ICFA mini-workshop on impedances and beam instabilities in particle accelerators – Benevento 2017

Vlasov solvers and macroparticle simulations

in the context of transverse coherent instabilities

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Introduction

- From a given machine & impedance model, there are two wide-spread techniques to compute **transverse coherent instabilities growth rates** & the effectiveness of possible mitigation techniques (e.g. damper, non-linearities):
  - Macroparticle tracking simulations

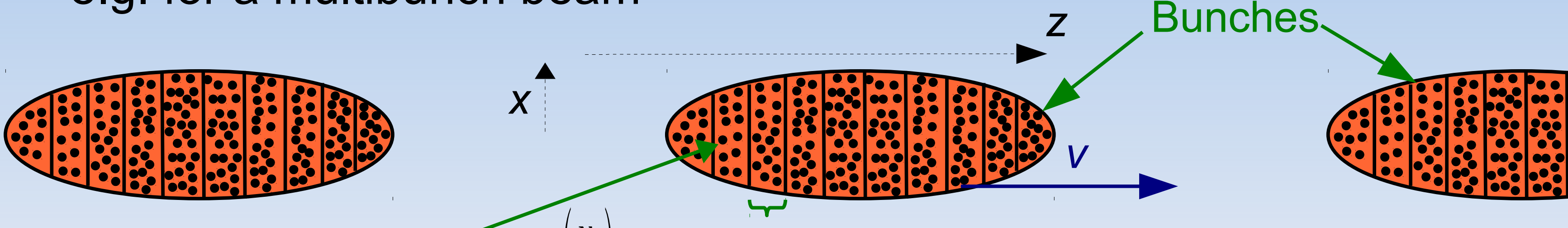
    → discretization of the beam into a discrete set of macroparticles which are tracked through the machine and on which kicks from **wake functions** are applied.
  - Vlasov equation solvers

    → beam distribution in phase space is taken as a whole, and one computes the possible (unstable) **modes** arising from **impedance**, by solving Vlasov equation.

Macroparticle simulations

- Example: **HEADTAIL** code (*G. Rumolo et al*, PRST-AB, 2002):

  e.g. for a multibunch beam



Each step (turn or fraction of turn):

- macropart. *i* receives kick from the wake of all preceding slices: $\begin{pmatrix} x_i \\ x_i' \end{pmatrix} \rightarrow \begin{pmatrix} x_i \\ x_i' + \Delta x_i'(x_S, x_{S_i}, z_S - z_{S_i}) \end{pmatrix}$
- then it is transported through the machine lattice: $\begin{pmatrix} x_i \\ x_i' \end{pmatrix} \rightarrow M \cdot \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$
- (similar treatment for the other components of the macroparticle y_i, z_i).

  - The kick results from the combined action of the wake functions of all the preceding bunches:

$$\Delta x_i' = \frac{e^2}{m_0 \gamma v^2} \sum_{z_S > z_{S_i}} n_S W_x \left(z_S - z_{S_i}, x_S, y_S, x_{S_i}, y_{S_i} \right),$$

with $W_x(z, x_S, y_S, x_{S_i}, y_{S_i}) = W_x^{dip}(z) x_S + W_{xy}^{dip}(z) y_S + W_x^{quad}(z) x_{S_i} + W_{xy}^{quad}(z) y_{S_i},$

n_S, x_S, y_S, z_S: number of particles and position of the slice *S*.

Macroparticle simulations

- What they can do:
  - Assess stability in complex, realistic situations (with e.g. localized impedance sources, feedback damper, transverse and longitudinal non-linearities, space-charge, even electron cloud)
- But...
  - they are computationally intensive, especially to evaluate the combined effect of coupled-bunch and intra bunch motion, and for slow instabilities,
  - they remain a time domain tool → essentially unable to predict what happens after an infinite time and we can never be 100% sure that a given configuration is stable (see examples later).
- Examples of macroparticle simulation codes: HEADTAIL, MTRISM, PTC-ORBIT ...

Vlasov solvers

[A. A. Vlasov, J. Phys. USSR 9, 25 (1945)]

- Vlasov equation expresses that the local phase space density does not change when one follows the flow of particles.
- In other words: local phase space area is conserved in time:

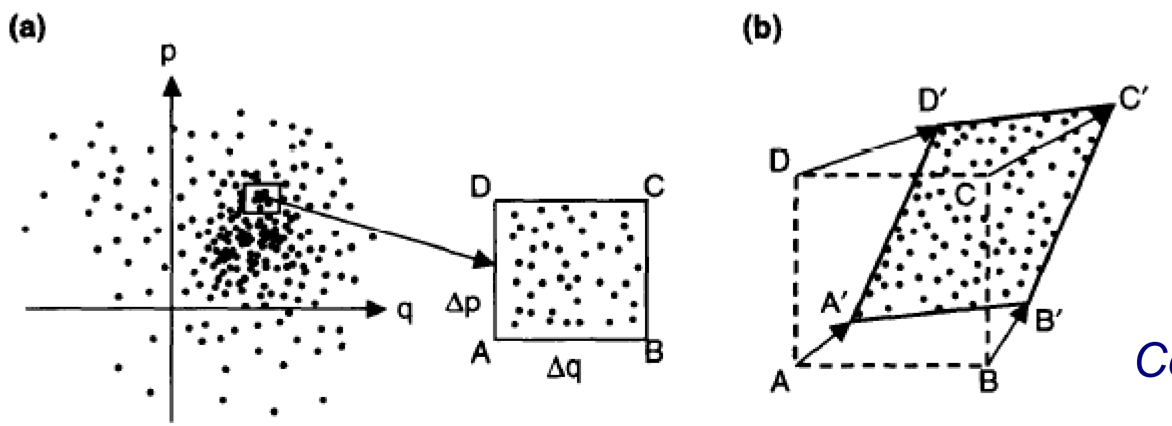


Figure 6.3. (a) Phase space distribution of particles at time t. A rectangular box $ABCD$ with area $\Delta q \, \Delta p$ is drawn and magnified. (b) At a later time, $t + dt$, the box moves and deforms into a parellelogram with the same area as $ABCD$. All particles inside the box move with the box.

Courtesy A. W. Chao

- Assumptions:
  - conservative & deterministic system (governed by Hamiltonian) – no damping or diffusion from external sources (no synchrotron radiation),
  - external forces (no discrete internal force or collision).
    → impedance seen as a *collective field from ensemble of particles.*

Vlasov equation

- Simplest expression: with ψ the general 6D phase space distribution density (and *t* the time),
$$\frac{d\psi}{dt} = 0,$$
- Following standard approach from A. W. Chao (*Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley & Sons (1993), chap. 6), assuming no x/y coupling (4D):
  - independent variable $s=v\,t$
  - transverse motion: $(y, p_y) \Leftrightarrow (J_y, \theta_y)$ (action/angle),
  - longitudinal motion: (z, δ)

$$\rightarrow \frac{\partial\psi}{\partial s} + J'_y \frac{\partial\psi}{\partial J_y} + \theta'_y \frac{\partial\psi}{\partial\theta_y} + z' \frac{\partial\psi}{\partial z} + \delta' \frac{\partial\psi}{\partial\delta} = 0.$$

Hamiltonian

- In the presence of a dipolar vertical impedance resulting in a force $F_y(z,s)$:

synchrotron freq. slippage factor

$$H = \frac{Q_y}{R} J_y - \frac{1}{2\eta}\left(\frac{\omega_s}{v}\right)^2 z^2 - \frac{\eta}{2}\delta^2 - \frac{y}{E} F_y(z,s)$$

machine radius transverse part longitudinal part (linear) dipolar wake fields

with $Q_y = Q_{y0} + Q'_y \delta,$

velocity=βc

total energy

chromaticity

and

unperturbed tune

$$J_y = \frac{1}{2}\left(\frac{Q_{y0}}{R} y^2 + \frac{R}{Q_{y0}} p_y^2\right),$$

$$y = \sqrt{2J_y \frac{R}{Q_{y0}}} \cos\theta_y,$$

$$p_y = \sqrt{2J_y \frac{Q_{y0}}{R}} \sin\theta_y.$$

→ important assumption : invariant (and action-angle variables) stay as in linear case.

Hamilton's equations

- Give derivatives of J_y, θ_y, z and δ w.r.t. s:

$$
\begin{aligned}
J'_y &= -\frac{\partial H}{\partial \theta_y} = \frac{\partial y}{\partial \theta_y} \frac{F_y(z,s)}{E}, \\
\theta'_y &= \frac{\partial H}{\partial J_y} = \frac{Q_y}{R} - \frac{\partial y}{\partial J_y} \frac{F_y(z,s)}{E}, \\
z' &= \frac{\partial H}{\partial \delta} = -\eta\delta + \cancel{\frac{Q'_y}{R} J_y}, \\
\delta' &= -\frac{\partial H}{\partial z} = \left(\frac{\omega_s}{v}\right)^2 \frac{z}{\eta} + \cancel{\frac{y}{E} \frac{\partial F_y}{\partial z}}.
\end{aligned}
$$

From dipolar wake fields

Neglected (Chao)

Neglected (Chao: OK when far from synchro-betatron resonances & small transverse beam size)

How to solve Vlasov equation ?

- Equation remains quite complicated: partial differential eq. for distribution function $\psi\,(s, J_y, \theta_y, z, \delta)$.
- To simplify the problem:
  - Assume a mode is developping in the bunch along the revolutions, with a certain (complex) frequency $\Omega = Q_c \omega_0$,
  - Assume we stay close to the stationary unperturbed distribution ψ_0, function of invariants J_y and $r = \sqrt{z^2 + \frac{\eta^2 v^2 \delta^2}{\omega_s^2}}$

$\rightarrow$ perturbation formalism:

$$\psi = f_0(J_y) g_0(r) + f_1(J_y, \theta_y) g_1(z, \delta) e^{\frac{j\Omega s}{v}}.$$

$\Delta\psi_1$: self-consistent perturbation to be found

and we write f_1 as $f_1(J_y, \theta_y) = f(J_y) e^{-j\theta_y}.$

while g_1 is expanded over azimuthal (headtail) modes $R_l(r)$.

Sacherer integral equation (extended)

- After lots of algebra, defining $\tau = r/v$ and generalizing to M equidistant bunches of intensity per bunch N with the usual assumption that they all oscillate in the same way, we get:

damper part (NHTVS, DELPHI)

$$\left(\Omega - Q_{y0}\omega_0 - l\omega_s\right) R_l(\tau) = - \kappa g_0(\tau) \sum_{l'=-\infty}^{+\infty} j^{l'-l} \int_0^{+\infty} \tau' R_l(\tau') \left[\frac{\mu}{\omega_0} J_l\left(-\omega_\xi \tau\right) J_{l'}\left(-\omega_\xi \tau'\right) \mathrm{d}\tau' + \sum_{p=-\infty}^{+\infty} Z_y(\omega_p) J_l\left((\omega_\xi - \omega_p)\tau\right) J_{l'}\left((\omega_\xi - \omega_p)\tau'\right) \right].$$

impedance part (Chao's book, Sacherer formula, Laclare, MOSES, NHTVS, DELPHI)

$Q'_y\, \omega_0 / \eta$

$$\kappa = -j \frac{N f_0 e^2 M}{2\gamma m_0 c Q_{y0}}, \qquad \frac{-\kappa\mu}{\omega_0} = j \frac{f_0}{n_d} \qquad \omega_p = \left(n + pM + [Q_{y0}]\right)\omega_0,$$

coupled-bunch mode

n° damping turns

tune fractional part

Solving Vlasov equation

- Using appropriate decompositions (e.g. over Laguerre polynomials of the radial functions, as in MOSES or DELPHI), Sacherer integral equation (extended or not) can be set as an eigenvalue problem:

$$\underbrace{(\Omega - Q_{y0}\omega_0)}_{\text{eigenvalue looked for}}\alpha_{ln} = \sum_{l'=-\infty}^{+\infty}\sum_{n'=0}^{+\infty} \alpha_{l'n'} \left(\delta_{ll'}\delta_{nn'} l\omega_s + \underbrace{\mathcal{M}_{ln,l'n'}}_{\text{impedance and/or damper matrix}}\right).$$

eigenvector

- Can be extended into a non-linear equation of Q_c to include also transverse non-linearities:

$$det\left(\left[\delta_{ll'}\delta_{nn'}\frac{\omega_0}{I_l(Q_c)}\right] + [\mathcal{M}_{ln,l'n'}]\right) = 0.$$

(DELPHI)

with the dispersion integral:

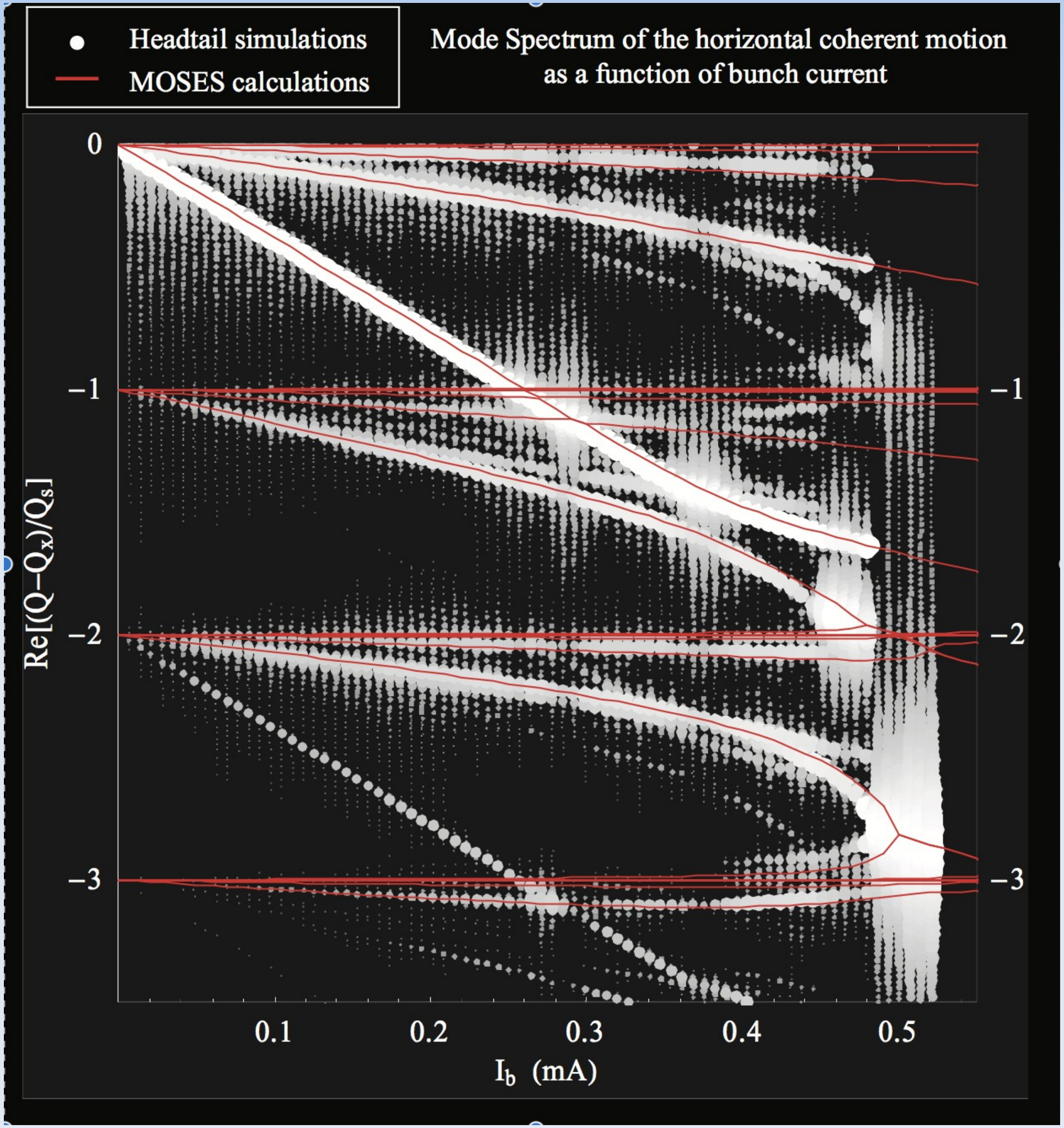
$$I_l(Q_c) = \iint_0^{+\infty} \mathrm{d}J_x \mathrm{d}J_y \frac{\frac{\partial f_0(J_x,J_y)}{\partial J_y} J_y}{Q_c - Q_{y0} - a_{yy}J_y - a_{yx}J_x - lQ_s},$$

Vlasov solvers

- What they can do:
  - predict growth rates from the knowledge of the beam-coupling impedance and (semi-) analytical formulas → very fast,
  - provides vision & understanding of the modes that can develop in a beam,
  - perform large parameters scans.
- But...
  - they rely on a number of reasonable but always questionable assumptions,
  - they work only in simplified & idealized situations,
  - any additional ingredient to be put in such a solver, typically requires pages of analytical derivations...
- Examples of Vlasov solvers: Laclare's formalism, MOSES (Y. Chin), NHTVS (A. Burov), DELPHI, ...

How do macroparticle simulations and Vlasov solvers compare?

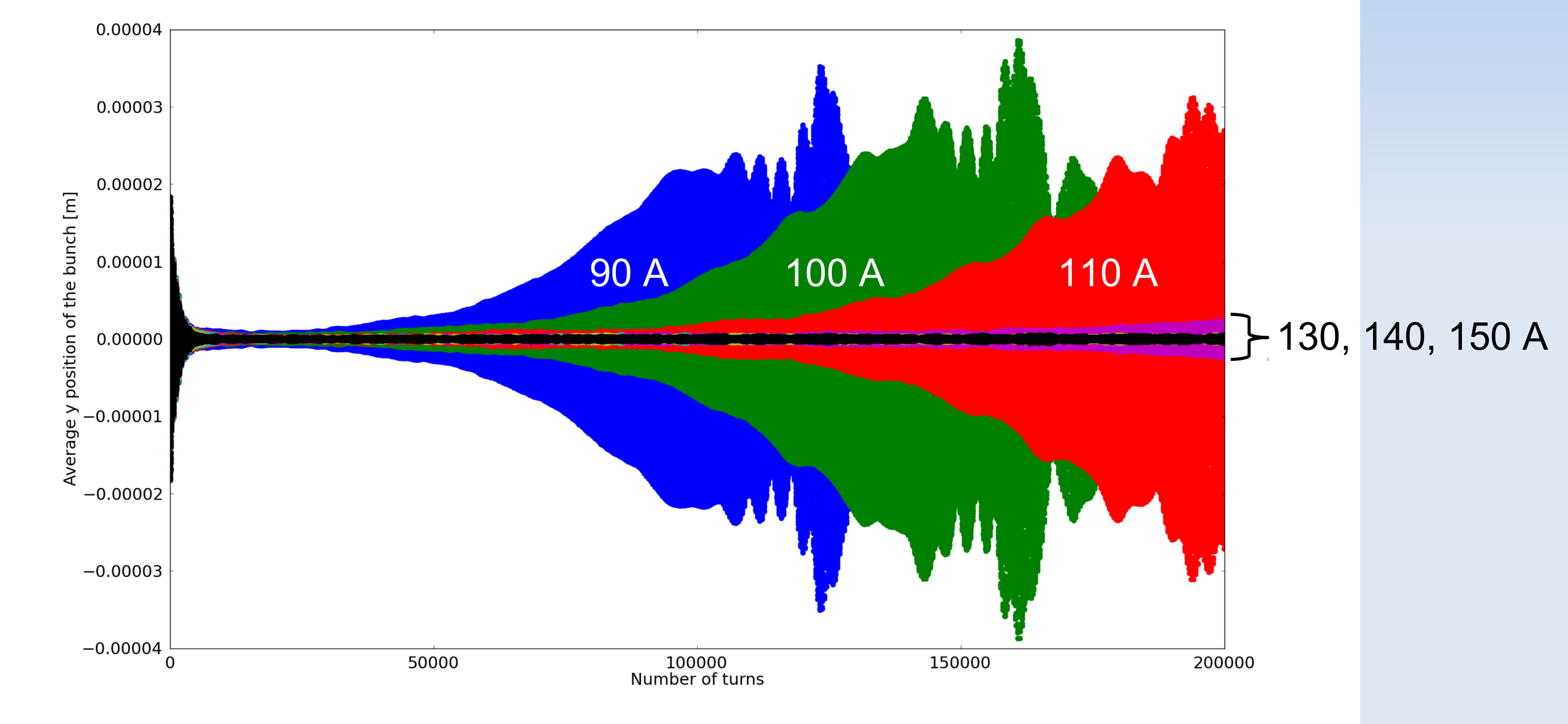
- MOSES vs HEADTAIL for the SPS impedance model:



From Benoît Salvant' PhD thesis, EPFL n°4585 (2010)

Example of macroparticle simulations: LHC single bunch with octupoles

- Bunch centroid from HEADTAIL simulations with various currents in the octupoles:



⇒ 120 A < stabilizing I_{oct} < 130 A?

⇒ But what if we simulate more turns?

Re-analysing LEP TMCI with the help of a Vlasov solver

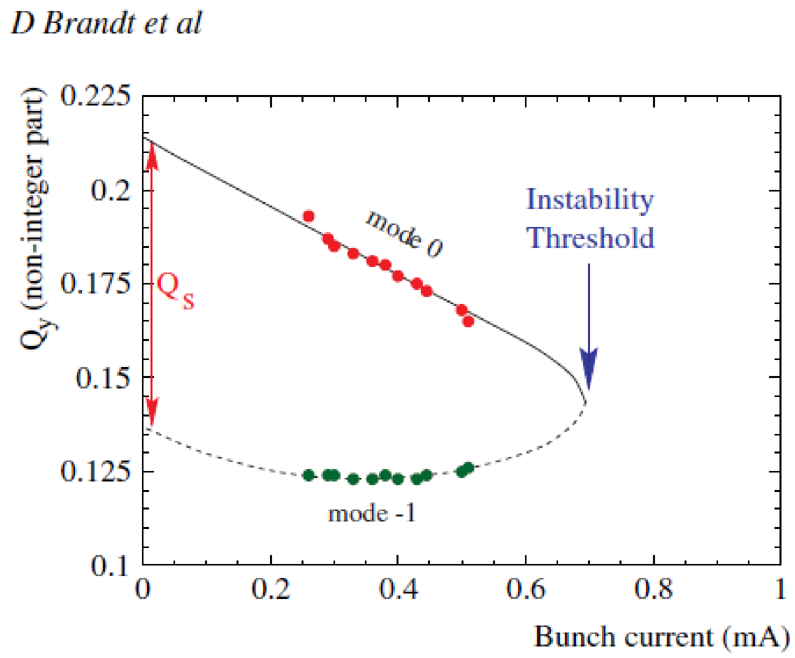


Figure 12. Measurement of the 0 and −1 modes of oscillation as a function of the bunch current at LEP for $Q_s = 0.082$. As the current increases the two modes approach until they merge at the instability threshold.

- Impedance model: two broad-band resonators (RF cavities + bellows), the rest is relatively small (<10%) [G. Sabbi, 1995].

  → experimental tune shifts and TMCI threshold (from simple formula) well reproduced,

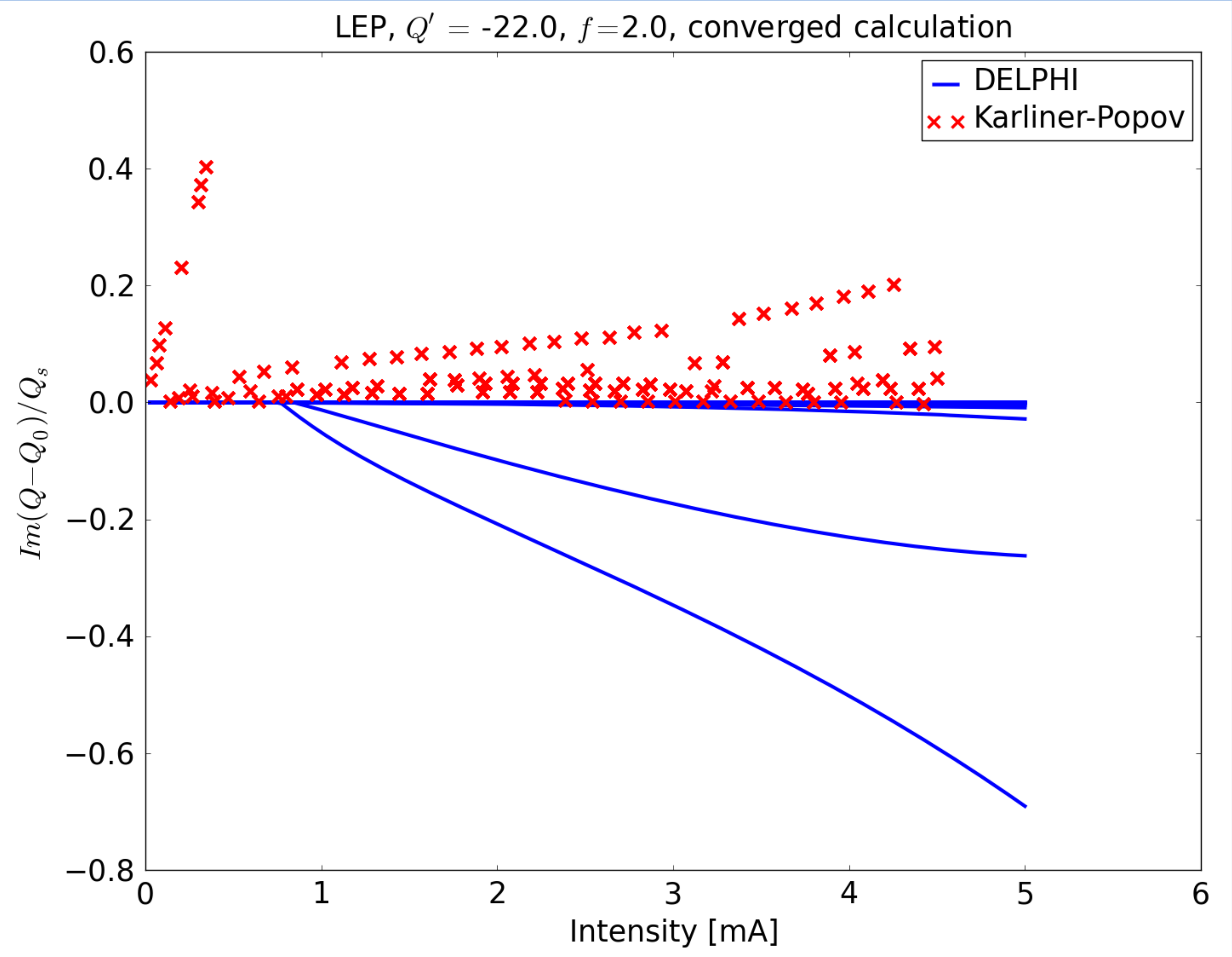
  → threshold slightly less than 1mA.

- Transverse feedback: several ideas
  - reactive feedback (prevent mode 0 to shift down and couple with mode -1) → not more than 5-10 % increase in threshold, despite several attemps and models developped [Danilov-Perevedentsev 1993, Sabbi 1996, Brandt et al 1995],
  - resistive feedback, first found ineffective [Ruth 1983], tried at LEP but never used in operation. More recently (2005) thought to be a good option by Karliner-Popov with a possible increase by factor ~5 of TMCI threshold.

LEP

- LEP with resistive damper (typical LEP2 parameters)

Imag. part, Q'=-22

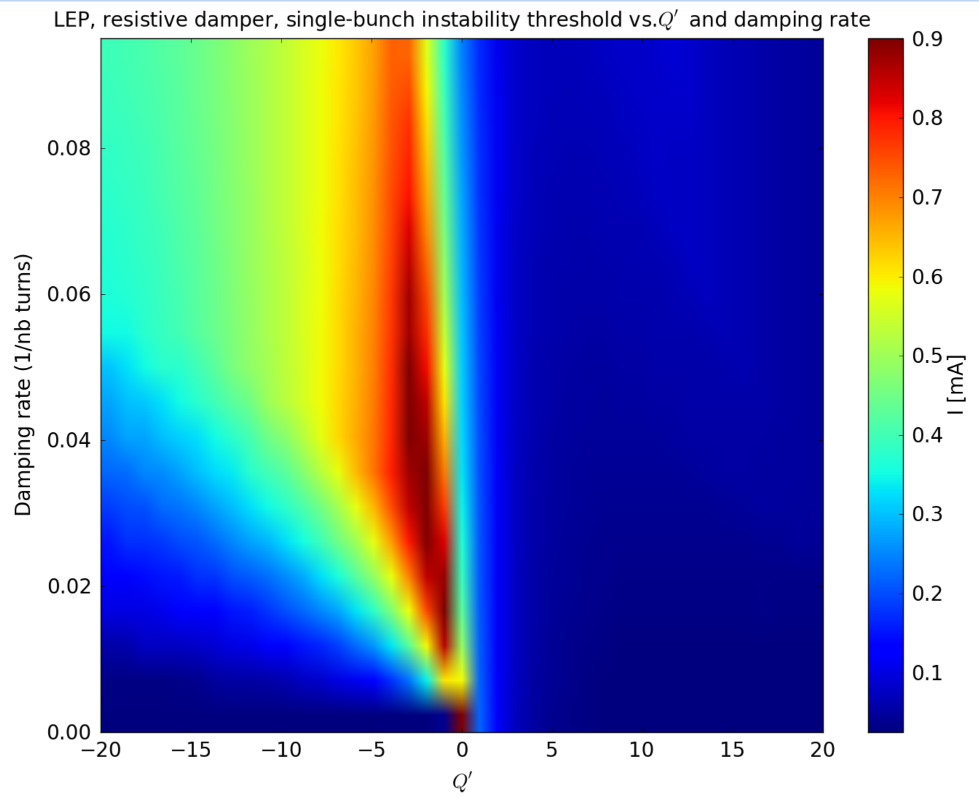


→ we cannot reproduce the result of Karliner & Popov.

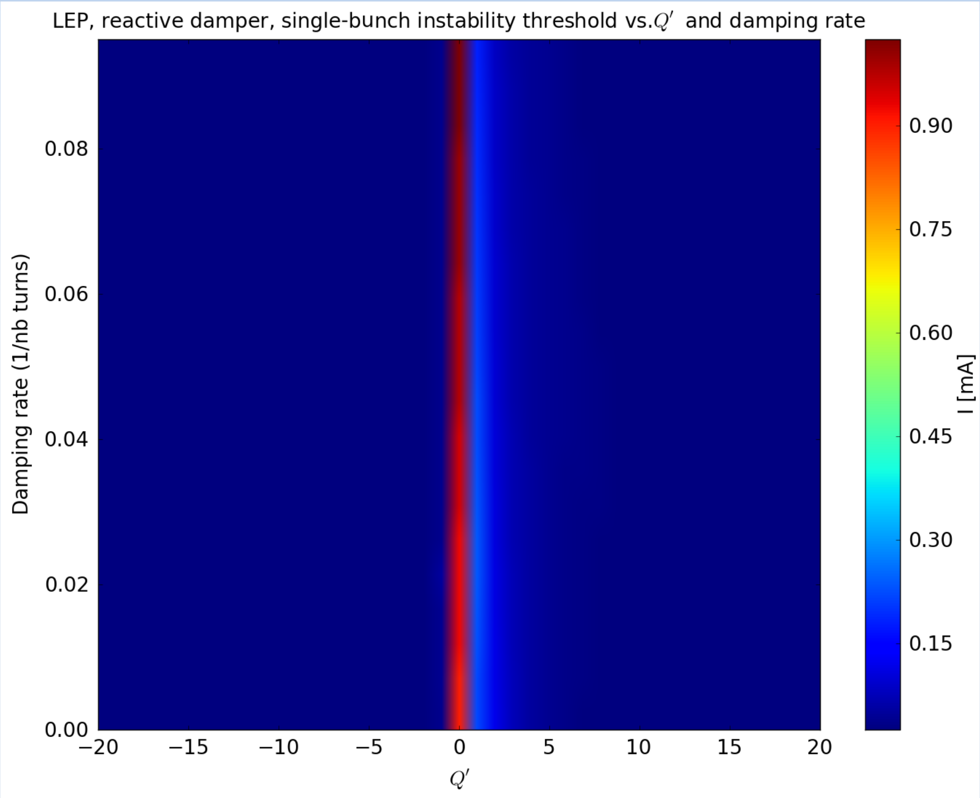
LEP: TMCI with damper

- Instability threshold vs. Q' and damper gain (up to 10 turns) with DELPHI:

Resistive damper



Reactive damper



Essentially, one cannot do better than the natural (i.e. without damper) TMCI threshold.

We can do a little better than the ”natural” TMCI.
→ seems to match (qualitatively) LEP observations.

Conclusions

- Vlasov solvers and macroparticle simulations are two equivalent ways to predict coherent instabilities; they give essentially the same results when they can be applied to the same situations.
- Macroparticle simulation codes are simple in essence, so easily extensible, while Vlasov solvers typically require complete re-derivations of complex analytical formulas when any ingredient has to be added.
- Macroparticle simulations can deal with very complex beam & machine configurations.
- Vlasov solvers are to be used in simplified and idealized problems, because of the assumptions needed to solve the complex Vlasov equation.
- Macroparticle simulations are typically much more computationally intensive than Vlasov solvers, which therefore can be used to make broad parameter scans. Macroparticle codes can then be used to refine the analysis.
- Macroparticle simulations are in any case *fundamentally* unable to know what happens after an *infinite time*. Vlasov solvers, on the contrary, can spot even very slowly growing modes.

Backup slides

Vlasov solvers

- Another possibility: Vlasov solver in ”mode domain”

  → solver that looks for all modes that can develop, among which one can easily spot the most critical (i.e. unstable).

- Idea is not new (non exhaustive list):
  - Laclare formalism [J. L. Laclare, CERN-87-03-V-1, p. 264],
  - MOSES [Y. Chin, CERN/SPS/85-2 & CERN/LEP-TH/88-05],
  - NHTVS [A. Burov, Phys. Rev. ST AB 17, 021007 (2014)].

Getting to Sacherer integral equation

- Outline:
  - Vlasov equation
  - Hamiltonian
  - Perturbative approach adopted
  - Impedance term

  → Sacherer integral equation for transverse modes.
- We follow here an approach largely inspired from A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley & Sons (1993), chap. 6.
- Note: here, unlike Chao we use ”engineer” convention for the Fourier transform → $e^{j\omega t}$ (unstable modes have imag. part<0). Also, SI units (c.g.s in Chao), and notations often different.

The trick is to find appropriate decompositions...

- Writing f_1 as a Fourier series

$$f_1(J_y, \theta_y) = \sum_{k=-\infty}^{+\infty} f_1^k(J_y) e^{jk\theta_y},$$

we can show that all f_1^k are zero except for $k=-1$ (this is exact except for $k=1$ for which it relies on $|Q_c - Q_y| << |Q_c + Q_y|$)

$$\rightarrow \quad f_1(J_y, \theta_y) = f(J_y) e^{-j\theta_y}.$$

- For g_1 decomposition is more subtle:

$$g_1(r, \phi) = e^{\frac{-jQ_y' z}{\eta R}} \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi},$$

→ azimuthal modes decomposition

to cancel some term in Vlasov eq.

Rewriting Vlasov equation

- Use polar coordinates in longitudinal:

$$\begin{aligned} z &= r\cos\phi, \\ \frac{\eta v}{\omega_s}\delta &= r\sin\phi. \end{aligned}$$

- After some algebra, neglecting second order terms proportional to $\Delta\psi_1$ F_y (wake field force assumed to be small):

$$e^{\frac{j\Omega s}{v}}\left(f_1 g_1 \frac{j\Omega s}{v} + \frac{Q_y}{R}\frac{\partial f_1}{\partial \theta_y} g_1 + \frac{\omega_s}{v} f_1 \frac{\partial g_1}{\partial \phi}\right) = \frac{\sin\theta_y}{E}\sqrt{\frac{2J_y R}{Q_{y0}}} F_y(z,s) g_0(r) f_0'(J_y).$$

$$Q_y = Q_{y0} + Q_y'\delta,$$

The trick is to find appropriate decompositions...

- After such decompositions, Vlasov eq. now looks like

$$\sum_{l=-\infty}^{+\infty} R_l(r) \underbrace{\left[\frac{f(J_y)(Q_c - Q_{y0} - lQ_s)}{f_0'(J_y)\sqrt{\frac{2J_yR}{Q_{y0}}}} \right]}_{\text{must be constant w.r.t } J_y} e^{-jl\phi} = \frac{Re^{-j\frac{Q_c s}{R}}}{2E} F_y(z,s) e^{\frac{jQ_y' z}{\eta R}}$$

must be constant w.r.t J_y → $f(J_y) = D f_0'(J_y)\sqrt{\frac{2J_yR}{Q_{y0}}}.$ → dipole mode

- Next step is to evaluate F_y:
  - written initially as a 4D integral (convolution of the wake in *z*, weighted by total distribution function):

$$F_y(z,s) = \frac{e^2}{2\pi R} \underbrace{\sum_{k=-\infty}^{+\infty} e^{\frac{jQ_c(s-2\pi kR)}{R}}}_{\text{multiturn sum (importance of self-consistency)}} \iiiint \mathrm{d}\tilde{z}\mathrm{d}\tilde{\delta}\mathrm{d}J_y\mathrm{d}\theta_y \underbrace{\psi(\tilde{z},\tilde{\delta},J_y,\theta_y)}_{\text{distribution}} y \underbrace{W_y(z-\tilde{z})}_{\text{wake f}^{\text{o}}}.$$

How to write the wake fields force

- Using the fact that the unperturbed distribution has no dipole moment, and the previous decompositions:

$$F_y(z,s) = \frac{e^2}{2Q_{y0}} e^{\frac{jQ_c s}{R}} \sum_{l=-\infty}^{+\infty} \left(D \int_0^{+\infty} \mathrm{d}J_y f_0'(J_y) J_y \right) \cdot \underbrace{\sum_{k=-\infty}^{+\infty} e^{-2\pi j Q_c k} \int \mathrm{d}\tilde{z} W_y(z-\tilde{z}) \int \mathrm{d}\tilde{\delta} e^{\frac{-jQ_y' \tilde{z}}{\eta R}} R_l(r) e^{-jl\phi}.}_{S(z)}$$

- After some tricks we get for an impedance (details in Chao):

$$S(z) = -\frac{vQ_s}{\eta R^2} j^{-l} \sum_{p=-\infty}^{+\infty} e^{-j(Q_c+p)\frac{z}{R}} Z_y\left[-\omega_0(Q_c+p)\right] \int_0^{+\infty} rR_l(r) J_l \left[(\omega_\xi - \omega_0(Q_c+p)) \frac{r}{v} \right] \mathrm{d}r.$$

$Q'_y\ \omega_0 / \eta$

and for an ideal damper (constant imag. wake, no multiturn):

$$S(z) \propto -\frac{vQ_s}{\eta R^2} j^{-l} \int_0^{+\infty} rR_l(r) J_l\left(\frac{\omega_\xi r}{v}\right) \mathrm{d}r.$$

How are we going to solve this ?

- Using a decomposition over Laguerre polynomials of the radial functions (idea from Besnier 1974, used then by Y. Chin in code MOSES – 1985):

$$R_l(\tau) = \left(\frac{\tau}{\tau_b}\right)^{|l|} e^{-b\tau^2} \sum_{n=0}^{+\infty} c_n^l L_n^{|l|}(a\tau^2),$$

Laguerre polynomial

$$g_0(\tau) = e^{-b\tau^2} \sum_{k=0}^{n_0} g_k L_k^0(a\tau^2),$$

→ in principle any long. distribution can be put in.

- Then the following integrals can be computed analytically:

$$\int_0^{+\infty} \tau^{1+|l|} L_n^{|l|}(a\tau^2) e^{(b-a)\tau^2} g_0(\tau) J_l(\omega\tau) \mathrm{d}\tau,$$

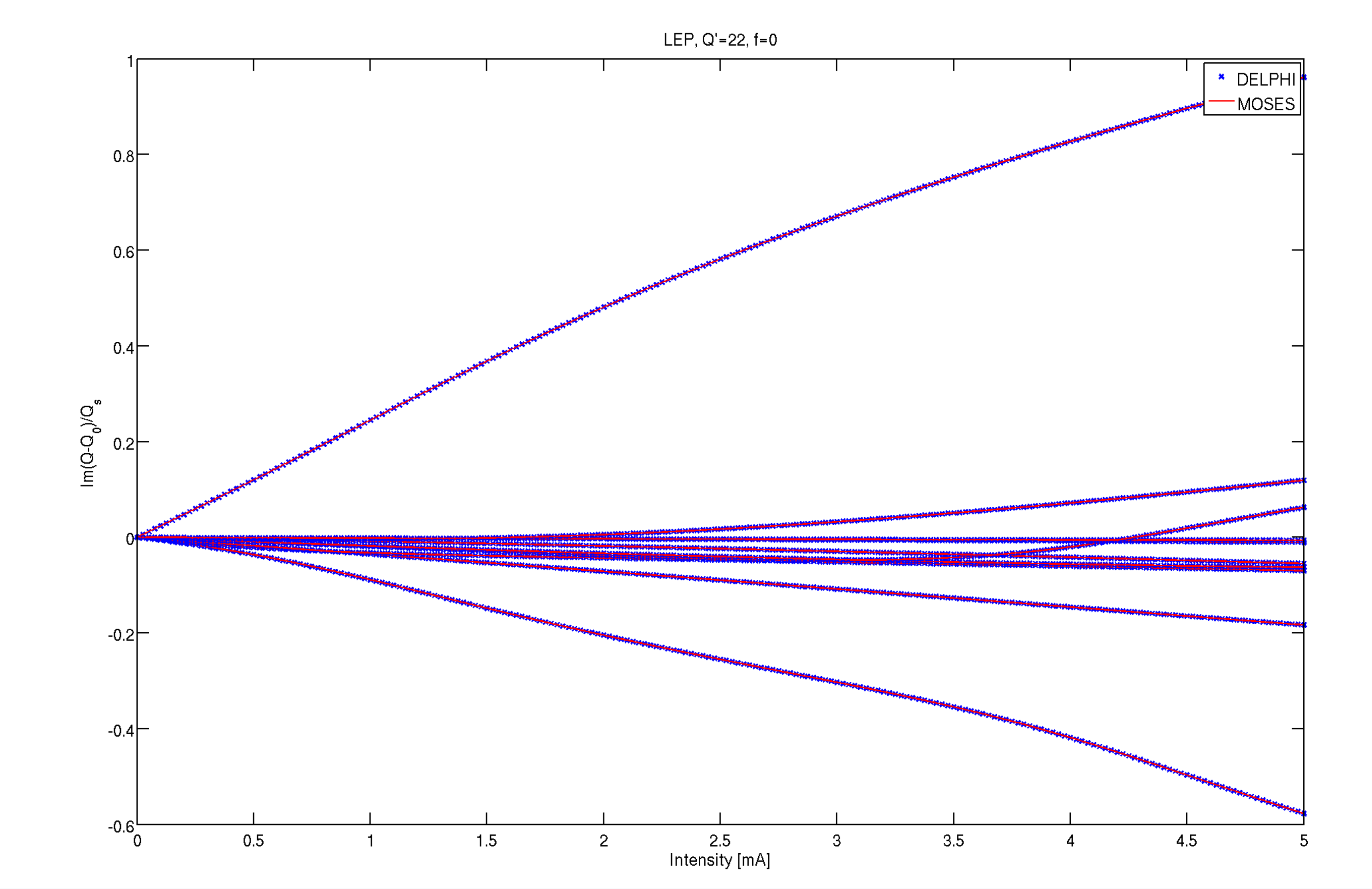
$$\int_0^{+\infty} \tau^{1+|l|} L_n^{|l|}(a\tau^2) e^{-b\tau^2} J_l(\omega\tau) \mathrm{d}\tau,$$

→ can also play with parameters *a* & *b*
→ exponentials make impedance sum convergence easy.

Benchmarks

- DELPHI vs MOSES, single-bunch without damper (LEP RF cavities modeled as a broadband resonator):

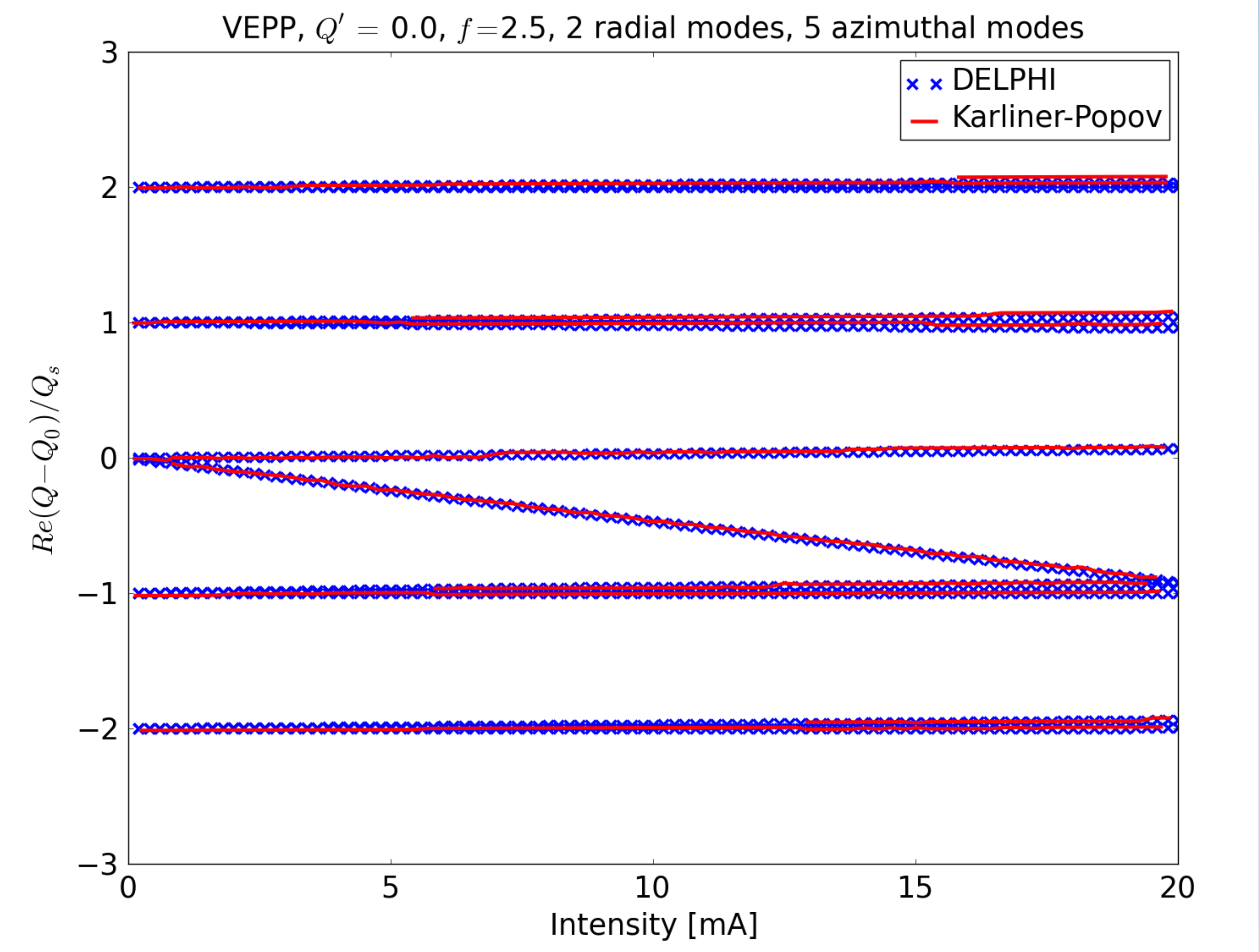
Imag. part, Q'=22



Benchmarks

- DELPHI vs Karliner-Popov, single-bunch with damper (VEPP-4, broadband resonator):

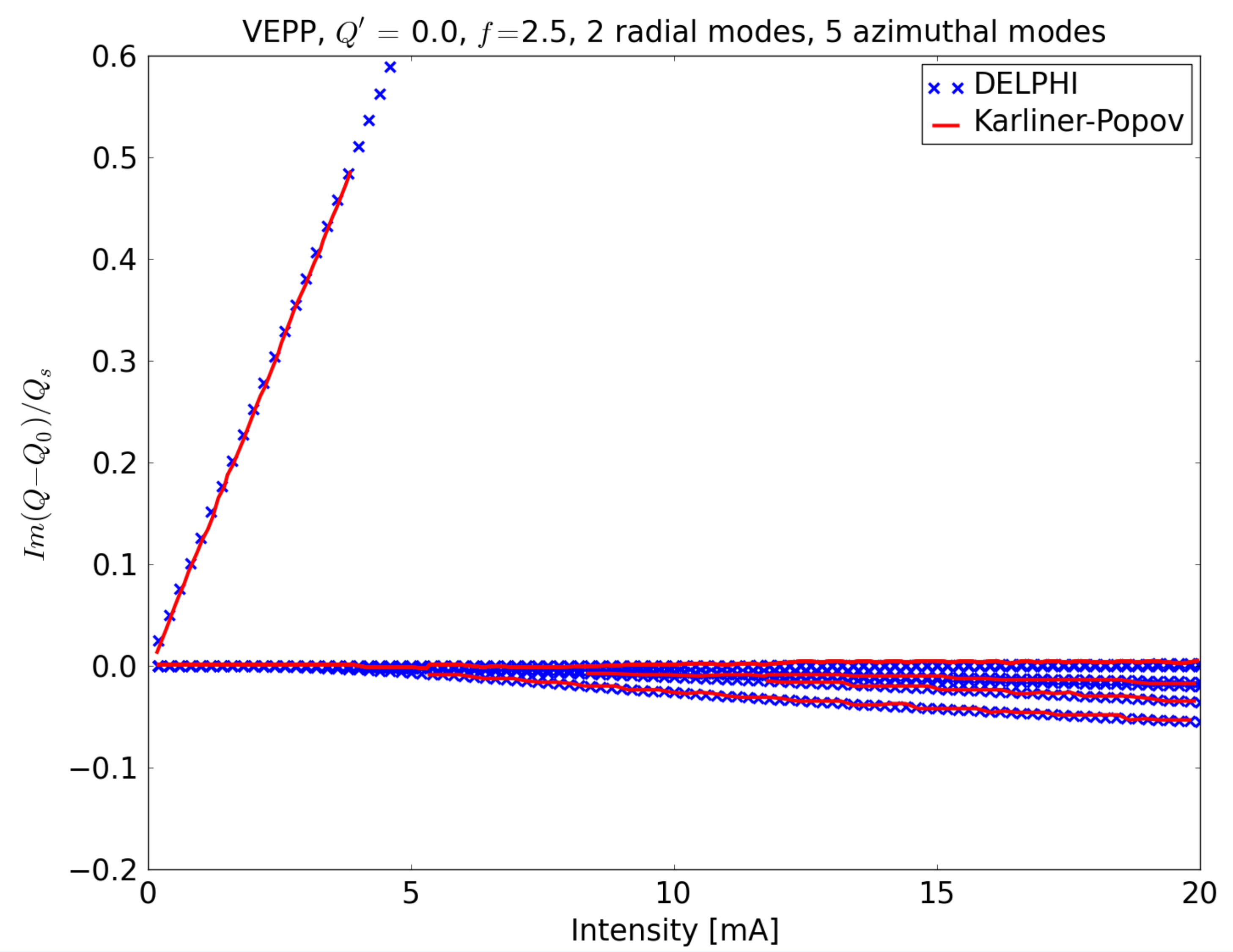
Real, part, Q'=0



Benchmarks

- DELPHI vs Karliner-Popov, single-bunch with damper (VEPP-4, broadband resonator):

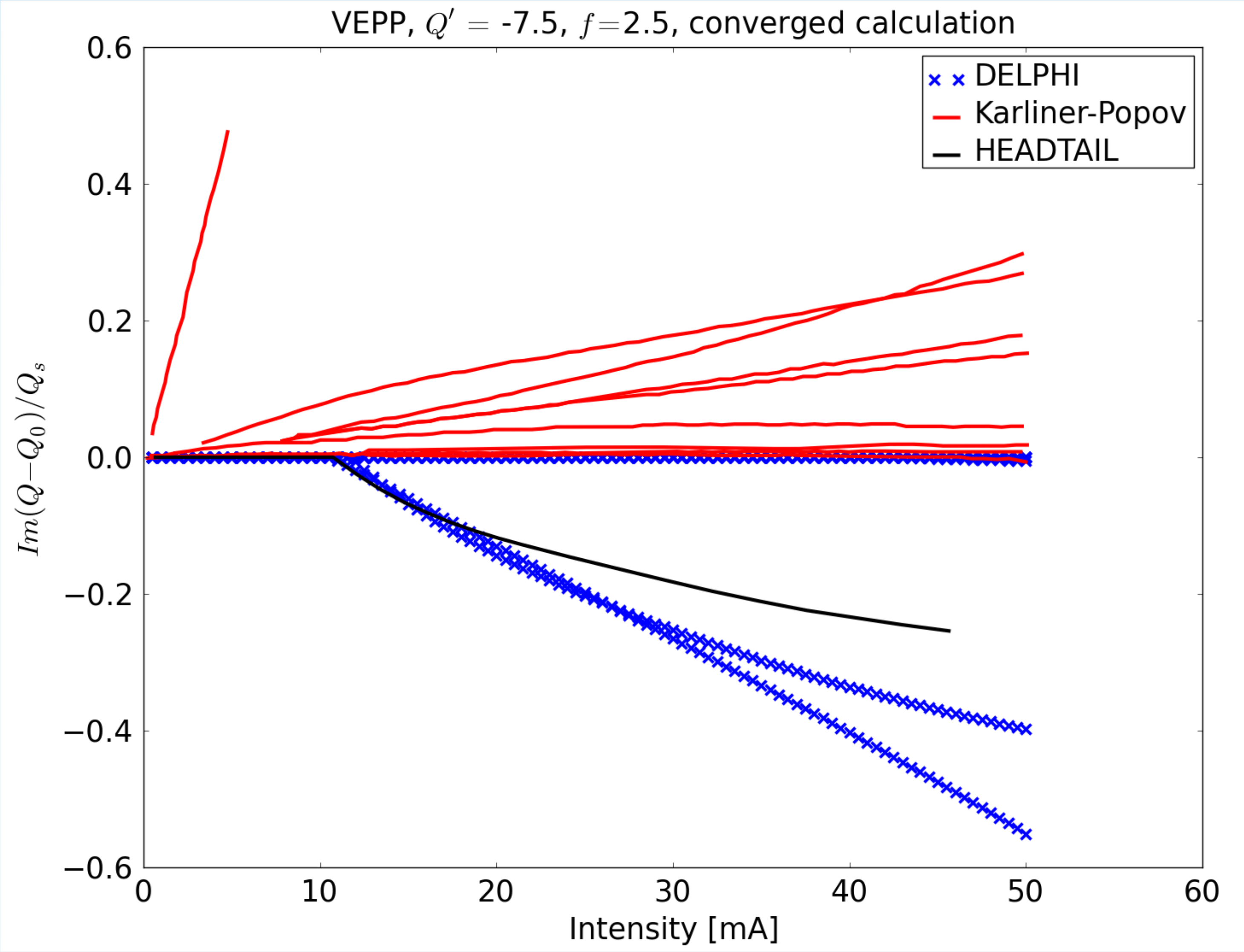
Imag. part, Q'=0



Benchmarks

- DELPHI vs Karliner-Popov and HEADTAIL (macroparticle simulation code – G. Rumolo et al), single-bunch with damper (VEPP-4, broadband resonator):

Imag, part, Q'=-7.5



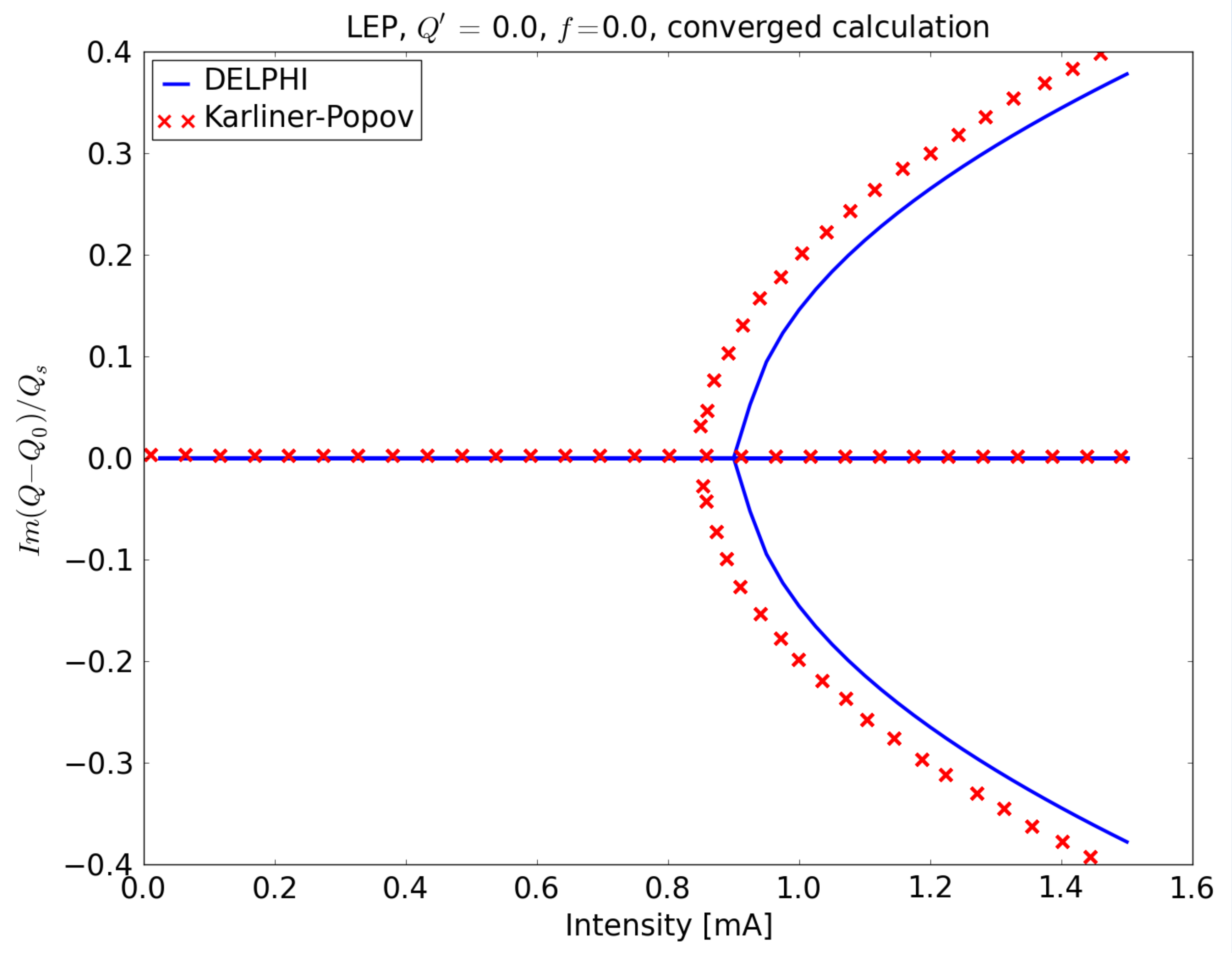
DELPHI is closer to HEADTAIL.

Karliner-Popov is more stable → due to their non flat damper ? (we cannot check because Karliner-P damper parameters are not provided).

LEP

- LEP without damper (typical LEP2 parameters)

Imag. part, Q'=0



Note: we had to change the bunch length (1.3cm instead of 1.8cm) to match Karliner-Popov's result.

Landau damping

- To include Landau damping, we simply need to replace the tune by

$$Q_y = Q_{y0} + Q'_y \delta + \frac{a_{yy}}{2} J_y + a_{yx} J_x,$$

- Then, assuming the transverse invariant stays ~ the same, the transverse part of the perturbation becomes:

$$f_l(J_x, J_y) = D_l \frac{\frac{\partial f_0(J_x,J_y)}{\partial J_y}\sqrt{\frac{2J_y R}{Q_{y0}}}}{Q_c - Q_{y0} - a_{yy}J_y - a_{yx}J_x - lQ_s}.$$

and the expression of the force from dipolar wake fields

$$F_y(z,s) = \frac{e^2}{2Q_{y0}} e^{\frac{jQ_c s}{R}} \sum_{l=-\infty}^{+\infty} \left(D_l \iint_0^{+\infty} \mathrm{d}J_x \mathrm{d}J_y \frac{\frac{\partial f_0(J_x,J_y)}{\partial J_y} J_y}{Q_c - Q_{y0} - a_{yy}J_y - a_{yx}J_x - lQ_s} \right) \cdot$$
$$\sum_{k=-\infty}^{+\infty} e^{-2\pi j Q_c k} \int \mathrm{d}\tilde{z} W_y(z-\tilde{z}) \int \mathrm{d}\tilde{\delta} e^{\frac{-jQ'_y \tilde{z}}{\eta R}} R_l(r) e^{-jl\phi}.$$

Landau damping

- We define the dispersion integral as

$$I_l(Q_c) = \iint_0^{+\infty} \mathrm{d}J_x \mathrm{d}J_y \frac{\frac{\partial f_0(J_x,J_y)}{\partial J_y} J_y}{Q_c - Q_{y0} - a_{yy}J_y - a_{yx}J_x - lQ_s},$$

which can be computed analytically for many transverse distributions (Gaussian, parabolic, and others).

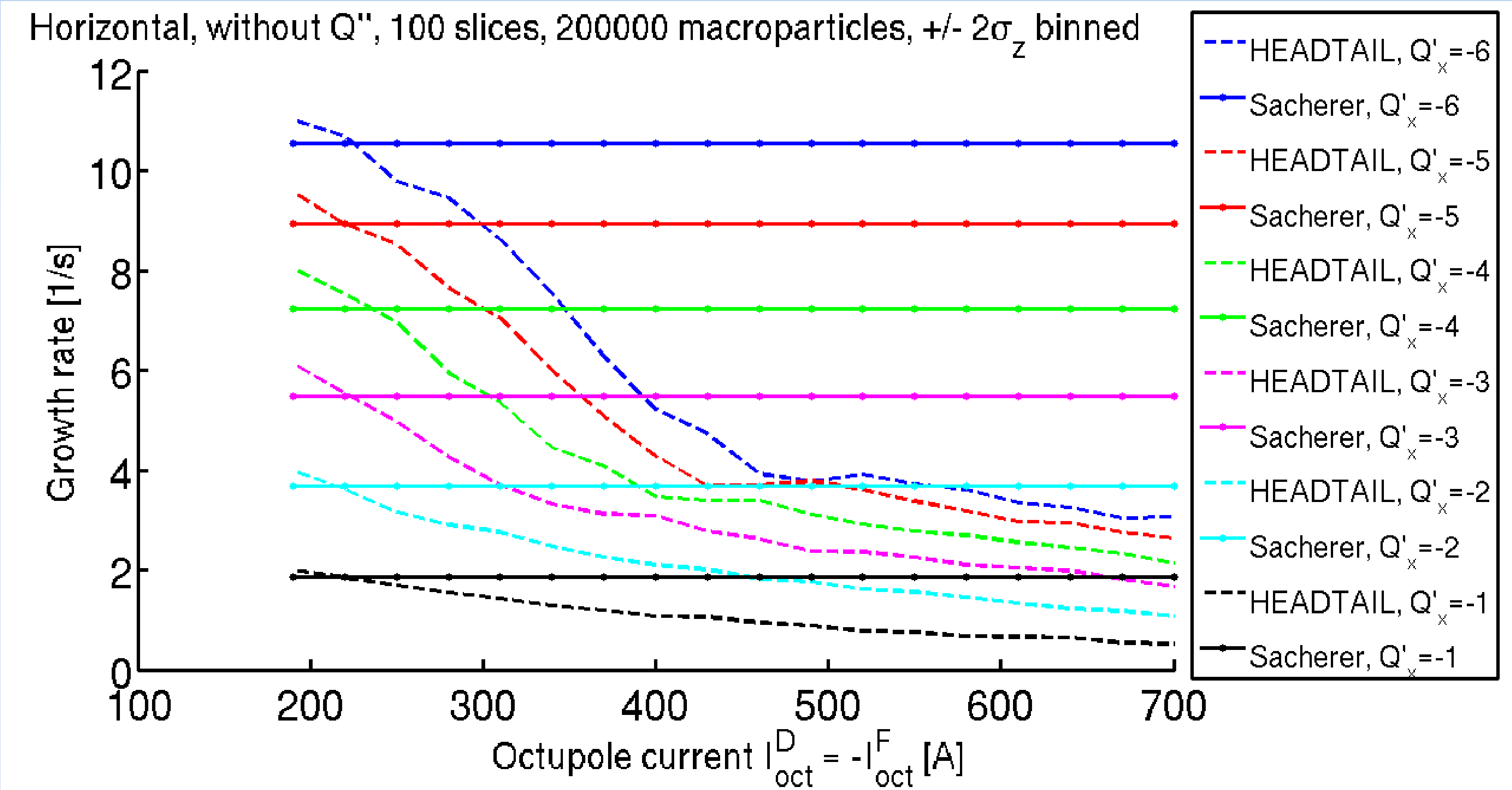
- Then the equation becomes:

$$det\left(\left[\delta_{ll'}\delta_{nn'}\frac{\omega_0}{I_l(Q_c)}\right] + [\mathcal{M}_{ln,l'n'}]\right) = 0.$$

This is a non-linear equation of the coherent (complex) tune shift Q_c, which can be solved numerically.

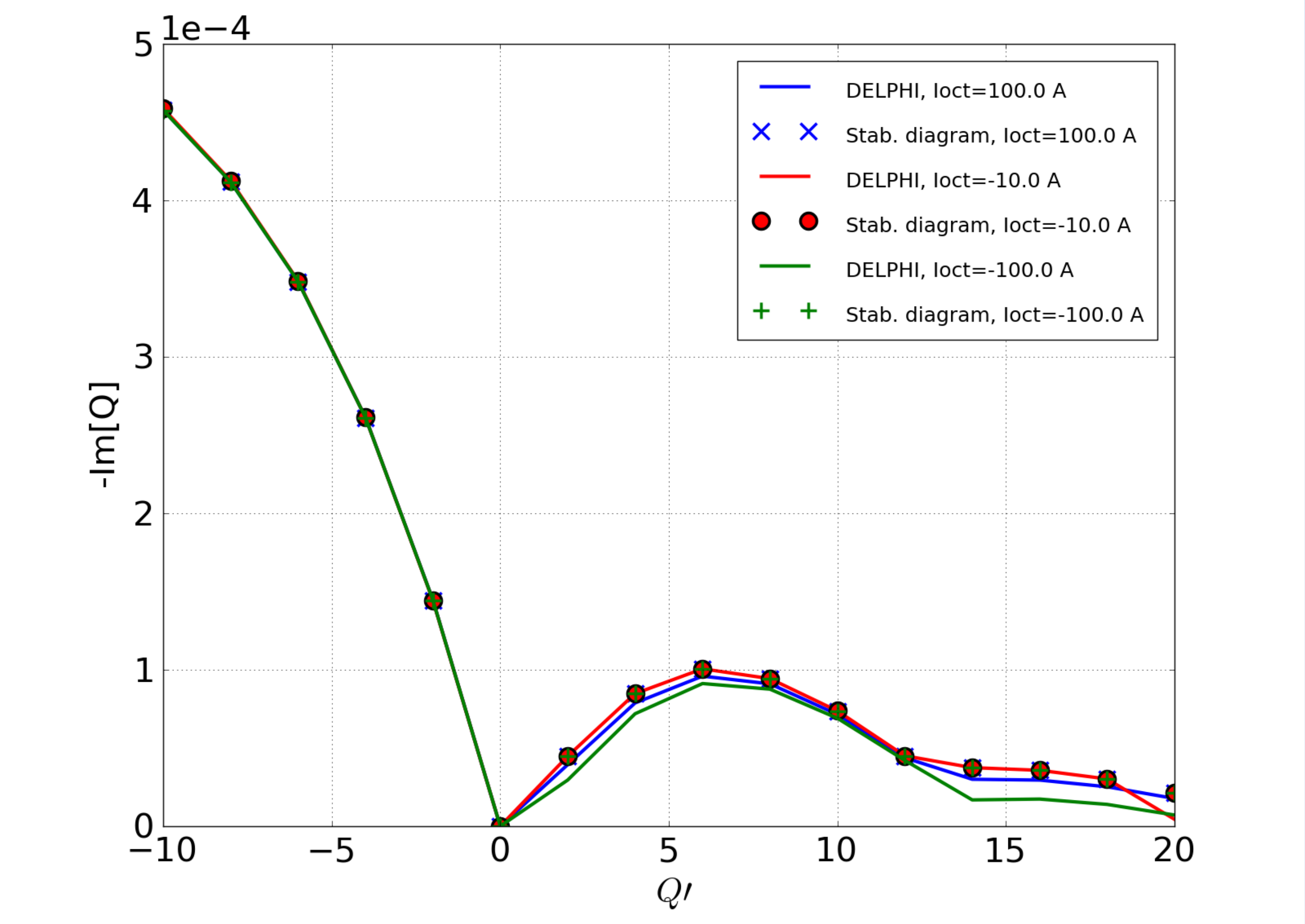
Macroparticles vs. Vlasov solver (time domain vs. mode domain)

- LHC single bunch with octupoles:



Landau damping with DELPHI: preliminary examples

- With the (old) LHC 2012 model, 5 azimuthal modes and 5 radial modes (non converged computation), no damper, single-bunch, N_b=3e11 p+/b: comparison with stability diagram theory



→ quite good agreement !