle-in-Cell simulations: sufficiently small macroparticle distribution placed at the zero crossing of the longitudinal electric field
- Cross-checked with a semi-analytic numerical model (SANA)*




*T. Mehrling et al, NIM A 2016.01.091 (2016)

# Scenario I – the formulas

- The basis is the differential equation for the transverse position of a single electron in a homogeneous ion-channel:

$$\frac{d^2x}{dt^2} + \omega_\beta^2 x = 0$$

- With the solution, ignoring energy gain

$$x(t) \simeq x_0 \cos[\varphi(t)] + \frac{p_{x,0}}{m\gamma_0\omega_{\beta,0}} \sin[\varphi(t)]$$

- With $\omega_{\beta,0} = \omega_p/\sqrt{2\gamma_0}$, and the phase advance defined as:

$$\varphi(t) = \omega_\beta t$$

# Scenario I – from particle to distribution

- Assuming a phase-space distribution $f_0(x_0, p_{x,0}, \gamma_0)$ without correlation between initial transverse offset $x_0$, the transverse momentum $p_{x,0}$ and energy:

$$f_0 = f_\perp(x_0, p_{x,0}) f_\gamma(\gamma_0), \quad \int f_0 \mathrm{d}x_0 \mathrm{d}p_{x,0} \mathrm{d}\gamma_0 = 1$$

- The relevant distributions can then be obtained by averaging over the phase-space together with the Gaussian energy distribution, $f_\gamma = (\sqrt{2\pi}\sigma_\gamma)^{-1} \exp(-\delta\gamma^2/2\sigma_\gamma^2)$

$$\langle x^2 \rangle(t) = \int_{-\infty}^{\infty} (x^2(t)) f_0 \mathrm{d}x_0 \mathrm{d}p_{x,0} \mathrm{d}\delta\gamma$$

$$\langle p_x^2 \rangle(t) = \int_{-\infty}^{\infty} (p_x^2(t)) f_0 \mathrm{d}x_0 \mathrm{d}p_{x,0} \mathrm{d}\delta\gamma$$

$$\langle x p_x \rangle(t) = \int_{-\infty}^{\infty} (x(t) p_x(t))\, f_0 \mathrm{d}x_0 \mathrm{d}p_{x,0} \mathrm{d}\delta\gamma$$

$$x = x' k_p$$

$$u_x = \frac{p_x}{m_e c}$$

$$\tilde{t} = t\omega_p$$

# Scenario I – a closer look at the emittance

$$\epsilon^2_{n,rms}(\tilde{t}) = \left(\frac{\overline{\gamma_0}}{8}\left\langle x_0^2\right\rangle^2 + \frac{1}{2\overline{\gamma_0}}\left\langle u_{x,0}^2\right\rangle^2\right)\left(1 - e^{-b\tilde{t}^2}\right) + \frac{1}{2}\left\langle x_0^2\right\rangle\left\langle u_{x,0}^2\right\rangle\left(1 + e^{-b\tilde{t}^2}\right) - \left\langle x u_{x,0}\right\rangle^2 e^{-b\tilde{t}^2}$$

- With $b = \Delta\gamma^2/2\gamma_0$, $\Delta\gamma = \sigma_\gamma/\gamma$, giving a time frame for emittance growth saturation as $\tilde{t}[\omega_p] \gg \frac{\sqrt{2\gamma_0}}{\Delta\gamma}$
- For FLASHForward parameters ($\gamma_0 = 2000, \Delta\gamma = 0.1\%$), $z \gg 1.0\mathrm{m}$
- With matching conditions for plasma environments, $\left\langle x u_x\right\rangle_m = 0, \left\langle x^2\right\rangle_m = \epsilon_0\sqrt{2/\overline{\gamma_0}}, \left\langle u_x^2\right\rangle_m = \epsilon_0\sqrt{\overline{\gamma_0}/2}$, based on $\hat{\alpha}_m = 0, \hat{\beta}_m = \sqrt{2\overline{\gamma}}/k_p, \hat{\gamma}_m = k_p/\sqrt{2\overline{\gamma}}$

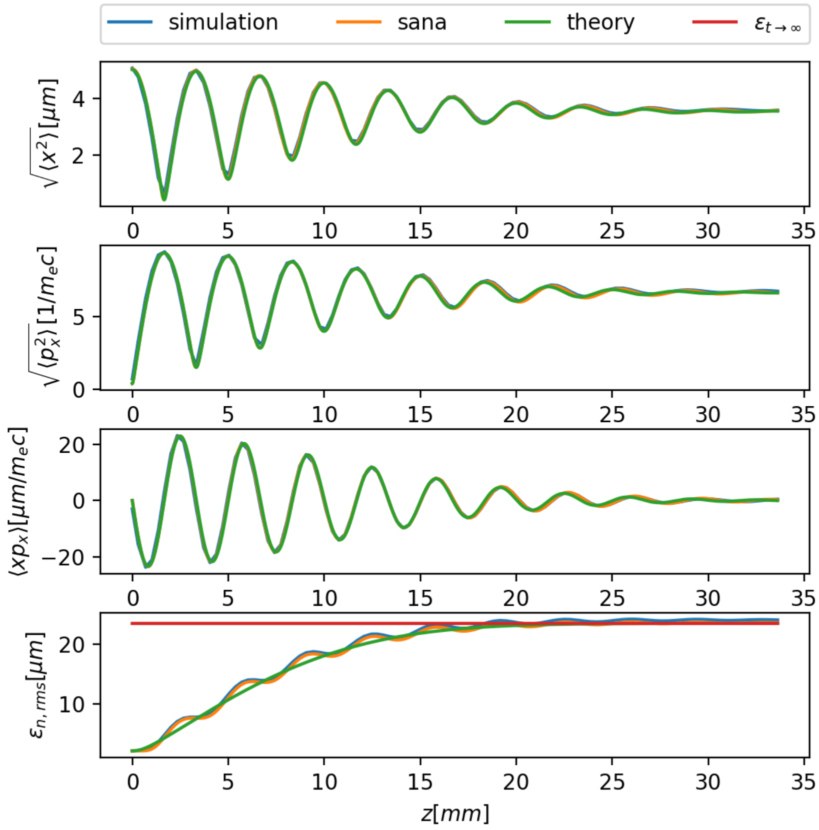
$$\frac{\epsilon^2_{n,rms}(\tilde{t})}{\epsilon_0^2} = \frac{1}{4}\left(\left(\frac{\left\langle x_0^2\right\rangle}{\left\langle x^2\right\rangle_m}\right)^2 + \left(\frac{\left\langle u_{x,0}^2\right\rangle}{\left\langle u_x^2\right\rangle_m}\right)^2\right)\left(1 - e^{-b\tilde{t}^2}\right) + \frac{1}{2}\frac{\left\langle x_0^2\right\rangle}{\left\langle x^2\right\rangle_m}\frac{\left\langle u_{x,0}^2\right\rangle}{\left\langle u_x^2\right\rangle_m}\left(1 + e^{-b\tilde{t}^2}\right) - \frac{\left\langle x u_{x,0}\right\rangle^2}{\epsilon_0^2} e^{-b\tilde{t}^2}$$

- An interesting property is the final emittance value,

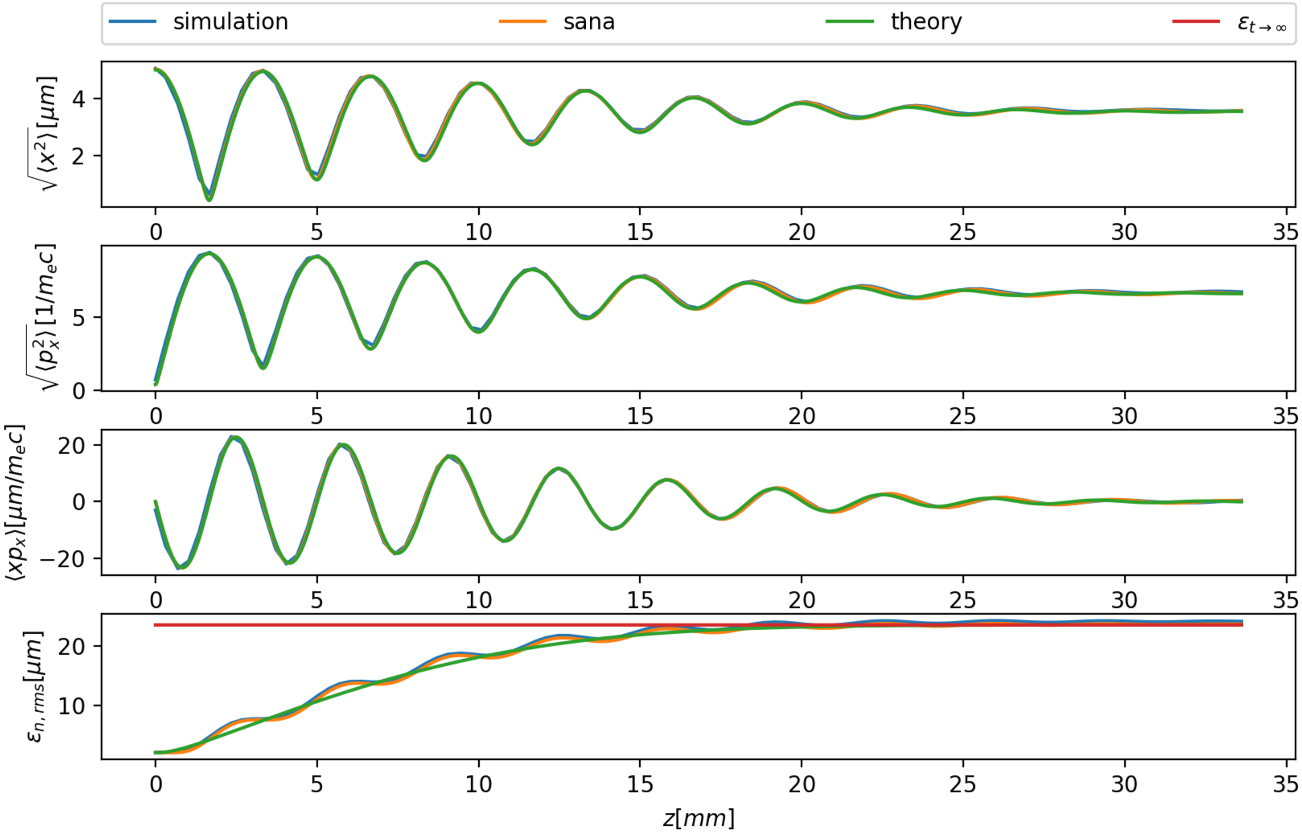
$$\lim_{t\to\infty}\epsilon^2_{n,rms}(\tilde{t}) = \frac{\overline{\gamma_0}}{8}\left\langle x_0^2\right\rangle^2 + \frac{1}{2\overline{\gamma_0}}\left\langle u_{x,0}^2\right\rangle^2 + \frac{1}{2}\left\langle x_0^2\right\rangle\left\langle u_{x,0}^2\right\rangle$$

# Scenario I – comparison between models

- PIC simulation with HiPACE:
  - Driver: I = 3 kA, Q = 240 pC, $\sigma_\gamma/\gamma$ =0.1%, E = 1.0 GeV, $\varepsilon_{n,rms}$ = 1.7 μm, $\sigma_\zeta$ = 10 μm
  - A thin witness slice of macroparticles ($\sigma_\gamma/\gamma$ =10%, E = 1.0 GeV, $\varepsilon_{n,rms}$ = 2 μm, $\sigma_{x,y}$ = 5 μm) at the zero-crossing of $E_z$ to avoid energy gain
  - A homogeneous plasma density distribution
- SANA calculation with same witness beam parameters, assuming $E_z$ = 0
- Analytical formula evaluation for same initial witness beam parameters

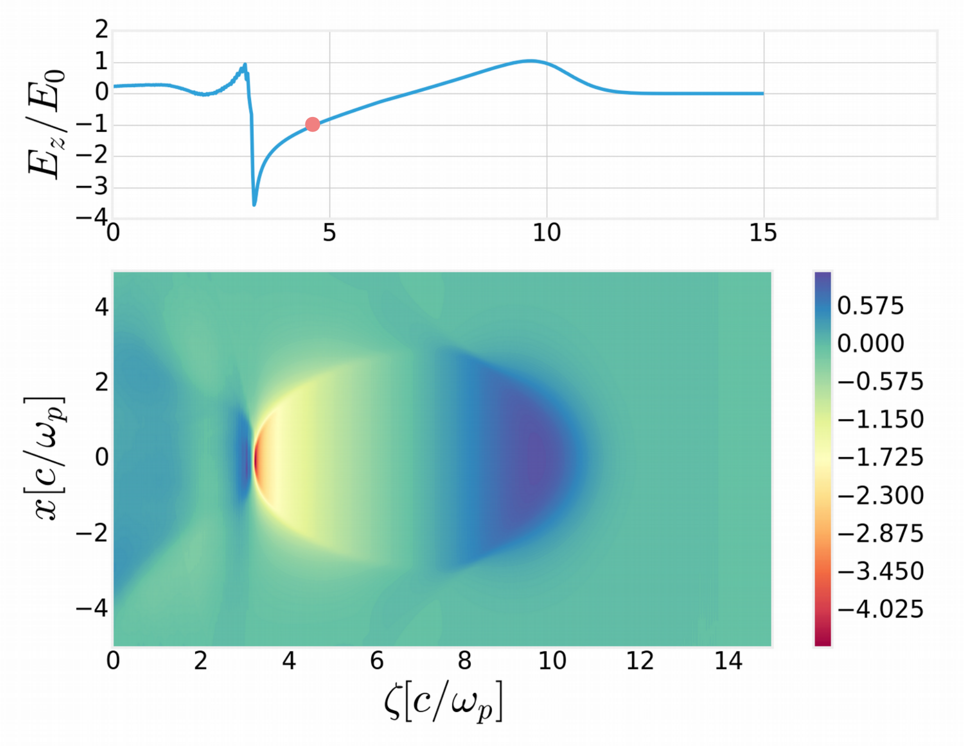


# Scenario I – comparison between models



# Scenario II – single slice with energy gain

- Single slice with predefined Twiss parameters moving through the ion channel
- Moderate energy gain
- Use field information obtained from Particle-in-Cell simulations: sufficiently small macroparticle distribution placed at $E_z = -0.5\ E_0$
- Cross-checked with the SANA model

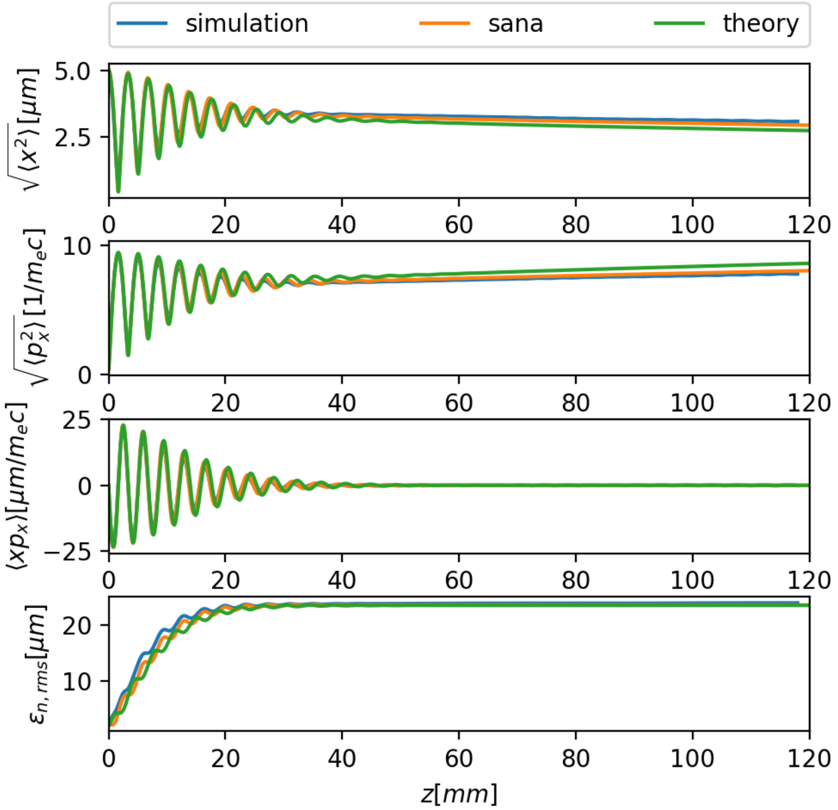


# Scenario II – revisiting the basic formula

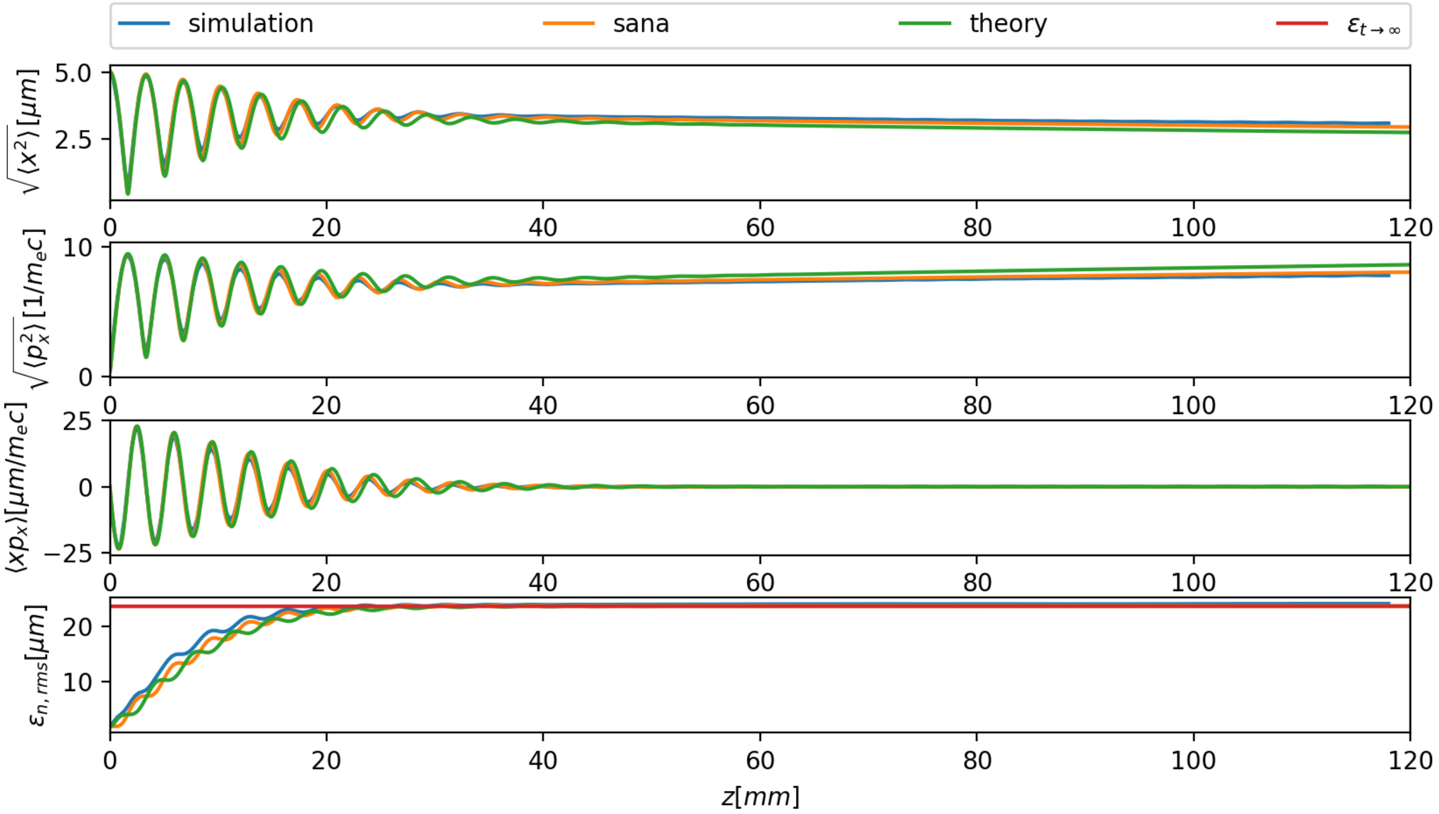
- Revisiting the DGL
$$\frac{d^2x}{dt^2}+\frac{\dot\gamma}{\gamma}\frac{dx}{dt}+\omega_\beta^2 x=0$$
- The single-particle solution now includes an amplitude term
$$x(t)\simeq x_0 A(t)\cos[\varphi(t)]+\frac{p_{x,0}}{m_e\gamma_0\omega_{\beta,0}}A(t)\sin[\varphi(t)]$$
- With $\omega_{\beta,0}=\omega_p/\sqrt{2\gamma_0}$, $A(t)=[\gamma_0/\gamma(t)]^{1/4}$, $\gamma(t)=\overline{\gamma_0}+\mathcal{E}t+\delta\gamma$, $\mathcal{E}=\omega_p E_z/E_0$
- Together with the additional element of acceleration we can introduce a distribution-based phase advance:
$$\varphi(t)=\int\omega_\beta \mathrm{d}t=\bar\varphi\left(1-\frac{\delta\gamma}{2\overline{\gamma_0}}\frac{\overline{\omega_\beta}}{\overline{\omega_{\beta,0}}}\right)$$

# Scenario II – comparison between models

- PIC simulation with HiPACE:
  - Driver: I = 3 kA, Q = 240 pC, $\sigma_\gamma/\gamma$ =0.1%, E = 1.0 GeV, $\varepsilon_{n,rms}$ = 1.7 µm, $\sigma_\zeta$ = 10 µm
  - A thin witness slice of macroparticles ($\sigma_\gamma/\gamma$ =10%, E = 1.0 GeV, $\varepsilon_{n,rms}$ = 2 µm, $\sigma_{x,y}$ = 5 µm) at $E_z$ = 0.5 $E_0$
  - A homogeneous plasma density distribution
- SANA calculation with same witness beam parameters, tuned so that $E_z$ = 0.5 $E_0$
- Analytical formula evaluation for same initial witness beam parameters



# Scenario II – comparison between models



# Summary & Outlook

- Analytical models were obtained, allowing to describe the evolution of beam moments and the emittance for two scenarios:
  - No energy gain for beam slice
  - Energy gain for beam slice
- The analytical descriptions are in agreement with the results provided by simulations and numerical models for the long-term emittance evolution, with minor deviations
- Possible next steps:
  - Include plasma density variations described by an analytical formula
  - Whole-beam description

# References

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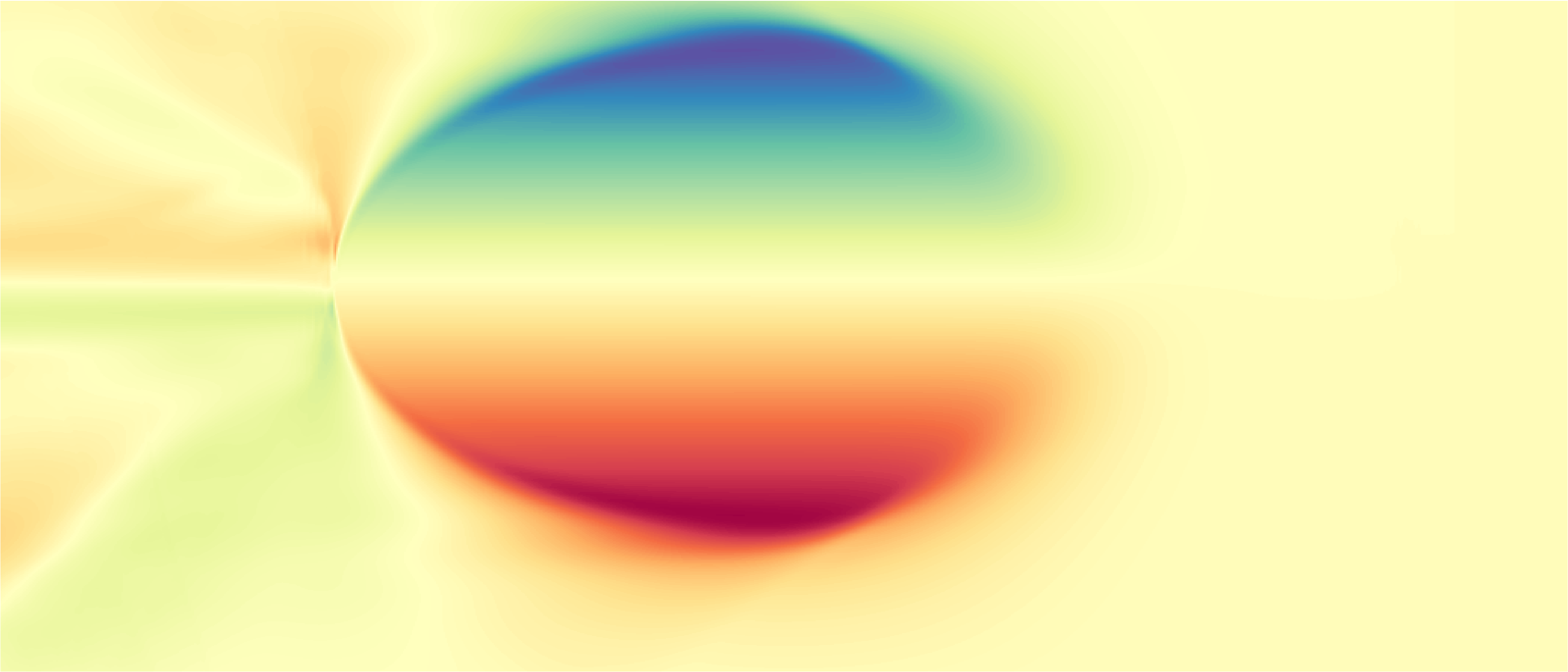
# Questions?



# The scenario and applicable parameters

- Trace-space variables: $x, x' = dx/dz = \dot{x}/\dot{z} = p_x/p_z$
- Courant-Snyder (Twiss) parameters:

$$\hat{\alpha} = -\frac{\langle xx' \rangle}{\hat{\epsilon}}, \quad \hat{\beta} = \frac{\left\langle x^2 \right\rangle}{\hat{\epsilon}}, \quad \hat{\gamma} = \frac{\left\langle x'^2 \right\rangle}{\hat{\epsilon}}$$

- Trace-space emittance: $\hat{\epsilon} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$
- Normalized transverse phase-space emittance:

$$\epsilon_n = \frac{1}{m_e c} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2}$$

# SANA – semi-analytic numerical approach

$$\partial_t \left\langle x^2 \right\rangle_k = 2 \left\langle x p_x \right\rangle_k / \gamma_k$$

$$\partial_t \left\langle x p_x \right\rangle_k = \left\langle p_x^2 \right\rangle_k / \gamma_k - \hat{k}_x(t) \left\langle x^2 \right\rangle_k$$

$$\partial_t \left\langle p_x^2 \right\rangle_k = -2\hat{k}_x(t) \left\langle x p_x \right\rangle_k$$

$$\partial_t \gamma_k = F_z(\zeta_k, t)$$

$$\hat{k}_x = \frac{d}{dx} \left(E_x - B_y\right)\Big|_{x=0} = \frac{n(z)}{2n_0}$$

$$F_z(\zeta, z) = \frac{dEz}{d\zeta} \left( \frac{n(z)}{n_0} (\zeta - \zeta_d) - \sqrt{\frac{n(z)}{n_0}} d_b \right)$$

*T. Mehrling et al, NIM A 2016.01.091 (2016)

# Scenario I – the formulas used

$$x(t) = \frac{p_{x,0}}{\overline{\gamma_0} m_e \overline{\omega_{\beta,0}}} \sin\left(t\left(1-\frac{\delta\gamma}{2\overline{\gamma_0}}\right)\overline{\omega_{\beta,0}}\right) + x_0 \cos\left(t\left(1-\frac{\delta\gamma}{2\overline{\gamma_0}}\right)\overline{\omega_{\beta,0}}\right)$$

$$p_x(t) = p_{x,0} \cos\left(t\left(1-\frac{\delta\gamma}{2\overline{\gamma_0}}\right)\overline{\omega_{\beta,0}}\right) - x_0 \overline{\gamma_0} m_e \overline{\omega_{\beta,0}} \sin\left(t\left(1-\frac{\delta\gamma}{2\overline{\gamma_0}}\right)\overline{\omega_{\beta,0}}\right)$$

# Scenario II – revisiting the basic formula

$$\frac{d^2x}{dt^2} + \frac{\dot{\gamma}}{\gamma}\frac{dx}{dt} + \omega_\beta^2 x = 0$$

$$x(t) \simeq x_0 A(t)\cos[\varphi(t)] + \frac{p_{x,0}}{m_e \gamma_0 \omega_{\beta,0}} A(t)\sin[\varphi(t)]$$

$$\omega_{\beta,0} = \omega_p/\sqrt{2\gamma_0}, \quad A(t) = [\gamma_0/\gamma(t)]^{1/4}, \quad \gamma(t) = \overline{\gamma_0} + \mathcal{E}t + \delta\gamma, \quad \mathcal{E} = \omega_p E_z/E_0$$

$$\varphi(t) = \int \omega_\beta \mathrm{d}t = \bar{\varphi}\left(1 - \frac{\delta\gamma}{2\overline{\gamma_0}}\frac{\overline{\omega_\beta}}{\overline{\omega_{\beta,0}}}\right)$$

$$\bar{\varphi} = \frac{2\left(\overline{\omega_{\beta,0}}/\overline{\omega_\beta} - 1\right)}{\epsilon}, \quad \overline{\omega_\beta} = \frac{\overline{\omega_{\beta,0}}}{\sqrt{\epsilon\overline{\omega_{\beta,0}}t + 1}}, \quad \epsilon = \frac{E_z}{E_0}\left(-\sqrt{\frac{2}{\overline{\gamma_0}}}\right)$$