TMD evolution
as
a double-scale evolution

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in collaboration with Ignazio Scimemi
based on [1803.11089]
Motivation

TMD evolution is a double-scale evolution

\[ F(x, b; \mu, \zeta) \]

This aspect has been completely overlooked. Its account reveals completely novel picture of TMD evolution.

Outlook

- Review of TMD evolution status
- Evolution plane and the general solution
- Induced path dependence of the solution
- Evolution potential
- \( \zeta \)-prescription and optimal TMD
TMD evolution is used for two practical purposes

- Compare different experiments
- Modeling TMD distribution

\[
\frac{d\sigma}{dX} \sim \int d^2 b e^{i(bq_T)} H_{f\rightarrow f'}(Q, \mu) F_{f\leftarrow h}(x_1, b; \mu, \zeta_1) F_{f'\leftarrow h}(x_2, b; \mu, \zeta_2)
\]
TMD evolution is used for two practical purposes

- Compare different experiments
- Modeling TMD distribution

\[ \frac{d\sigma}{dX} \sim \int d^2 b e^{i(bqT)} H_{f'f}(Q, \mu) F_f \rightarrow h(x_1, b; \mu, \zeta_1) F_{f'} \rightarrow h(x_2, b; \mu, \zeta_2) \]

Minimize \( \ln(Q/\mu) \)
\[ \mu = Q \]

\[ \zeta_1 \zeta_2 = Q^4 \]
or
\[ \zeta_1 = \zeta_2 = Q^2 \]
TMD evolution is used for two practical purposes

- Compare different experiments
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\[ \frac{d\sigma}{dX} \sim \int d^2b \, e^{i(bq_T)} \, H_{f'f}(Q, \mu) F_f \leftarrow h(x_1, b; \mu, \zeta_1) F_{f'} \leftarrow h(x_2, b; \mu, \zeta_2) \]

Minimize \( \ln(Q/\mu) \)

\( \mu = Q \)

\( \zeta_1 \zeta_2 = Q^4 \)

or

\( \zeta_1 = \zeta_2 = Q^2 \)

Minimize \( L_\mu, L_{\sqrt{\zeta}} \)

\( \mu \sim \sqrt{\zeta} \sim b^{-1} \)

\[ F(x, b; \mu, \zeta) \sim C(x, b; \mu, \zeta) \otimes \text{PDF}(x, \mu) \]

Typical model for TMD includes matching
TMD evolution is used for two practical purposes

- Compare different experiments
- Modeling TMD distribution

\[
\frac{d\sigma}{dX} \sim \int d^2 b e^{i(bqT)} H_{f,f'}(Q,\mu) F_f \leftarrow h(x_1, b; \mu, \zeta_1) F_{f'} \leftarrow h(x_2, b; \mu, \zeta_2)
\]

\[
\text{Minimize } \ln\left(\frac{Q}{\mu}\right) \\
\mu = Q
\]

\[
\zeta_1 \zeta_2 = Q^4 \\
\text{or} \\
\zeta_1 = \zeta_2 = Q^2
\]

\[
F(x, b; \mu_f, \zeta_f) = R[b, (\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)] F(x, b; \mu_i, \zeta_i)
\]

Typical model for TMD includes matching

\[
F(x, b; \mu, \zeta) \sim C(x, b; \mu, \zeta) \otimes \text{PDF}(x, \mu)
\]
TMD evolution equations

\[ \mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x, b; \mu, \zeta) = \frac{\gamma_F^f(\mu, \zeta)}{2} F_{f\leftarrow h}(x, b; \mu, \zeta), \]  
(1)

\[ \zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x, b; \mu, \zeta) = -D^f(\mu, b) F_{f\leftarrow h}(x, b; \mu, \zeta), \]  
(2)

Solution: \[ F(x, b; \mu_f, \zeta_f) = R[b; (\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)]F(x, b; \mu_i, \zeta_i) \]

- \( \gamma_F \) – TMD anomalous dimension
- \( D \) – rapidity anomalous dimension \((= -\frac{\tilde{K}}{2} \text{[Collins' book]}, = K \text{[Bacchetta, at al,1703.10157]})\)
- Anomalous dimensions are universal, i.e. depend only on flavor (gluon/quark).
\[ D(\mu, b) = D_{\text{perp}}(\mu, b^*) + d_{NP}(b) \]

**Perturbative part**

- Soft/rapidity correspondence \( D \leftrightarrow \gamma_s \) [AV, PRL 118(2017)]
- Everything at NNLO (+\( \Gamma \)-cusp at N^3LO [Vogt et al., 1808.08981])

\( d_{NP} \) is a universal non-perturbative function. In many aspects more fundamental than TMDs.
Global extraction of $F_1$ at NNLO by [Scimemi & AV,1706.01473]

TMD evolution is a key element
\[ \frac{\chi^2_{\text{global}}}{d.o.f.} \approx 1.25 \]

Here:
- 3-loop evolution
- 2-loop coefficient function
- 2-loop matching
- $\zeta$-prescription

plots from [1706.01473]
Global extraction of $F_1$ at NNLO by [Scimemi & AV,1706.01473]

Drell-Yan at $Q = 5 - 6\text{GeV}$

ATLAS 8TeV
46–66 GeV
model 2 NNLO
$\chi^2/\text{points}=1.91$

$\chi^2/\text{points}=1.21$

$\chi^2/\text{points}=1.01$

Drell-Yan at $Q = 116 - 150\text{GeV}$

$\chi^2/\text{points}=1.91$

$\chi^2/\text{points}=1.21$

$\chi^2/\text{points}=1.01$

TMD evolution is a key element

$\frac{\chi^2_{\text{global}}}{\text{d.o.f.}} \approx 1.25$

Here:

3-loop evolution
2-loop coefficient function
2-loop matching
$\zeta$-prescription

$\zeta$-prescription consistently separates the TMD evolution from TMDs, and makes all the theory elements work together.

It is the consequence of 2D nature of TMD evolution.

plots from [1706.01473]
TMD evolution is two-dimensional

\[
\begin{pmatrix}
\mu^2 \frac{d}{d\mu^2} \\
\zeta \frac{d}{d\zeta}
\end{pmatrix} F = \begin{pmatrix}
\frac{\gamma F}{2} \\
-\mathcal{D}
\end{pmatrix} F
\]

\[\nabla F = \vec{E} F\]

\(\vec{E}\) is 2D evolution field in \(\vec{v} = (\ln \mu^2, \ln \zeta)\) coordinates
TMD evolution is two-dimensional

\[
\begin{pmatrix}
\mu^2 \frac{d}{d\mu^2} \\
\zeta \frac{d}{d\zeta}
\end{pmatrix}
F = \begin{pmatrix}
\frac{\gamma F}{2} \\
-D
\end{pmatrix}
F
\]

\[\vec{\nabla} F = \vec{E} F\]

\(\vec{E}\) is 2D evolution field in 2D = (ln \(\mu^2\), ln \(\zeta\)) coordinates

Solution

\[R[(\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)] = \exp \left( \int_P d\vec{v} \cdot \vec{E} \right)\]
TMD evolution is two-dimensional

\[
\begin{pmatrix}
\mu^2 \frac{d}{d\mu^2} \\
\zeta \frac{d}{d\zeta}
\end{pmatrix} \mathcal{F} = \left( \gamma_F \right) \frac{2}{-D} \mathcal{F}
\]

\[
\vec{\nabla} \mathcal{F} = \vec{E} \mathcal{F}
\]

\( \vec{E} \) is 2D evolution field in \( \vec{\nu} = (\ln \mu^2, \ln \zeta) \) coordinates

Solution

\[
R[(\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)] = \exp \left( \int_P d\vec{\nu} \cdot \vec{E} \right)
\]

The integration path is unimportant!
Examples

\[ \ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - D(\mu_i, b) \ln \left( \frac{\zeta_f}{\zeta_i} \right) \]

[Collins’ textbook], [Aybat, Rogers, 1101.5057], ....

99% popular

\[ \times \exp \left\{ \ln \frac{\sqrt{\xi_A}}{\mu_b} R(b_\ast; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[ \nu_D(g(\mu'); 1) - \ln \frac{\sqrt{\xi_A}}{\mu'} \gamma_K(g(\mu')) \right] \right\}. \]

(13.70)
Examples

Solution 1

\[ \ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - D(\mu_i, b) \ln \left( \frac{\zeta_f}{\zeta_i} \right) \]

[Collins’ textbook], [Aybat, Rogers, 1101.5057], ...

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Solution 2

\[ \ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_i) - D(\mu_f, b) \ln \left( \frac{\zeta_f}{\zeta_i} \right) \]
Examples

Solution 1

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[Collins’ textbook], [Aybat, Rogers, 1101.5057],...

99% popular

Solution 2

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Examples

**Solution 1**

\[
\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - \mathcal{D}(\mu_i, b) \ln \left( \frac{\zeta_f}{\zeta_i} \right)
\]

[Carrison’s textbook], [Aybat, Rogers, 1101.5057],...

99% popular

**Solution 2**

\[
\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_i) - \mathcal{D}(\mu_f, b) \ln \left( \frac{\zeta_f}{\zeta_i} \right)
\]

**Solution 3**

\[
\ln R = \int_0^1 \left( \gamma_F(\mu(t), \zeta(t)) \right) \frac{\mu_f - \mu_i}{(\mu_f - \mu_i)t + \mu_i} - \mathcal{D}(\mu(t), b) \frac{\zeta_f - \zeta_i}{(\zeta_f - \zeta_i)t + \zeta_i} dt
\]
Unique solution

Solution exist only if integrability condition holds
\[ \zeta \frac{d\gamma F}{d\zeta} = -\mu^2 \frac{dD}{d\mu^2} \]

\[ \vec{\nabla} \times \vec{E} = 0 \]

\( \vec{E} \) is conservative field
Unique solution

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\[ \zeta \frac{d \gamma_F}{d \zeta} = -\mu^2 \frac{d \mathcal{D}}{d \mu^2} \]

\[ \nabla \times \mathbf{E} = 0 \]

\( \mathbf{E} \) is conservative field

**Integrability condition** is trivially satisfied due to collinear overlap of divergences

\[ \zeta \frac{d}{d \zeta} \gamma_F(\mu, \zeta) = -\Gamma(\mu), \quad \mu^2 \frac{d}{d \mu^2} \mathcal{D}(\mu, b) = \Gamma(\mu) \]
Unique solution

Solution exist only if integrability condition holds

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\[ \vec{E} \] is conservative field

**Integrability condition** is trivially satisfied due to collinear overlap of divergences

\[ \zeta \frac{d}{d \zeta} \gamma_F(\mu, \zeta) = -\Gamma(\mu), \quad \mu^2 \frac{d}{d \mu^2} \mathcal{D}(\mu, b) \neq \Gamma(\mu) \]

In fixed order PT integrability condition is violated.

The restoration procedure is ambiguous

(large impact at large-\(b\))

See extended dicussion in [Scimemi, AV; 1803.11089]
Numerical effect of path dependence

Evolution from $M_Z$ to $1/b^*$

Without Log-resummation $\mathcal{O}(a_s^{n+1} L^n)$

With Log-resummation $\mathcal{O}(a_s^{n+1} L)$

There are methods to eliminate path-dependence by adding higher-PT terms in anomalous dimension. For detailed discussion see [1803.11089].
Evolution potential

Solution exist only if integrability condition holds
\[ \zeta \frac{d\gamma_F}{d\zeta} = -\mu^2 \frac{dD}{d\mu^2} \]

\[ \nabla \times \mathbf{E} = 0 \]
\( \mathbf{E} \) is conservative field

Conservative field is determined by a potential
\[ \mathbf{E} = \nabla U \]

Evolution is a difference between potentials
\[ R[(\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)] = \exp (U_f - U_i) \]
Evolution potential

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Conservative field is determined by a potential

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Evolution is a difference between potentials

\[ R[(\mu_f, \zeta_f) \to (\mu_i, \zeta_i)] = \exp (U_f - U_i) \]

This absolutely standard picture contains an important message.
TMD distribution is not defined by a scale \((\mu, \zeta)\). It is defined by an equipotential line.

The scaling is defined by a difference between scales and a difference between potentials.
TMD distribution is not defined by a scale \((\mu, \zeta)\).
It is defined by an equipotential line.

The scaling is defined by a difference between scales
a difference between potentials

Evolution factor to both points
is the same
although the scales are different by \(10^2\text{GeV}^2\)
TMD distributions on the same equipotential line are equivalent.

We can enumerate them by a lines not by \((\mu, \zeta)\)

\[
F(x, b; \mu, \zeta) \rightarrow F(z, b; \text{line})
\]
TMD distributions on the same equipotential line are equivalent.

We can enumerate them by a lines not by \((\mu, \zeta)\)

\[
F(x, b; \mu, \zeta) \rightarrow F(z, b; \text{line})
\]

Initially in [Scimemi,AV,1706.01473] we call it \(\zeta\)-prescription, which is, probably, not the best name.
In $\zeta$-prescription we set

$$\zeta \to \zeta_\mu(\nu)$$

- TMDs are "enumerated" by $\nu$ (the number of line)
- TMDs are "naive" scale-independent

$$\mu \frac{d}{d\mu} F(x, b; \mu, \zeta_\mu) = 0 \quad \Rightarrow \text{No double-logs in the matching.}$$
In $\zeta$-prescription we set

$$\zeta \to \zeta_\mu(\nu)$$

- TMDs are "enumerated" by $\nu$ (the number of line)
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$$\mu \frac{d}{d\mu} F(x, b; \mu, \zeta_\mu) = 0 \quad \Rightarrow \text{No double-logs in the matching.}$$

TMD distribution depends only on the "number" of equipotential line

$$F(x, b; \mu, \zeta) \to F(x, b; \nu)$$

$$\frac{dF(x, b; \nu)}{d\nu} = \frac{dU(b; \nu)}{d\nu} F(x, b; \nu)$$

$$\Updownarrow$$

$$F(x, b; \nu) = e^{U(b; \nu) - U(b; \nu_0)} F(x, b; \nu_0)$$
The simplest way to measure the difference between potentials

$$R = \left( \frac{\zeta_f}{\zeta_{\mu_f}} \right)^{-D(\mu_f, b)}$$

- Numerically simple (and fast).
  Compare to
  $$\times \exp \left\{ \ln \frac{\sqrt{\lambda_h}}{\mu_b} K(b_h; \mu_b) + \int_{\mu}^{\mu'} \frac{d\mu'}{\mu'} \left[ \gamma_0(g(\mu')/1) - \ln \frac{\sqrt{\lambda_h}}{\mu'} \gamma_k(g(\mu')) \right] \right\}.$$ (13.70)

- $\mu_f = Q$ thus $a_s$ is small
- It is different representation of the Sudakov exponent.
The simplest way to measure the difference between potentials

\[ R = \left( \frac{\zeta_f}{\zeta_{\mu_f}} \right)^{-D(\mu_f, b)} \]

- Numerically simple (and fast).
- Compare to

\[ \times \exp \left\{ \ln \frac{\sqrt{\epsilon}}{\mu_b} K(b, \mu_b) + \int_{\mu_s}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_0(\mu') \gamma_1(\mu') - \ln \frac{\sqrt{\epsilon}}{\mu'} \gamma_2(\mu') \right] \right\}. \]

- \( \mu_f = Q \) thus \( a_s \) is small
- It is different representation of the Sudakov exponent.

Different solutions converge with increase of PT order
There is a unique line which passes through all $\mu$'s.

The optimal TMD distribution

$$F(x, b) = F(x, b; \mu, \zeta_\mu)$$

where $\zeta_\mu$ is the special line.
\[
\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2 b}{4\pi} e^{i (b \cdot q_T)} H_{ff'}(Q) \{ \tilde R_f[b; Q] \}^2 \tilde F_f \leftarrow h(x_1, b) \tilde F_{f'} \leftarrow h(x_2, b),
\]

with \( \zeta_f = \mu_f^2 = Q^2 \)

\[
\tilde R_f[b; Q] = (Qb)^{-\mathcal{D}^f(Q,b)} \exp\{-\mathcal{D}^f(Q,b)v_f(Q,b)\}
\]

- \( v \) is given by the perturbative series, \( v = \frac{3}{2} + a_s \ldots \)
- \( \tilde F \) is TMD in the "naive" \( \zeta \)-prescription

- There are only \( (\mu_f, \zeta_f) \) scales and no solution dependence.
- **Clear separation of TMD evolution from the model for TMD distribution.**
**Evolution with $b$-dependent scale (CSS-like)**

$$(Q, Q^2) \rightarrow (\mu_b, \mu_b^2)$$

Here $\mu_b = \frac{C_0}{b^*}$ with $b_{\text{max}} = 1.2\text{GeV}^{-1}$

**Analogy in DIS**

Scale depends on parameter $\leftrightarrow d\sigma = C(Q) R[Q \rightarrow \text{ch}(x)] f(x, \text{ch}(x))$

PDF $f(x, \text{ch}(x))$ has no interpretation, no sense, and depends on the order of evolution in use.
Optimal version

\((Q, Q^2) \rightarrow (Q, \zeta_Q)\)

Analogy in DIS

Scale (potential) is fixed \(\leftrightarrow d\sigma = C(Q)R[Q \rightarrow 2\text{GeV}]f(x, 2\text{GeV})\)

PDF \(f(x, 2\text{GeV})\) is just a model and is dependent on the order of evolution in use.
The evolution potential depends on $b$. Relative position of its elements (saddle-point, special lines) dictates the shape of evolution factor.

\begin{align*}
\mu &= 1 \text{GeV} \\
\mu &= 3 \text{GeV} \\
\mu &= 10 \text{GeV}
\end{align*}
Variety of evolutions
- LO, NLO, NNLO
- No restriction for NP models
- Fast code
- DY cross-sections
- SIDIS cross-sections (not tuned yet)
- Theory uncertainty bands

https://teorica.fis.ucm.es/artemide/
Main message:
TMD evolution is a double scale evolution. Therefore, it should be considered with care, and then it grants many simplifications.

TMD distributions on a same equipotential line are equivalent. Enumerate them with lines!
- Universal for all quantum numbers
- Very simple practical formula (no integrations!)
- Guaranteed absence of (large) logarithms in the matching coefficient
- TMD model is independent on evolution order.
  E.g. You can use NNLO unpolarized and LO Sivers together, without theory tensions

Double-scale evolution is not unique for TMD case. It also appears in $k_T$-resummation, joint resummation, DPDs, etc.
Some non-interesting singularities at $\mu, \zeta \to \infty$

Landau pole at $\mu = \Lambda$

Saddle point (blue dot)

\[
D(\mu_{\text{saddle}}, b) = 0, \quad \gamma_M(\mu_{\text{saddle}}, \zeta_{\text{saddle}}, b) = 0
\]