Iso-vector Transversity Quark Distributions from Lattice QCD

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Outline

- Introduction
- Quark distributions and quark quasi-distributions
- Extracting quark distributions from the quasi-distributions
- Computation of the matrix elements using lattice QCD
- Renormalization
- The $x$ dependence of the iso-vector quark distributions
- Summary
Introduction

Complete set of twist-2 parton distribution functions
Cross sections are measured

Totally inclusive

Have access too to the chiral-even distributions $f_1(x)$ and $g_1(x)$

Semi-inclusive

Have access to the chiral-odd distribution $h_1(x)$

It is naturally more difficult to have info on $h_1(x)$
Two recent extractions

Can we have a first principles calculation of $h_1^q(x)$?
Light-cone quark distributions

The most general form of the matrix element is:

\[ \langle P|O^{\mu_1\mu_2\cdots\mu_n}|P\rangle = 2a_n^{(0)} \Pi^{\mu_1\mu_2\cdots\mu_n} \]

\[ \Pi^{\mu_1\mu_2\cdots\mu_n} = \sum_{j=0}^{k} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \{g \cdots gP \cdots P\}_{k,j} (P^2)^j \]

We use the following four-vectors

\[ P = (P_0,0,0,P_3) \quad \lambda = (1,0,0,-1)/\sqrt{2} \]

\[ \lambda \cdot P = (P_0 + P_3)/\sqrt{2} = P_+ \]

\[ \lambda_{\mu_1}\lambda_{\mu_2}\left\langle P \left| O^{\mu_1 \mu_2} \right| P \right\rangle = 2a_n^{(0)} \left( P^+ P^+ - \lambda^2 \frac{M^2}{4} \right) = 2a_n^{(0)} P^+ P^+ \]

In general, we have

\[ \lambda_{\mu_1} \cdots \lambda_{\mu_n} \Pi^{\mu_1 \cdots \mu_n} = (P^+)^n \]

\[ \langle P|O^{\cdots +}|P\rangle = 2a_n^{(0)} (P^+)^n \]

Matrix elements projected on the light-cone are protected from target mass corrections.
Taking the inverse Mellin transform

\[ a_n^{(0)} = \int dx\ x^{n-1}q(x) \quad q(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dn\ x^{-n}a_n^{(0)} \]

Using \[ a_n^{(0)} = \langle P|O^{++}\cdots|P\rangle / 2(P^+)^n \]

\[ q(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+W(\xi^-,0)\psi(0)|P\rangle \]

\[ W(\xi^-,0) = e^{-ig\int_0^{\xi^-} A^+(\eta^-)d\eta^-} \quad \text{(Wilson line)} \]

- Light cone correlations
- Equivalent to the distributions in the Infinite Momentum Frame
- Light cone dominated \( \xi^2 = t^2 - z^2 \sim 0 \)
- Not calculable on Euclidian lattice \( t^2 + z^2 \sim 0 \)
Suppose we project outside the light-cone:

\[ \lambda = (0,0,0,-1) \quad P = (P_0,0,0,P_3) \quad \lambda \cdot P = P_3 \]

For example, for n=2

\begin{equation}
\langle P | O^{33} | P \rangle = 2\tilde{\alpha}_n^{(0)}(P^3 P^3 - \lambda^2 P^2 / 4) = 2\tilde{\alpha}_n^{(0)}((P^3)^2 + P^2 / 4)
\end{equation}

After the inverse Mellin transform,

\[ \tilde{q}(x, P_3) = \int_{-\infty}^{+\infty} \frac{dz}{4\pi} e^{-ixP_3} \langle P | \bar{\psi}(z) \gamma^3 W(z, 0) \psi(0) | P \rangle + O \left( \frac{M^2}{P_3^2}, \frac{\Lambda_{QCD}^2}{P_3^2} \right) \]

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice

Quasi Distributions

The light cone distributions:

\[ x = \frac{k^+}{P^+} \]

\[ 0 \leq x \leq 1 \]

Distributions can be defined in the infinite momentum frame: \( P_3, P^+ \to \infty \)

Quasi distributions:

\( P_3 \) large but finite

Usual partonic interpretation is lost

\[ x < 0 \text{ or } x > 1 \text{ is possible} \]

But they can be related to each other!


Infinite momentum: 

\[ p_3 \to \infty \]  

(before integrating over the quark transverse momentum \( k_T \))

\[
q(x, \mu) = q_{bare}(x) \left\{ 1 + \frac{\alpha_s}{2\pi} Z_F(\mu) \right\} + \frac{\alpha_s}{2\pi} \int_x^1 \Gamma \left( \frac{x}{y}, \mu \right) q_{bare}(y) \frac{dy}{y} + \mathcal{O}(\alpha_s^2)
\]

Finite momentum: 

\[ p_3 \text{ fixed} \]

\[
\tilde{q}(x, P_3) = q_{bare}(x) \left\{ 1 + \frac{\alpha_s}{2\pi} \tilde{Z}_F(P_3) \right\} + \frac{\alpha_s}{2\pi} \int_{x/y_c}^1 \tilde{\Gamma} \left( \frac{x}{y}, P_3 \right) q_{bare}(y) \frac{dy}{y} + \mathcal{O}(\alpha_s^2)
\]

\[ \tilde{q}(\pm y_c) = 0 \]

In principle, \( y_c \to \infty \)
One-loop correction using pQCD

Vertex: $\Gamma$ or $\tilde{\Gamma}$

Self-energy: $Z_F$ or $\tilde{Z}_F$

For the transversity operator:

X. Xiong, X. Ji, J. H. Zhang and Y. Zhao, PRD 90 014051 (2014), in a cut-off scheme

C. Alexandrou et al., 1807.00232, in the $\overline{MS}$ scheme
The two equations can be solved for the quark distributions, resulting in a matching equation:

\[ q(x, \mu) = \tilde{q}(x, p_3) - \frac{\alpha_s}{2\pi} \tilde{q}(x, p_3) \delta Z_F \left( \frac{\mu}{p_3}, x_c \right) - \frac{\alpha_s}{2\pi} \int_{-|x|/y_c}^{-x_c} \delta \Gamma \left( y, \frac{\mu}{p_3} \right) \tilde{q} \left( \frac{x}{y}, p_3 \right) \frac{dy}{|y|} \]

\[ - \frac{\alpha_s}{2\pi} \int_{+|x|/y_c}^{+x_c} \delta \Gamma \left( y, \frac{\mu}{p_3} \right) \tilde{q} \left( \frac{x}{y}, p_3 \right) \frac{dy}{|y|} \]

Matching equation

Where

\[ \delta \Gamma = \tilde{\Gamma} - \Gamma \]

\[ \delta Z_F = \tilde{Z}_F - Z_F \]

are calculated using perturbation theory in the continuum.

The integral in \( x \) in the quasi-quark self-energy, \( \tilde{Z}_F \), is left unintegrated, hence the dependence on the limits of integration, \( \pm x_c \). At the end, \( x_c \to \infty \) in \( \delta Z_F \).

Because quasi-quark vertex correction, \( \tilde{\Gamma} \), only vanishes at the infinity, the range of integration in the vertex also extends to zero in the convolution as \( y_c \to \infty \).
Main steps of the procedure:

1. Compute the matrix elements between proton states with finite $P_3$;
2. Non-perturbative renormalization of the matrix elements;
3. Fourier transform to obtain the quasi-PDF $\tilde{q}(x, P_3, \mu)$;
4. Matching procedure to obtain the light-cone PDF $q(x, \mu)$;
5. Apply Target Mass Corrections (TMCs) to correct for the powers of $M^2/P_3^2$. 
Computation of matrix elements using the lattice QCD

\[
\frac{C^{3pt}(T_s, \tau, 0; P_3)}{C^{2pt}(T_s, 0; P_3)} \propto M_{h_1}(P_3, z), \quad 0 \ll \tau \ll T_s
\]

With the 3 point function given by:

\[
C^{3pt}(t, \tau, 0) = \langle N_\alpha(\vec{P}, t) \mathcal{O}(\tau) \overline{N_\alpha}(\vec{P}, 0) \rangle
\]

And

\[
\mathcal{O}(z, \tau, Q^2 = 0) = \sum_{y} \overline{\psi}(y + z) \sigma_{3i} W (y + z, y) \psi(y)
\]

Where the matrix elements (ME) are:

\[
M_{h_1}(P_3, z) = \langle P | \overline{\psi}(z) \sigma_{3i} W (z, 0) \psi(0) | P \rangle
\]

Setup:

\[
N_f = 2, \quad \beta = \frac{6}{g_0^2} = 2.10, \quad a = 0.0938(3)(2) \text{ fm}
\]

\[
48^3 \times 96, \quad L = 4.5 \text{ fm}, \quad m_\pi = 0.1304(4) \text{ GeV}, \quad m_\pi L = 2.98(1)
\]

\[
P_3 = \frac{6\pi}{L}, \frac{8\pi}{L}, \frac{10\pi}{L} = 0.84, 1.11, 1.38 \text{ GeV}
\]
Computation made for the transversity ($\sigma_{3i}$) distributions

6 directions of Wilson line: $\pm x, \pm y, \pm z$

16 source positions

Separation $T_s \approx 1.1$ fm as the lowest safe choice

<table>
<thead>
<tr>
<th></th>
<th>$\frac{6\pi}{L}$ (0.83GeV)</th>
<th>$\frac{8\pi}{L}$ (1.11GeV)</th>
<th>$\frac{10\pi}{L}$ (1.38GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{conf}$</td>
<td>100</td>
<td>425</td>
<td>811</td>
</tr>
<tr>
<td>$N_{meas.}$</td>
<td>9600</td>
<td>38250</td>
<td>72990</td>
</tr>
</tbody>
</table>

Effectively, we compute the following matrix elements

$$M_{h_1}^{u-d} = \langle P | \bar{\psi}(z) \frac{\sigma_{31} + \sigma_{32}}{2} W(z, 0) \tau^3 \psi(0) | P \rangle$$
The bare matrix elements \( \langle P | \bar{\psi}(z) \frac{\sigma_{31} + \sigma_{32}}{2} W(z, 0) \tau^3 \psi(0) | P \rangle \), however, contain divergences:

Renormalization is necessary!
Renormalization

\[ M_{h_1}^{R,u-d} = Z_{h_1} M_{h_1}^{u-d} = (Re[Z_{h_1}] + i Im[Z_{h_1}])(Re[M_{h_1}^{u-d}] + i Im[M_{h_1}^{u-d}]) \]

\( Z_{h_1} \) renormalizes both the usual log divergence and the linear divergence associated with the Wilson line.

Nonperturbative renormalization using the RI'-MOM to remove both divergences

C. Alexandrou et al., NPB 923 (2017) 394 (Frontier Article)
J-W. Chen et al., PRD 97 014505 (2018)
C. Alexandrou et al., 1807.00232

Convert the ME from RI'-MOM to \( \overline{MS} \) using 1-loop perturbation theory

M. Constantinou, H. Panapoulos, PRD (2017)054506

We present results for the \( \overline{MS} \) scheme
Renormalization factor for transversity

RI’-MOM scheme at the scale \( \bar{\mu}_0 = 3 \) GeV

Perturbative conversion to \( \overline{MS} \) scheme at the scale \( \sqrt{2} \) GeV

The linear divergence associated with the Wilson line makes \( Z_{h_1} \) to grow very fast for large \( z \);

That makes the renormalized ME to have amplified errors at large \( z \);
Renormalized ME for the transversity

\[ Re \left[ h_{1}^{u-d} \right] = Re [M_{h_{1}}^{u-d}] Re [Z_{h_{1}}] - Im [M_{h_{1}}^{u-d}] Im [Z_{h_{1}}] \]

\[ Im \left[ h_{1}^{u-d} \right] = Re [M_{h_{1}}^{u-d}] Im [Z_{h_{1}}] + Im [M_{h_{1}}^{u-d}] Re [Z_{h_{1}}] \]
The $x$ dependence of the distributions

Once we have the ME, we compute the qPDF:

$$\tilde{h}_1(x, \mu^2, P_3) = \int \frac{dz}{4\pi} e^{-ixP_3 z} \langle P | \bar{\psi}(z) \sigma_3 i W(z, 0) \psi(0) | P \rangle$$

And then apply the matching plus target mass corrections to obtain the light-cone PDF:

$$h_1(x, \mu) = \int_{-\infty}^{+\infty} \frac{d\xi}{|\xi|} \delta C \left( \xi, \frac{\xi \mu}{xP_3} \right) \tilde{h}_1 \left( \frac{x}{\xi}, \mu, P_3 \right)$$

C. Alexandrou et al., 1803.02685
Comparing with phenomenology:

\[ g_T = \int_{-1}^{1} dx h_1^{u-d} = 1.10(34) \]

This should be compared to: \( g_T = 1.06(1) \) from dedicated lattice QCD calculation

C. Alexandrou et al., PRD95, 114514 (2017)

\( g_T = 1.0(1) \) from Monte Carlo global analysis


\( g_T = 0.53(25) \) from global analysis of \( ep \) and \( pp \) data

Radici and Bacchetta PRL 120, 192001 (2018)
Summary

We have shown an *ab initio* computation of the $x$ dependence of the iso-vector transversity PDF at the physical point;

Excited state contamination suppressed for the source-sink separation ($T_s \approx 1.12$ fm) and momentum $P_3 = 1.38$ GeV used here;

Enormous progress over the last couple of years:

- a complete non-perturbative prescription for the ME has emerged;
- a perturbative conversion from RI’-MOM and $\overline{MS}$ has been developed;
- the matching equations relating the quasi-PDFs to the light-cone PDFs for the transversity have been developed.

New computer architectures can lead to higher values of $P_3$;

Discretization effects need to be addressed: at least 3 different lattice spacings needed;

Volume effects also need to be addressed;

We have ahead of us the exciting possibility to calculate PDFs in general from first principles.
And the final matching can be written taking $x_c \to \infty$

$$
\delta \Gamma^R \left( y, \frac{\mu}{p_3} \right) = \begin{cases} 
-\frac{2y}{1-y} \ln \frac{y-1}{y} + \frac{2}{y} & y > 1 \\
-\frac{2y}{1-y} \ln \frac{y^2 \mu^2}{4p_3^2 y(1-y)} - \frac{2y}{1-y} & 0 < y < 1 \\
-\frac{2y}{1-y} \ln \frac{y}{y-1} + \frac{2}{1-y} & y < 0
\end{cases}
$$

$$
\delta Z_F^R \left( \frac{\mu}{p_3} \right) = \begin{cases} 
\int_{-\infty}^{+\infty} d\eta \left( \frac{2\eta}{1-\eta} \ln \frac{\eta-1}{\eta} - \frac{2}{\eta} \right) & \eta > 1 \\
\int_{-\infty}^{+\infty} d\eta \left( \frac{1+\eta^2}{1-\eta} \ln \frac{y^2 \mu^2}{4p_3^2 \eta(1-\eta)} + \frac{2\eta}{1-\eta} \right) & 0 < \eta < 1 \\
\int_{-\infty}^{+\infty} d\eta \left( \frac{2\eta}{1-\eta} \ln \frac{\eta}{\eta-1} - \frac{2}{1-\eta} \right) & \eta < 0
\end{cases}
$$

Regions outside the physical region, $0 < x < 1$, are not equal to zero!

Infrared divergences are the same in the quasi and light-cone PDFs: same splitting functions;

Automatically preserves quark number in all stages of the computation;