Effective Lagrangian for mesons and axion

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Plan of the talk

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- **Finding the minimum.**
- **Spectrum of pseudoscalar mesons.**
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- **Strong CP violating mesonic amplitudes.**
- **Strong CP violating amplitudes with baryons.**
- **Including the axion.**
- **Conclusions**
- **Outlook**

References

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1 INTRODUCTION

- **Strong interactions are described by the Lagrangian of QCD:**

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi - \theta q(x)$$

Mass matrix can always be put in a diagonal form

$$m_{ij} = m_i \delta_{ij} \quad ; \quad i, j = 1 \ldots N_f$$

Topological charge density

$$q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad ; \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Topological charge

$$\int d^4x \; q(x) = n = integer$$

For instanton configurations $n \neq 0$.
Physics is invariant under $\theta \to \theta + 2\pi$.

- **Gauge theory with $SU(3)$ color gauge group: $N_c = 3$.**

- **It depends on the following parameters:**

$$g \to \Lambda_{QCD}, \quad \theta \quad ; \quad m_i \quad , \quad N_c \quad ; \quad N_f = 2, 3$$

Dimensional transmutation: $\Lambda \sim 250 MeV$ instead of g.
Only dimensional parameter for massless quarks.

- **If the quarks are massless the transformations (A and B are $N_f \times N_f$ unitary matrices)**

$$\Psi_R^i \to A^{ij}\Psi_R^j \quad ; \quad \Psi_L^i \to B^{ij}\Psi_L^j \quad ; \quad \Psi_{R,L} = \frac{1 \pm \gamma^5}{2}\Psi$$

are a symmetry of the QCD Lagrangian:

$$U(N_f) \times U(N_f) \text{ chiral invariance}$$

- **This symmetry is spontaneously broken to the vectorial $U(N_f)$ generated by the transformations for which $A = B$.**

- **Pseudoscalar mesons are Goldstone bosons associated to the spontaneous breaking of chiral symmetry.**

- **They get a non-zero mass from the quark mass matrix ($m \neq 0$).**

- **But the split in the quark masses:**

$$\frac{m_u}{m_d} = 0.56 \quad ; \quad \frac{m_s}{m_d} = 20.1 \quad ; \quad \bar{m}_d|_{\mu=2GeV} = (3.1 \pm 1)MeV$$

is not sufficient to explain the mass spectrum of the pseudoscalar mesons:

$$m_\pi = 139MeV \,;\, m_\eta = 547MeV \,;\, m_{\eta'} = 957MeV \,;\, m_K = 498MeV$$

This problem was called $U(1)$-problem.

- **$U(1)$ axial anomaly:**

$$\partial_\mu \left[\bar{\Psi}\gamma^\mu\gamma^5\Psi\right] = 2N_f q(x) + \ Mass\ Matrix$$

- **How can we incorporate the effect of the anomaly in the meson mass matrix?**

2 Effective Lagrangian for the light d.o.f

- **At low energy ($E << \Lambda$) and low quark masses ($m_i << \Lambda$) we can neglect all degrees of freedom except the Goldstone bosons (pseudoscalar mesons).**
- **They are described by the following chiral Lagrangian:**

$$L = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(M(U+U^\dagger)\right) \quad ; \quad UU^\dagger = \frac{F_\pi^2}{2}$$

- **The constraint implies:**

$$U(x) = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \quad ; \quad \Phi(x) = \Pi^a\tau^a + \frac{S}{\sqrt{N_f}} \; ; \; Tr[\tau^a\tau^b] = \delta^{ab}$$

- **For $N_f = 3$ Φ corresponds to the octet of pseudoscalar mesons:**

$$\Pi^a\tau^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^0 + \eta_8/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta_8/\sqrt{3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\eta_8/\sqrt{3} \end{pmatrix}$$

- **As the quark mass matrix, also M can be chosen to be diagonal:**

$$M_{ij} = \mu_i^2 \delta_{ij}$$

- **<u>Gell-Mann-Oakes-Renner relation</u>**

$$\mu_i^2 F_\pi^2 = -2m_i < \bar{\Psi}_i \Psi_i >$$

In first approximation the ratio m_i/μ_i^2 is independent on i.

- **If $M = 0$ the previous Lagrangian is invariant under $U(N_f) \times U(N_f)$ transformations:**

$$U \to AUB^\dagger \quad ; \quad U^\dagger \to BU^\dagger A^\dagger \quad ; \quad A^{-1} = A^\dagger \quad ; \quad B^{-1} = B^\dagger$$

- **It has the same global symmetries as QCD with massless quarks,**
  but it does not take care of the $U(1)$ axial anomaly !!

- **We have to add a term that is invariant under $SU(N_f) \times SU(N_f) \times U(1)_V$ and trasforms under the $U(1)_A$ to reproduce the axial anomaly:**

$$L \to L + 2N_f q(x)\alpha \quad ; \quad U \to e^{-2i\alpha} U$$

- **This brings us to the following modified Lagrangian:**

$$L = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(M(U + U^\dagger)\right) +$$

$$+\frac{i}{2} q(x) Tr\left(\log U - \log U^\dagger\right)$$

- **In general we could add a generic term of the form:**

$$\sum_{i=0}^{\infty} L_{2i}(U, U^\dagger)[q(x)]^{2i}$$

  preserving parity and $U(N_f) \times U(N_f)$.

- **It turns out that, for large N_c, only one term of the previous sum contributes.**

- **The one with $i = 1$ and $L_2 = \frac{1}{aF_\pi^2}$.**

- **This brings us to:**

$$L(U,U^\dagger,q) = \frac{1}{2}Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(M(U+U^\dagger)\right) +$$

$$+\frac{i}{2}q(x)Tr\left(\log U - \log U^\dagger\right) + \frac{q^2}{aF_\pi^2} - \theta q(x)$$

Added also the θ parameter for studying the dependence of physical quantities on θ.

- **The equation of motion for q gives:**

$$q(x) = \frac{aF_\pi^2}{2}\left[\theta - \frac{i}{2}q(x)Tr\left(\log U - \log U^\dagger\right)\right]$$

- **When inserted back into the Lagrangian it gives:**

$$L = \frac{1}{2}Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(M(U+U^\dagger)\right) +$$

$$-\frac{aF_\pi^2}{4}\left[\theta - \frac{i}{2}Tr\left(\log U - \log U^\dagger\right)\right]^2$$

- **Our aim in the following is**

  1. **to show how the $U(1)$ problem is solved**
  2. **to determine the dependence of physical quantities on the θ parameter**

3 Finding the minimum

- **We have to find the value of $< U >$ that minimizes the potential that follows from the previous Lagrangian.**
- **Since $UU^\dagger \sim 1$ and the meson mass matrix is diagonal we can take:**

$$< U_{ij} >= e^{-i\phi_i}\delta_{ij}\frac{F_\pi}{\sqrt{2}}$$

- **It is convenient to introduce the following quantity:**

$$U_{ij} = V_{ij}e^{-i\phi_i} \quad ; \quad < V_{ij} >= \frac{F_\pi}{\sqrt{2}}\delta_{ij}$$

- **and one obtains $(M_{ij}(\phi_i) = \mu_i^2 \cos\phi_i \delta_{ij})$:**

$$L = \frac{1}{2}Tr(\partial_\mu V \partial_\mu V^\dagger) + \frac{aF_\pi^2}{16}\left[Tr\left(\log V - \log V^\dagger\right)\right]^2 +$$

$$+\frac{F_\pi}{2\sqrt{2}}Tr\left(M(\phi_i)\left[(V+V^\dagger) - \frac{2F_\pi}{\sqrt{2}}\right]\right) +$$

$$+\frac{F_\pi^2}{2}\sum_{i=1}^{N_f}\mu_i^2\cos\phi_i - \frac{aF_\pi^2}{4}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)^2 +$$

$$+i\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)\frac{aF_\pi}{2\sqrt{2}}Tr\left[\frac{F_\pi}{\sqrt{2}}(\log V - \log V^\dagger) - (V - V^\dagger)\right]$$

- **The parametrs ϕ_i are determined by minimizing the potential:**

$$\mathcal{V} = \frac{F_\pi^2}{2}\left[\frac{a}{2}(\theta - \sum_{i=1}^{N_f}\phi_i)^2 - \sum_{i=1}^{N_f}\mu_i^2\cos\phi_i\right]$$

- **This implies the following equations:**

$$\mu_i^2 \sin\phi_i = a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right) \quad ; \quad i = 1 \ldots N_f$$

- **Finally in terms of Φ we get:**

$$L = \frac{1}{2}Tr(\partial_\mu V \partial_\mu V^\dagger) - \frac{aN_f}{2}S^2 + \frac{F_\pi^2}{2}Tr\left[M(\theta)\left(\cos\frac{\sqrt{2}\Phi}{F_\pi} - 1\right)\right] +$$

$$+\frac{aF_\pi}{\sqrt{2}}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right) Tr\left[\frac{F_\pi}{\sqrt{2}}\sin\frac{\sqrt{2}\Phi}{F_\pi} - \Phi\right]$$

where

$$V = \frac{F_\pi}{\sqrt{2}}e^{i\sqrt{2}\Phi(x)/F_\pi} \quad ; \quad \Phi = \tau^a \Pi^a + \frac{S}{\sqrt{N_f}} \quad ; \quad M_{ij}(\theta) = \mu_i^2 \cos\phi_i \delta_{ij}$$

- **We proceed as follows:**

  1. **First we solve the minimization equations that determine ϕ_i as functions of θ, a and μ_i^2.**
  2. **Then we insert them in L that will in general be a function of θ, a and μ_i^2.**

- **Physical quantities are invariant under $\theta \to \theta + 2\pi$!!**
If we have found a solution $\phi_i(\theta)$ of the minimization equations, then the following is also a solution:

$$\phi_1(\theta + 2\pi) = \phi_1(\theta) + 2\pi \quad ; \quad \phi_i(\theta + 2\pi) = \phi_i(\theta) \quad ; \quad i = 2 \ldots N_f$$

But the physical quantities depend only on $e^{i\phi_i}$ and therefore are invariant under a shift of 2π of θ.

4 Spectrum of pseudoscalar mesons

- **The quadratic part of the previous Lagrangian is:**

$$L_2 = \frac{1}{2} Tr\left(\partial_\mu \Phi \partial^\mu \Phi\right) - \frac{a}{2} Tr\left(\Phi\right) Tr\left(\Phi\right) - \frac{1}{2} Tr\left(M(\theta)\Phi^2\right)$$

- **It is convenient to decompose the matrix Φ as follows:**

$$\Phi_{ij} = \tilde{\Pi}^{\alpha\beta} \tilde{\tau}_{ij}^{\alpha\beta} + v_i \delta_{ij}$$

$\tilde{\tau}_{ij}^{\alpha\beta}$ **are the** $N_f(N_f - 1)$ **non-diagonal generators of** $SU(N_f)$**.**

- **One gets:**

$$< \tilde{\Pi}^{\alpha\beta}(x) \tilde{\Pi}^{\gamma\delta}(y) >^{F.T.} = \frac{i\delta^{\alpha\gamma}\delta^{\beta\delta}}{p^2 - M^2_{\alpha\beta}} \quad ; \quad M^2_{\alpha\beta} = \frac{\mu^2_\alpha(\theta) + \mu^2_\beta(\theta)}{2}$$

$$< v_i(x) v_j(y) >^{F.T.} = i A^{-1}_{ij}(p^2)$$

$$A_{ij}(p^2) = (p^2 - \mu_i^2(\theta))\delta_{ij} - a \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ . & . & . & . & . \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

- **The masses $M_i^2(\theta)$ of the physical states are determined by the following equation:**

$$\det A = \prod_{i=1}^{N_f} (p^2 - M_i^2(\theta)) = \prod_{i=1}^{N_f} (p^2 - \mu_i^2) \left[1 - a \sum_{j=1}^{N_f} \frac{1}{p^2 - \mu_j^2} \right] = 0$$

- **In the case with three flavours and in the limit $\mu_1^2, \mu_2^2 << \mu_3^2$ one gets the following masses for η and η':**

$$M^2_\pm = m^2_K + \frac{3}{2} a \pm \frac{1}{2} \sqrt{(2m^2_K - 2m^2_\pi - a)^2 + 8a^2}$$

$$\tan\phi = \sqrt{2} - \frac{3}{2\sqrt{2}} \cdot \frac{m_\eta^2 - m_\pi^2}{m_K^2 - m_\pi^2} \quad ; \quad |\eta\rangle = \cos\phi|8\rangle + \sin\phi|1\rangle$$

- **We get a from the sum of the masses:**

$$a = \frac{m_\eta^2 + m_{\eta'}^2 - 2m_K^2}{3} \sim 0.24(GeV)^2$$

- **Using this value of a and neglecting the square term in the square root we get:**

$$m_\eta^2 \sim m_K^2 + \frac{3 - 2\sqrt{2}}{2} a = 0.27(GeV)^2 \quad ; \quad [Exp.\ 0.30]$$

$$m_{\eta'}^2 \sim m_K^2 + \frac{3 + 2\sqrt{2}}{2} a = 0.95(GeV)^2 \quad ; \quad [Exp.\ 0.92]$$

and

$$\phi \sim 14\ [Exp.\ 11]$$

- **a is a parameter that appears in the effective Lagrangian as the coefficient of the q^2 term:**

$$L(U, U^\dagger, q) = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(M(U + U^\dagger)\right) +$$

$$+ \frac{i}{2} q(x) Tr\left(\log U - \log U^\dagger\right) + \frac{q^2}{a F_\pi^2} - \theta q(x)$$

- **How can we extract it from the underlying QCD?**
- **In the large N_c limit quark loops can be neglected and this corresponds, in the effective Lagrangian, to neglect the dependence on U.**

- **Therefore, for large N_c, a can be extracted from the topological susceptibility of pure Yang-Mills theory:**

$$\lim_{q\to 0}(-i)\int d^4y\ \mathrm{e}^{iqx} < q(x)q(y) >_{YM}= \frac{1}{2}aF_\pi^2 \sim (180MeV)^4$$

$$[Exp.\ Lattice] = (194(5)MeV)^4$$

- **This has to be distinguished from the topological susceptibility computed in large N_c QCD or, equivalently, from the previous effective Lagrangian:**

$$\lim_{q\to 0}(-i)\int d^4x \mathrm{e}^{iqx}\langle q(x)q(0)\rangle_{QCD} \equiv \chi_{QCD} = \frac{\chi_{YM}}{1 + a\sum_{i=1}^{N_f}\frac{1}{\mu_i^2}}$$

- **It follows from the fact that there is a two-point coupling between the singlet $S(x)$ and $q(x)$ and from the mixing of $S(x)$ with the octect through the mass term.**

- **It vanishes in the chiral limit, when at least one of the $\mu_i^2 = 0$, for reasons that will become clear soon.**

- **Using**

$$\mu_1^2 = 0.7m_\pi^2 \quad ; \quad \mu_2^2 = 1.3m_\pi^2 \quad ; \quad a = \frac{m_\eta^2 + m_{\eta'}^2 - 2m_K^2}{3} = 0.24(GeV)^2$$

we get

$$\chi_{QCD} = (78.5MeV)^4$$

5 The Witten-Veneziano relation

- **In order to get the WV relation we have to consider the theory without quarks:**

$$L^{noferm.} = \frac{q^2}{aF_\pi^2} - \theta q - iqJ$$

with a source term J.

- **From it we can compute the partition function:**

$$Z(J,\theta) \equiv e^{-iW(J,\theta)} = e^{-iV_4 aF_\pi^2(\theta+iJ)^2/4}$$

and the vacuum energy:

$$E(\theta) \equiv \frac{W(0,\theta)}{V_4} = \frac{aF_\pi^2}{4}\theta^2$$

- **From it we get:**

$$\frac{d^2E(\theta)}{d\theta^2}|_{\theta=0} = \frac{aF_\pi^2}{2} \quad ; \quad M_S^2 = aN_f$$

- **They imply the WV relation:**

$$M_S^2 = \frac{2N_f}{F_\pi}\frac{d^2E(\theta)}{d\theta^2}|_{\theta=0}$$

6 Strong CP violating mesonic amplitudes

- **CP is conserved if $\theta - \sum_{i=1}^{N_f} \phi_i = 0$. This happens when:**
  1. **$\theta = 0$ that implies that $\phi_i = 0$**
  2. **the mass of a quark flavour is zero**
  3. **and also sometimes if $\theta = \pi$**

- **Consider the minimization equations for two flavours with $a >> \mu_1^2, \mu_2^2$:**

$$\theta = \phi_1 + \phi_2 \quad ; \quad \mu_1^2 \sin\phi_1 = \mu_2^2 \sin(\theta - \phi_1)$$

- **Their solution:**

$$\sin\phi_1 = \frac{\mu_2^2 \sin\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta}} \quad ; \quad \sin\phi_2 = \frac{\mu_1^2 \sin\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta}}$$

and

$$\cos\phi_1 = \frac{\mu_1^2 + \mu_2^2 \cos\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta}} \quad ; \quad \cos\phi_2 = \frac{\mu_2^2 + \mu_1^2 \cos\theta}{\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta}}$$

- **The corresponding potential**

$$V(\theta) = -\frac{F_\pi^2}{2}\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta}$$

- **For equal masses $(\mu_1 = \mu_2 = \mu)$ we get:**

$$V(\theta) = -F_\pi^2 \mu^2 \left|\cos\frac{\theta}{2}\right|$$

Notice that both Eq.s are periodic of period 2π in θ.

- **Expanding around the previous solution and including terms of order $\frac{\mu_1\mu_2}{a}$**

$$\phi_{1,2} = \bar{\phi}_{1,2} + \epsilon\delta\phi_{1,2} + \mathcal{O}(\epsilon^2) \quad ; \quad \epsilon = \frac{\mu_1\mu_2}{a}$$

in the minimization equations

$$\mu_1^2 \sin\phi_1 = \mu_2^2 \sin\phi_2 = a(\theta - \phi_1 - \phi_2)$$

- **One gets:**

$$\phi_1 = \bar{\phi}_1 - \epsilon\frac{\sin\theta}{R^3}\left(\frac{\mu_2^2 + \mu_1^2\cos\theta}{\mu_1^2}\right) \quad ; \quad \phi_2 = \bar{\phi}_2 - \epsilon\frac{\sin\theta}{R^3}\left(\frac{\mu_1^2 + \mu_2^2\cos\theta}{\mu_2^2}\right)$$

where

$$\bar{\phi}_1 + \bar{\phi}_2 = \theta \quad ; \quad R = \sqrt{\frac{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}{\mu_1^2\mu_2^2}}$$

- **Compute the coefficient of the CP violating term:**

$$\theta - \phi_1 - \phi_2 = \epsilon\frac{\sin\theta}{R} = \frac{\mu_1^2\mu_2^2\sin\theta}{a\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}}$$

- **It is vanishing if $\theta = 0$ or if μ_1^2 and/or μ_2^2 are equal to zero.**
If $\mu_1 \neq \mu_2$ it is also zero for $\theta = \pi$. But if $\mu_1 = \mu_2 \equiv \mu$ we get:

$$\theta - \phi_1 - \phi_2 = \frac{\mu^2}{a} \neq 0$$

In conclusion if $\mu_1 = \mu_2$ then CP is violated at $\theta = \pi$.

- **From the CP violating term extract a cubic term in the fields of the pseudoscalar mesons:**

$$-\frac{a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)}{3\sqrt{2}F_\pi}Tr(\Phi^3) \Longrightarrow -\frac{a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)}{\sqrt{3}F_\pi}\pi^+\pi^-\eta_8$$

- **Decay amplitude $\eta_8 \to \Pi^+\Pi^-$ given by:**

$$T(\eta \to \pi^+\pi^-) = \frac{a\left(\theta - \sum_{i=1}^{N_f} \phi_i\right)}{\sqrt{3}F_\pi} = \frac{2m_\pi^2(\theta)}{\sqrt{3}F_\pi} \cdot \frac{\mu_1^2\mu_2^2 \sin\theta}{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}$$

$$m_\pi^2(\theta) = \frac{\mu_1^2\cos\phi_1 + \mu_2^2\cos\phi_2}{2} = \frac{1}{2}\sqrt{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2\cos\theta}$$

- **For small values of θ we get**

$$T(\eta \to \pi^+\pi^-) \sim \frac{2m_\pi^2}{\sqrt{3}F_\pi}\frac{\theta}{\left(\sqrt{\frac{m_1}{m_2}} + \sqrt{\frac{m_2}{m_1}}\right)^2}$$

m_1, m_2 **are the quark masses.**

- **This implies that**

$$\Gamma(\eta \to \pi^+\pi^-)) = \theta^2 \cdot (135\ KeV) \quad : \quad \frac{\Gamma(\eta \to \pi^+\pi^-)}{\Gamma_{tot}} = 159\ \theta^2$$

From experiments we get:

$$\frac{\Gamma(\eta \to \pi^+\pi^-)}{\Gamma_{tot}} < 3 \cdot 10^{-4}$$

that gives an upper limit to the value of $\theta < 10^{-3}$

- **The decay amplitude of $\eta \to \pi^+\pi^-$ is zero for $\theta = 0, \pi$ if $\mu_1^2 \neq \mu_2^2$**
if $\mu_1^2 = \mu_2^2$ it is not vanishing anymore at $\theta = \pi$.

- **Masses of the pseudoscalar mesons as a function of the angle θ:**

$$m^2_{\pi^0,\pi^\pm} = \frac{\mu_1^2 \cos\phi_1 + \mu_2^2 \cos\phi_2}{2} \quad ; \quad m^2_{k^\pm} = \frac{\mu_1^2 \cos\phi_1 + \mu_3^2 \cos\phi_3}{2}$$

and

$$m^2_{k^0;\bar{k}^0} = \frac{\mu_2^2 \cos\phi_1 + \mu_3^2 \cos\phi_3}{2}$$

- **They imply:**

$$R(\theta) \equiv \frac{m^2_{k^0} - m^2_{k^+} - m^2_{\pi^0} + m^2_{\pi^+}}{m^2_\pi} = \frac{\mu_2^2 \cos\phi_2 - \mu_1^2 \cos\phi_1}{\mu_2^2 \cos\phi_2 + \mu_1^2 \cos\phi_1} =$$

$$= \frac{(\mu_2^2 - \mu_1^2)(\mu_2^2 + \mu_1^2)}{\mu_1^4 + \mu_2^4 + 2\mu_1^2\mu_2^2 \cos\theta}$$

and in particular

$$R(\theta = 0) = \frac{\mu_2^2 - \mu_1^2}{\mu_2^2 + \mu_1^2} \quad ; \quad R(\theta = \pi) = \frac{\mu_2^2 + \mu_1^2}{\mu_2^2 - \mu_1^2}$$

- **Experimentally $R = 0.3$ that is consistent with $\theta = 0$.**

7 Strong CP violating amplitudes with baryons

- **The baryons belong to an octet of $SU(3)$:**

$$\begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & 2\frac{\Lambda}{\sqrt{6}} \end{pmatrix}$$

- **Under the chiral $U(3) \times U(3)$ the baryons transform as follows:**

$$R \equiv \frac{1+\gamma_5}{2} B \rightarrow A R B^\dagger \quad ; \quad L \equiv \frac{1-\gamma_5}{2} B \rightarrow B L A^\dagger$$

- **The Lagrangian involving baryons can be written as follows:**

$$L_{bar} = Tr\left[\bar{B} i \gamma^\mu \partial_\mu B\right] - \frac{\sqrt{2}\alpha}{F_\pi} Tr\left[\bar{L} U R U + \bar{R} U^\dagger L U^\dagger\right] +$$

$$+\delta Tr\left[\bar{L} U R M + \bar{R} U^\dagger L M^\dagger\right] + \gamma Tr\left[\bar{L} M R U + \bar{R} M^\dagger L U^\dagger\right]$$

- **As before we introduce**

$$V_{ij} = U_{ij} e^{i\phi_j} \quad ; \quad R_{ij} = e^{i\phi_i} R'_{ij} \quad ; \quad \bar{L}_{ij} = e^{i\phi_i} \bar{L}'_{ij}$$

- **We get**

$$L_{bar} = Tr\left[\bar{B}' i \gamma^\mu \partial_\mu B'\right] - i\frac{\sqrt{2}\alpha}{F_\pi} Tr\left[\bar{L}' V R' V + \bar{R}' V^\dagger L' V^\dagger\right] +$$

$$+\delta Tr\left[\left(\bar{L}' V R' + \bar{R}' V^\dagger L'\right) M(\theta)\right] + \gamma Tr\left[\bar{L}' M(\theta) R' V + \bar{R}' M(\theta) L' V^\dagger\right] +$$

$$+i\left(\theta - \sum_i \phi_i\right)\left[\delta Tr\left(\bar{L}' V R' - \bar{R}' V^\dagger L'\right) + \gamma Tr\left(\bar{L}' R' V + \bar{R}' L' V^\dagger\right)\right]$$

Same structure as before + CP violating term.

- **Determine α, γ and δ in terms of the baryon masses:**

$$m_N = \frac{F_\pi}{\sqrt{2}}\left(\alpha - \delta\mu_3^2 - \gamma\mu^2\right) \quad ; \quad m_\Sigma = \frac{F_\pi}{\sqrt{2}}\left(\alpha - (\gamma+\delta)\mu^2\right)$$

$$m_\Xi = \frac{F_\pi}{\sqrt{2}}\left(\alpha - \delta\mu^2 - \gamma\mu_3^2\right) \quad ; \quad m_\Lambda = \frac{F_\pi}{\sqrt{2}}\left(\alpha - \frac{1}{3}(\gamma+\delta)(\mu^2 + 2\mu_3^2)\right)$$

- **We get**

$$\alpha = \frac{\sqrt{2}}{F_\pi}\left[m_\Sigma + \frac{3\mu^2}{2(\mu_3^2 - \mu^2)}(m_\Sigma - m_\Lambda)\right]$$

$$\gamma = \frac{\sqrt{2}}{2F_\pi(\mu_3^2 - \mu^2)}\left[\frac{3}{2}(m_\Sigma - m_\Lambda) - (m_\Xi - m_N)\right]$$

$$\delta = \frac{\sqrt{2}}{2F_\pi(\mu_3^2 - \mu^2)}\left[\frac{3}{2}(m_\Sigma - m_\Lambda) + (m_\Xi - m_N)\right]$$

- **The baryon masses satisfy the Gell-Mann-Okubo mass formula:**

$$3m_\Lambda + m_\Sigma = 2(m_\Xi + m_N)$$

- **Introduce the octet of baryons, couple them with the mesons and extract the πN coupling constants:**

$$\sqrt{2}\bar{N}\left[i\gamma_5 g_{\pi NN} + \bar{g}_{\pi NN}\right]\pi^i\tau^i N$$

obtaining the Goldeberger-Treiman relation

$$F_\pi g_{\pi NN} = m_N$$

and the CP violating pion-nucleon coupling constant

$$\bar{g}_{\pi NN} = \frac{m_1 m_2 \theta}{2F_\pi(m_1+m_2)(m_3-m)}\times$$

$$\times\left[\frac{3}{2}(m_\Sigma - m_\Lambda) - (m_\Xi - m_N)\right]$$

- **They can be used to estimate the electric dipole moment of the neutron (CP violating):**

$$D_n = \frac{1}{4\pi^2 m_N} \cdot g_{\pi NN} \bar{g}_{\pi NN} \log \frac{m_N}{m_\pi} = 3.6 \cdot 10^{-16} \theta cm$$

in units where the electric charge $e = 1$**.**

- **The experimental limit is:**

$$D_n < 6 \cdot 10^{-26} \Longrightarrow \theta < 10^{-9}$$

8 Including the axion

- θ is very small and actually consistent with zero.
- Can we make it to be zero in a natural way?
- The vanishing of m_u would be a way because it allows to rotate θ away.
  But m_u seems to be $\neq 0$.
- The Peccei-Quinn solution of the strong CP problem uses a similar mechanism.
- It includes in the matter sector of QCD some new d.o.f. with an extra $U(1)_{PQ}$ symmetry that is broken by an anomaly exactly as $U(1)_A$.
- The would be Goldstone boson gets a mass with the same mechanism as the singlet.
- Denoting with $\alpha_{PQ}$the coefficient of the $U(1)_{PQ}$ anomaly and with F_α the scale of its spontaneous breaking, we can extend our previous Lagrangian as follows:

$$L = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{2} Tr(\partial_\mu N \partial_\mu N^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(M(U + U^\dagger)\right) +$$

$$-\theta q + \frac{q^2}{aF_\pi^2} + \frac{i}{2} q(x) \left(Tr(\log U - \log U^\dagger) + a_{PQ}(\log N - \log N^\dagger)\right)$$

where

$$U(x) = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \quad ; \quad N(x) = \frac{F_\alpha}{\sqrt{2}} e^{i\sqrt{2}\alpha(x)/F_\alpha}$$

- Under the two $U(1)$ transformations:

$$U \rightarrow e^{i\beta} U \quad ; \quad N \rightarrow e^{i\gamma} N \,,$$

  the effective Lagrangian transforms as follows:

$$L \rightarrow L - (N_f \beta + a_{PQ} \gamma)\, q(x)$$

- **It is invariant if we choose $N_f\beta + a_{PQ}\gamma = 0$.**
  This is an anomaly-free $U(1)$ subgroup, whose spontaneous and explicit breaking (by quark masses) implies a new, pseudo-Goldstone boson, the (Peccei-Quinn-Weinberg-Wilczek) axion.

- **Proceeding as before**

$$< U_{ij} >= e^{-i\phi_i}\delta_{ij}F_\pi/\sqrt{2} \quad ; \quad < N >= e^{-i\phi}F_\alpha/\sqrt{2}$$

  we have to minimize the potential

$$\mathcal{V} = \frac{F_\pi^2}{2}\left[\frac{a}{2}(\theta - \sum_{i=1}^{N_f}\phi_i - \phi)^2 - \sum_{i=1}^{N_f}\mu_i^2\cos\phi_i\right]$$

- **We get**

$$a\left(\theta - \sum_{i=1}^{N_f}\phi_i - \phi\right) = \mu_i^2\sin\phi_i \quad ; \quad \theta - \phi - \sum_{i=1}^{N_f}\phi_i = 0$$

- **They imply:**

$$\phi_i = 0 \quad , \quad i = 1 \dots N_f \quad ; \quad \theta - \phi = 0$$

  No dependence on θ and no CP violation in analogy with the case $m_u = 0$.

- **The mass matrix involving the axion and the components of Φ in the Cartan subalgebra of $U(N_f)$ ($\Phi_{ij} = v_i\delta_{ij}$) is given by:**

$$-\frac{1}{2}\left[\sum_{i=1}^{N_f}\mu_i^2 v_i^2 + a\left(\sum_{i=1}^{N_f}v_i + b\alpha\right)^2\right] \quad ; \quad b \equiv a_{PQ}\frac{F_\pi}{F_\alpha}$$

- **The masses of the neutral mesons and of the axion are given by setting to zero the determinant of the following matrix:**

$$\begin{pmatrix} b^2 a - \lambda & ba & ba & ba & \dots & ba \\ ba & \mu_1^2 + a - \lambda & a & a & \dots & a \\ ba & a & \mu_2^2 + a - \lambda & a & \dots & a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ ba & ba & ba & ba & \dots & \mu_{N_f}^2 + a - \lambda \end{pmatrix}$$

- **That is by solving the equation:**

$$\lambda \left[\frac{1}{a} + \sum_{i=1}^{N_f} \frac{1}{\mu_i^2 - \lambda} \right] = b^2$$

- **Since $b << 1$ the lowest eigenvalue, corresponding to the mass of the axion, is given by:**

$$m_\alpha^2 = \frac{b^2}{\frac{1}{a} + \sum_{i=1}^{3} \frac{1}{\mu_i^2}} \Longrightarrow m_\alpha^2 F_\alpha^2 = \frac{\alpha_{PQ}^2 F_\pi^2}{\frac{1}{a} + \sum_{i=1}^{3} \frac{1}{\mu_i^2}} = 2\alpha_{PQ}^2 \chi_{QCD}$$

χ_{QCD} **is the topological susceptibility in QCD.**

- **In terms of the masses of the pseudoscalar mesons, one gets:**

$$\frac{1}{a} \sim 4.2(GeV)^{-2} \ ; \quad \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} = 112.1(GeV)^{-2} \ ; \ \frac{1}{\mu_3^2} = 2(GeV)^{-2}$$

- **Neglecting the term with $1/a$, one gets the current algebra relation:**

$$m_\alpha^2 F_\alpha^2 = 2\alpha_{PQ}^2 m_\pi^2 F_\pi^2 \cdot \frac{m_1 m_2}{(m_1 + m_2)^2}$$

in terms of the mass of the up and down quarks.

- **Consistency with experiments requires:**

$$F_\alpha \geq 10^9 GeV \Longrightarrow m_\alpha < 0.01 eV$$

9 Conclusions

- **The topological susceptibility in pure (large N_c) YM theory determines the spectrum of the pseudoscalar mesons:**

$$\lim_{q\to 0}\int d^4x e^{iqx}\langle q(x)q(0)\rangle_{YM} \equiv i\chi_{YM} = i\frac{aF_\pi^2}{2}$$

- **The spectrum of the pseudoscalar mesons in (large N_c) QCD is fixed by the vanishing of the determinant of the mass matrix:**

$$\prod_{i=1}^{N_f}(q^2-\mu_i^2)\left[1-a\sum_{i=1}^{N_f}\frac{1}{q^2-\mu_i^2}\right]=\prod_{i=1}^{N_f}(q^2-M_i^2)=0$$

M_i **are the physical masses of the pseudoscalar mesons.**

- **Chiral limit ($N_f=3$): massless octect and $m_S=3a$.**
- **Two-point correlator in (large N_c) QCD**

$$(-i)\int d^4x e^{iqx}\langle q(x)q(0)\rangle_{QCD} = \frac{\chi_{YM}}{1-a\sum_{i=1}^{N_f}\frac{1}{q^2-\mu_i^2}} = \chi_{YM}\prod_{i=1}^{N_f}\frac{q^2-\mu_i^2}{q^2-M_i^2}$$

- **The topological susceptibility in (large N_c) QCD determines the mass of the axion:**

$$m_\alpha^2F_\alpha^2 = 2\alpha_{PQ}^2\chi_{QCD}$$

- **and it is related to χ_{YM} by**

$$\lim_{q\to 0}\int d^4x e^{iqx}\langle q(x)q(0)\rangle_{QCD} \equiv i\chi_{QCD} = i\frac{\chi_{YM}}{1+a\sum_{i=1}^{N_f}\frac{1}{\mu_i^2}}$$

that vanishes, as expected, in the chiral limit (no θ dependence in the chiral limit).

- **Including the axion one gets:**

$$q^2\prod_{i=1}^{N_f}(q^2-\mu_i^2)\left[1-a\left(\sum_{i=1}^{N_f}\frac{1}{q^2-\mu_i^2}+\frac{b^2}{q^2}\right)\right]=\prod_{i=1}^{N_f}(q^2-M_i^2)(q^2-m_\alpha^2)=0$$

and

$$(-i)\int d^4x e^{iqx}\langle q(x)q(0)\rangle_{QCDA}=\chi_{YM}\frac{q^2}{q^2-m_\alpha^2}\prod_{i=1}^{N_f}\frac{q^2-\mu_i^2}{q^2-M_i^2}\Longrightarrow\chi_{QCDA}=0$$

- **In agreement with the fact that the axion is there to eliminate the θ dependence of the physical quantities.**

- **Actually, the effective Lagrangian is a simple way to implement the Ward identities of the underlying microscopic theory (QCD):**

$$-i\int d^4x e^{iqx}\langle q^\mu J_{5\mu}^{(i)}(x)O(y)\rangle=2\int d^4x e^{iqx}\langle q(x)O(y)\rangle$$
$$+2m_i\int d^4x e^{iqx}\langle P_i(x)O(y)\rangle+\langle[Q_5^{(i)},O(y)]\rangle$$

where $O(y)=q(x),m_iP_i(x)$ **with** $P_i=i\psi_i\gamma_5\psi_i$**.**

- **For instance, the Ward identity with** $O(y)=q(y)$ **is satisfied by the following two-point functions:**

$$\int d^4x e^{iqx}\langle J_{5\mu}^{(i)}(x)q(y)\rangle=\frac{2iq_\mu}{q^2-\mu_i^2}\frac{aF_\pi^2}{2}\prod_{i=1}^{N_f}\frac{q^2-\mu_i^2}{q^2-M_i^2}$$

$$\int d^4x e^{iqx}\langle 2m_iP_i(x)q(y)\rangle=\frac{2\mu_i^2}{q^2-\mu_i^2}\frac{aF_\pi^2}{2}\prod_{i=1}^{N_f}\frac{q^2-\mu_i^2}{q^2-M_i^2}$$

and

$$\int d^4x e^{iqx}\langle q(x)q(y)\rangle=\frac{aF_\pi^2}{2}\prod_{i=1}^{N_f}\frac{q^2-\mu_i^2}{q^2-M_i^2}$$

with $[Q_5^{(i)},q(y)]=0$**.**

10 Outlook

- **Extend our results at finite temperature.**
- **Can we say something at finite N_c?**
- **Can lattice QCD say something about the dependence of the topological susceptibility on the quark masses in the chiral limit (also at finite temperature)?**
- **Can lattice QCD determine the momentum dependence of the two-point function:**

$$(-i)\int d^4x e^{iqx}\langle q(x)q(0)\rangle = \frac{\chi_{YM}}{1 - a\sum_{i=1}^{N_f}\frac{1}{q^2-\mu_i^2}} = \chi_{YM}\prod_{i=1}^{N_f}\frac{q^2-\mu_i^2}{q^2-M_i^2}\ ?$$