Low-*t* reactions:
π^0 and η production in Primakoff processes

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Outline

Introduction & motivation

- Primakoff reactions: $\gamma\pi^- \to \ldots$
- From chiral perturbation theory to dispersion relations
- Input to hadronic light-by-light scattering

Extracting the chiral anomaly

- $\ldots$ from $\gamma\pi^- \to \pi^-\pi^0$

Understanding the decay $\eta \to \pi^+\pi^-\gamma$

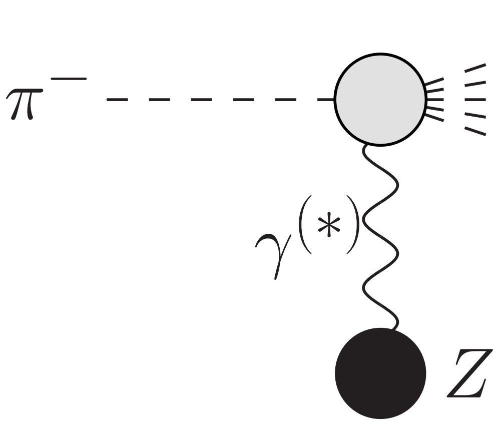
- $\ldots$ and why $\gamma\pi^- \to \pi^-\eta$ might be important for that

Summary / Outlook

Primakoff reactions: $\gamma\pi^- \to \ldots$

- pion beam at small momentum transfer:

  photon exchange $\propto 1/t \gg$ hadronic reactions



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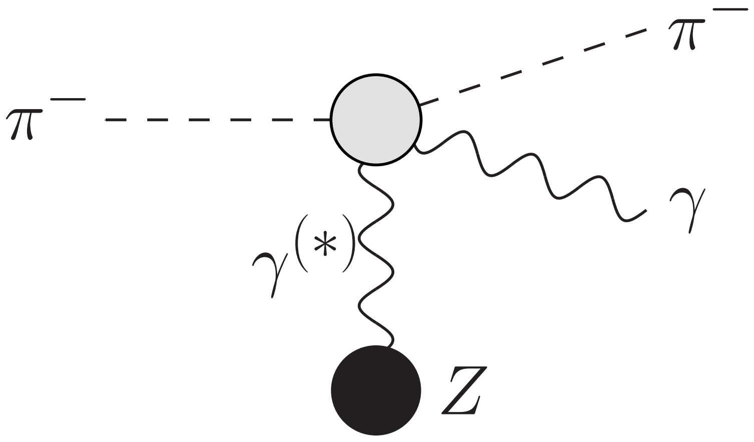

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  $\longrightarrow$ pion polarisabilities

  $\longrightarrow$ fundamental information on pion structure

  COMPASS 2015

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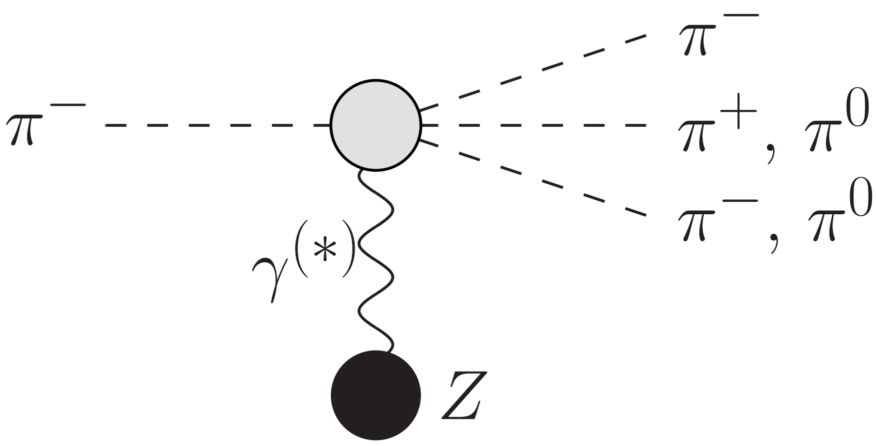

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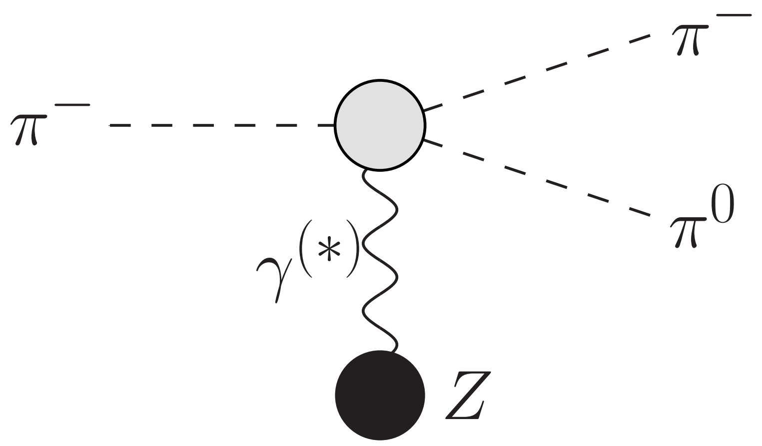

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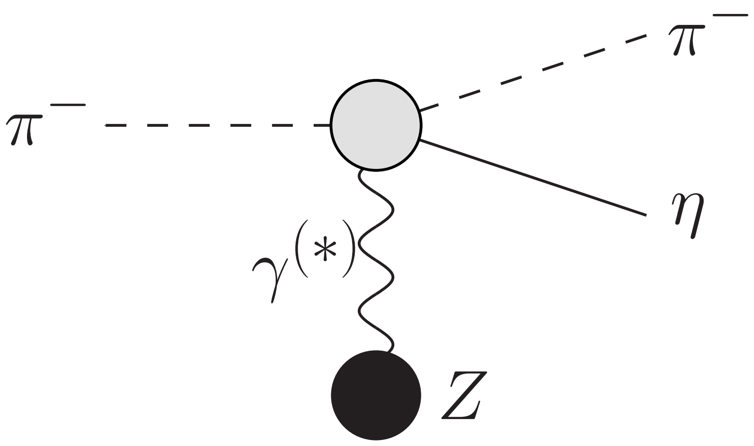

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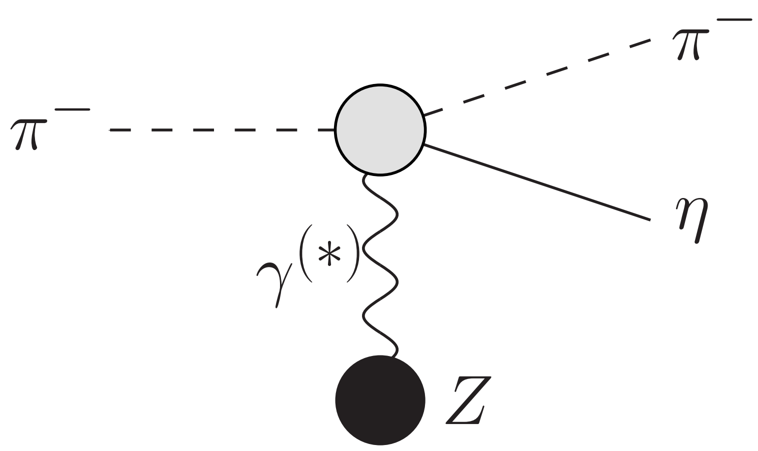

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$\longrightarrow$ many of these motivated by chiral perturbation theory

$\longrightarrow$ more fundamental interest of anomalous processes:

link to anomalous magnetic moment of the muon

Light mesons without modeling

Chiral perturbation theory. . .

- Effective field theory: simultaneous expansion in

  quark masses + small momenta

  - ▷ systematically improvable
  - ▷ well-established link to QCD: all symmetry constraints
  - ▷ interrelates many different observables

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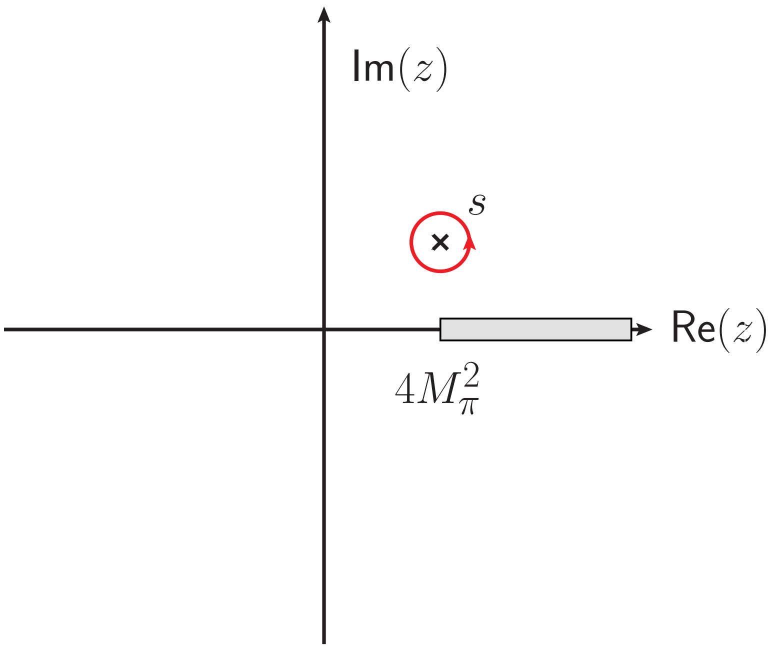
. . . and its limitations

- strong final-state interactions render corrections large
- physics of light pseudoscalars (π, K, η) only
  - ▷ (energy) range limited by resonances: $f_0(500)$, $\rho(770)$ . . .
  - ▷ unitarity ($\simeq$ probabilty cons.) only perturbatively fulfilled

$\longrightarrow$ find effective ways to resum rescattering / restore unitarity

$\longrightarrow$ dispersion relations

Dispersion relations for pedestrians



analyticity ($\simeq$ causality)

& Cauchy's theorem:

$$T(s) = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{T(z)dz}{z-s}$$

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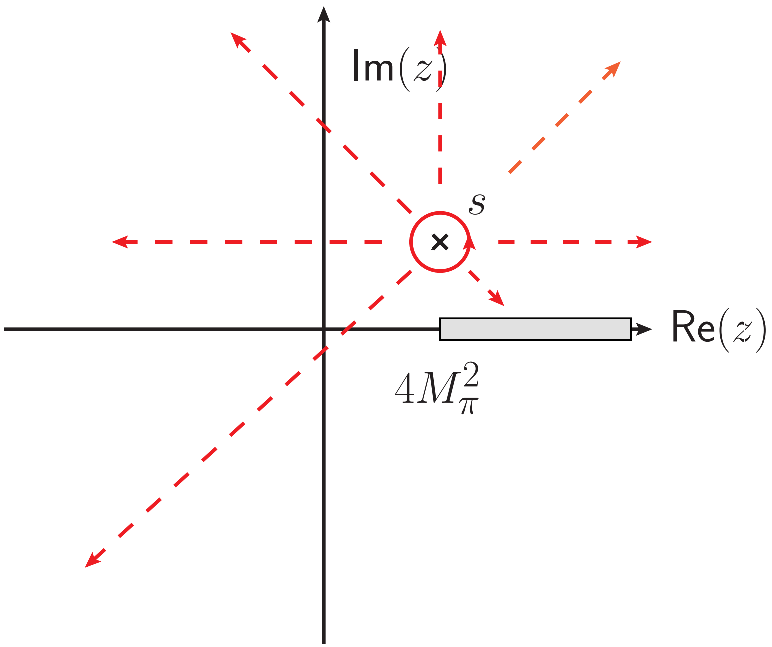

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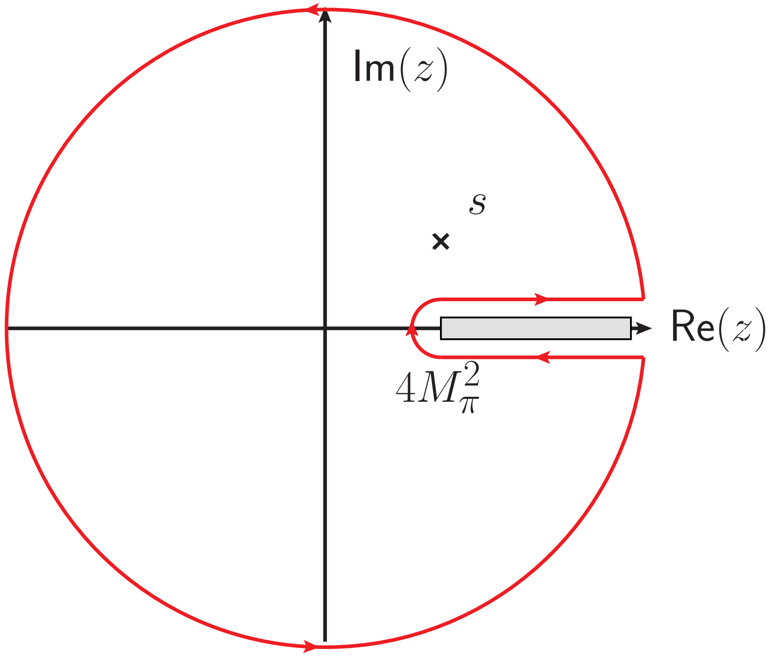

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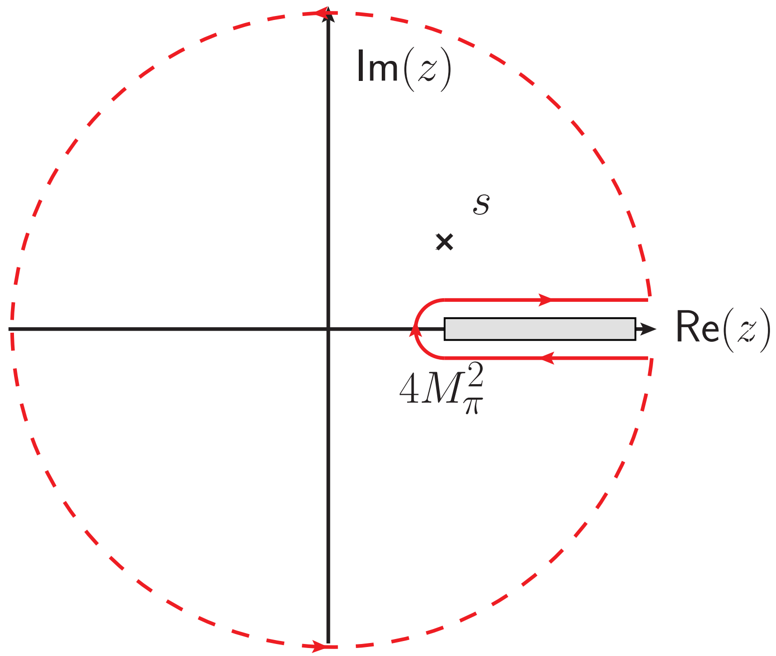

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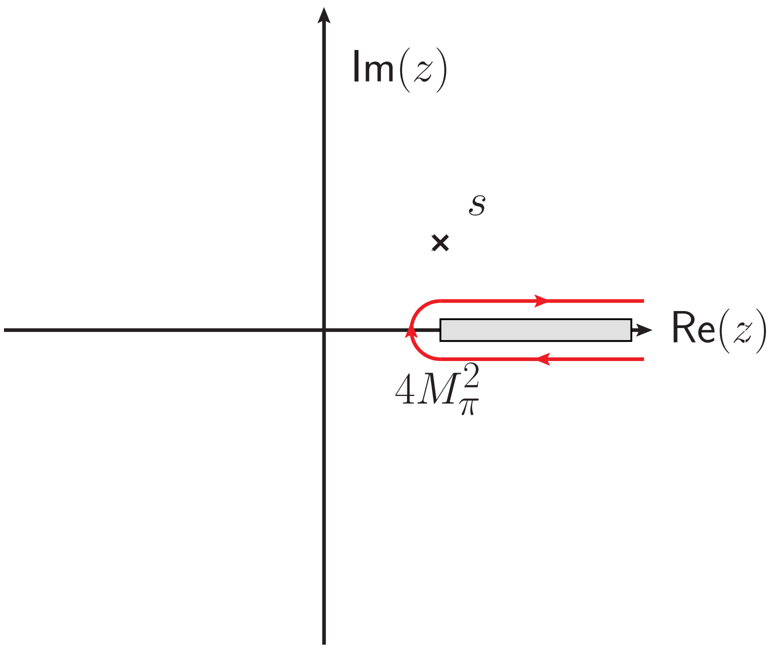

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$$\longrightarrow \frac{1}{2\pi i} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{disc}\, T(z)dz}{z-s}$$

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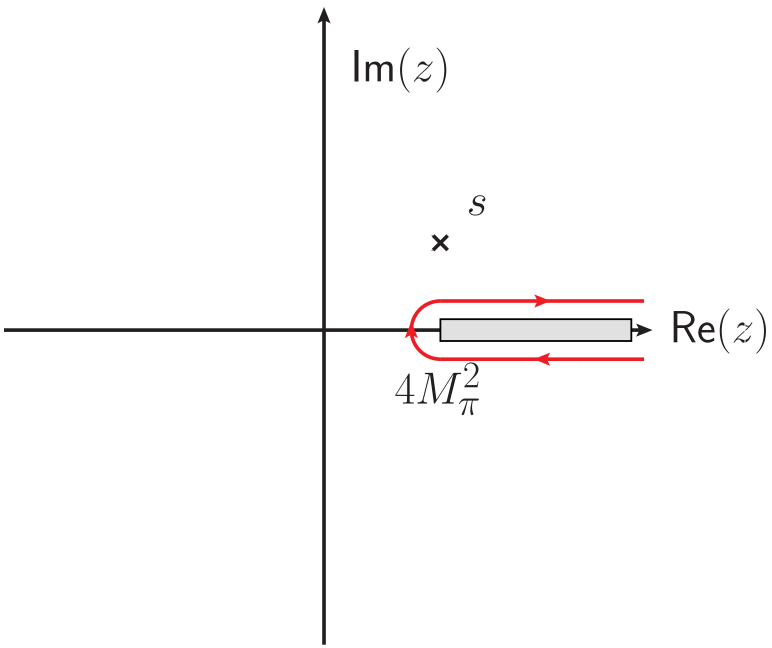

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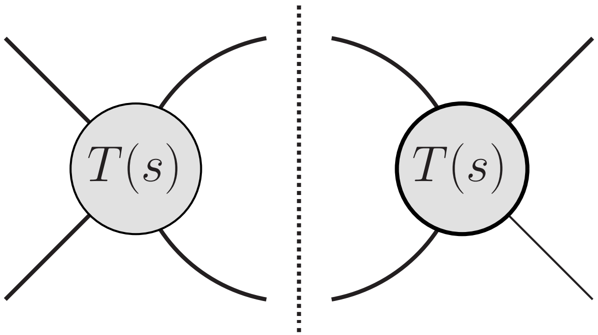

e.g. if $T(s)$ is a $\pi\pi$ partial wave $\longrightarrow$

$$\frac{\operatorname{disc} T(s)}{2i} = \operatorname{Im} T(s) = \frac{2q_\pi}{\sqrt{s}} \theta(s - 4M_\pi^2)|T(s)|^2$$

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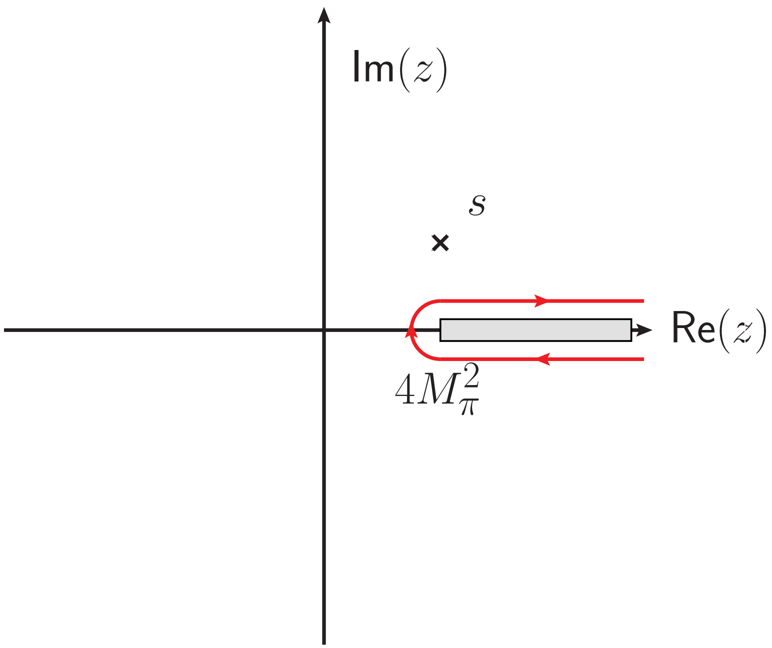

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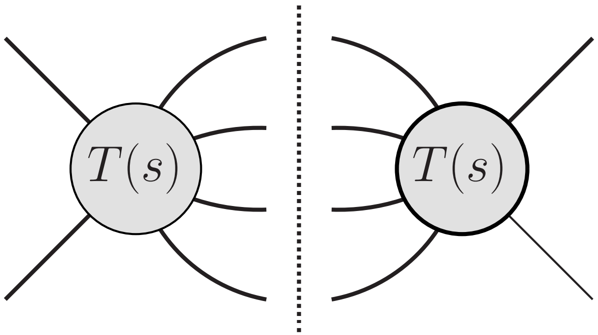
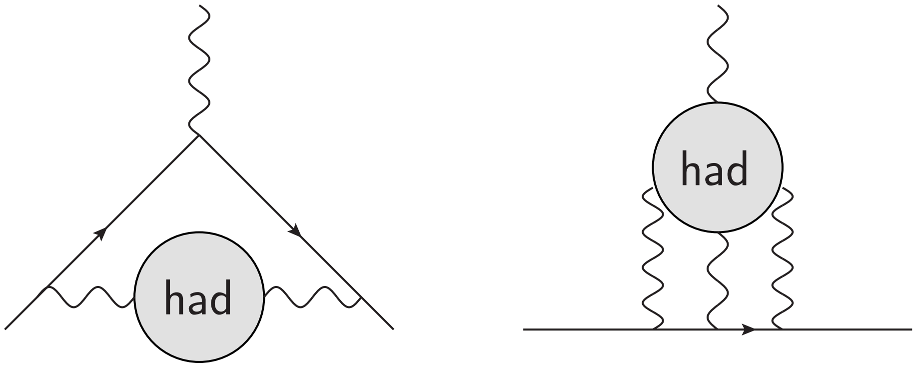


inelastic intermediate states ($K\bar{K}$, 4π)
suppressed at low energies
$\longrightarrow$ will be neglected in the following

Meson transition form factors and $(g-2)_\mu$

Czerwiński et al., arXiv:1207.6556 [hep-ph]

- leading and next-to-leading hadronic effects in $(g-2)_\mu$:



$\longrightarrow$ hadronic vacuum polarisation: $e^+e^- \to$ hadrons

$\longrightarrow$ hadronic light-by-light soon dominant uncertainty

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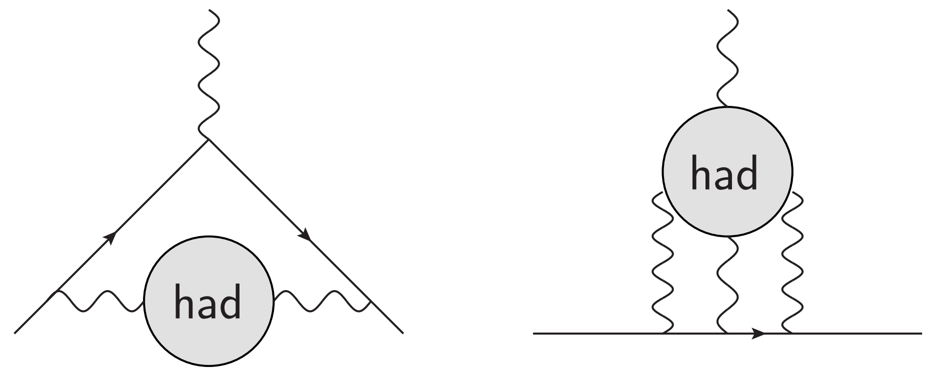

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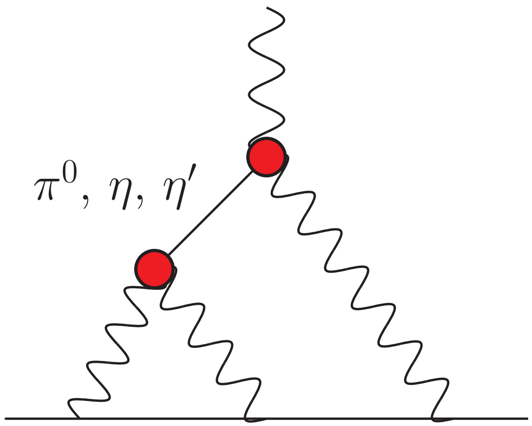
- important contribution: pseudoscalar pole terms

  singly / doubly virtual form factors

  $F_{P\gamma\gamma^*}(q^2,0)$ and $F_{P\gamma^*\gamma^*}(q_1^2,q_2^2)$



Dispersive analysis of $\pi^0/\eta \to \gamma^*\gamma^*$

- isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{vs}(q_1^2,q_2^2) + F_{vs}(q_2^2,q_1^2)$$

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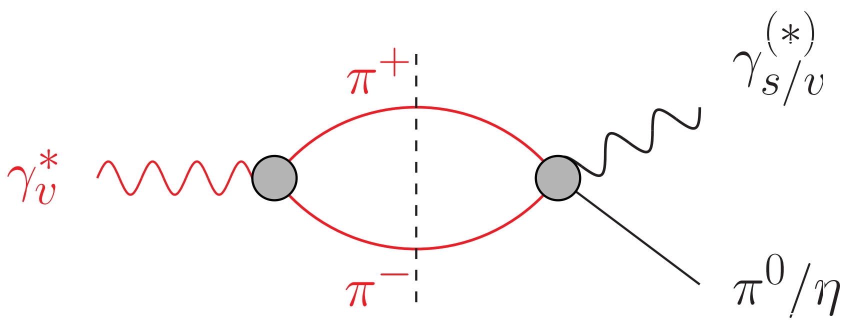
Hanhart et al. 2013, Hoferichter et al. 2014



▷ isovector photon: 2 pions

$\propto$ pion vector form factor $\times$ $\gamma\pi \to \pi\pi$ / $\eta \to \pi\pi\gamma$

all determined in terms of pion–pion P-wave phase shift

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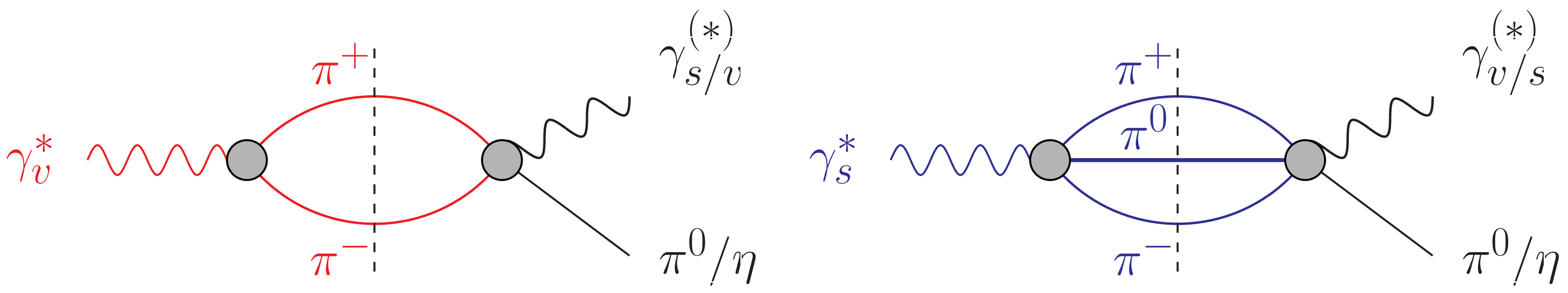

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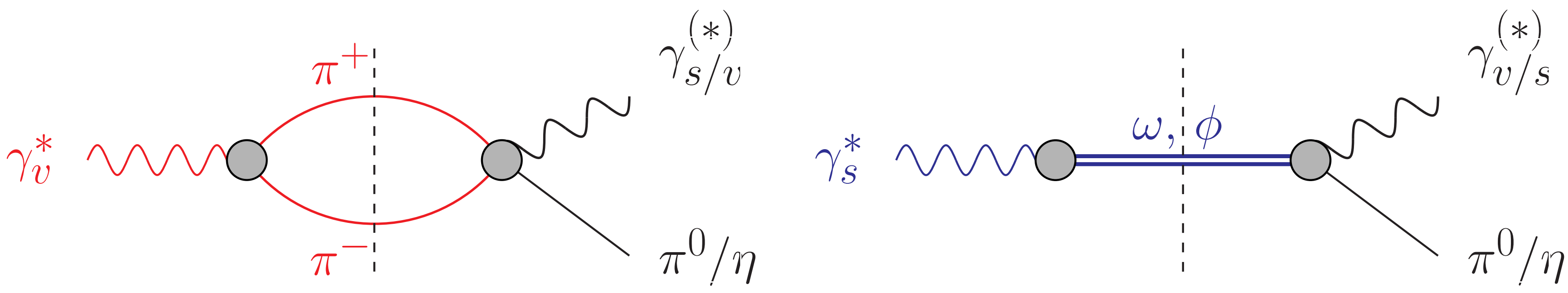

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▷ isoscalar photon: 3 pions $\longrightarrow$ dominated by narrow $\omega,\ \phi$

$\leftrightarrow \omega/\phi$ transition form factors; very small for the η

Testing the Wess–Zumino–Witten chiral anomaly

- controls low-energy processes of odd intrinsic parity
- π^0 decay $\pi^0 \to \gamma\gamma$: $F_{\pi^0\gamma\gamma} = \dfrac{e^2}{4\pi^2 F_\pi}$

  F_π: pion decay constant $\longrightarrow$ measured at 1.5% level PrimEx 2011

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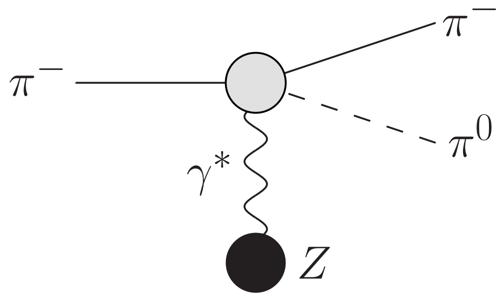
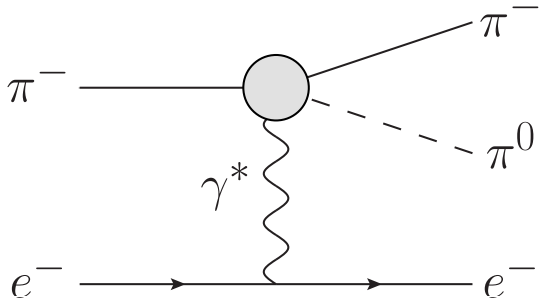


$F_{3\pi} = (10.7 \pm 1.2)$ GeV^{-3}

Serpukhov 1987, Ametller et al. 2001

$\pi^- e^- \to \pi^- e^- \pi^0$



$F_{3\pi} = (9.6 \pm 1.1)$ GeV^{-3}

Giller et al. 2005

$\longrightarrow$ $F_{3\pi}$ tested only at 10% level

Chiral anomaly: Primakoff measurement

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- idea: use dispersion relations to exploit all data below 1 GeV for anomaly extraction
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  Hoferichter, BK, Sakkas 2012

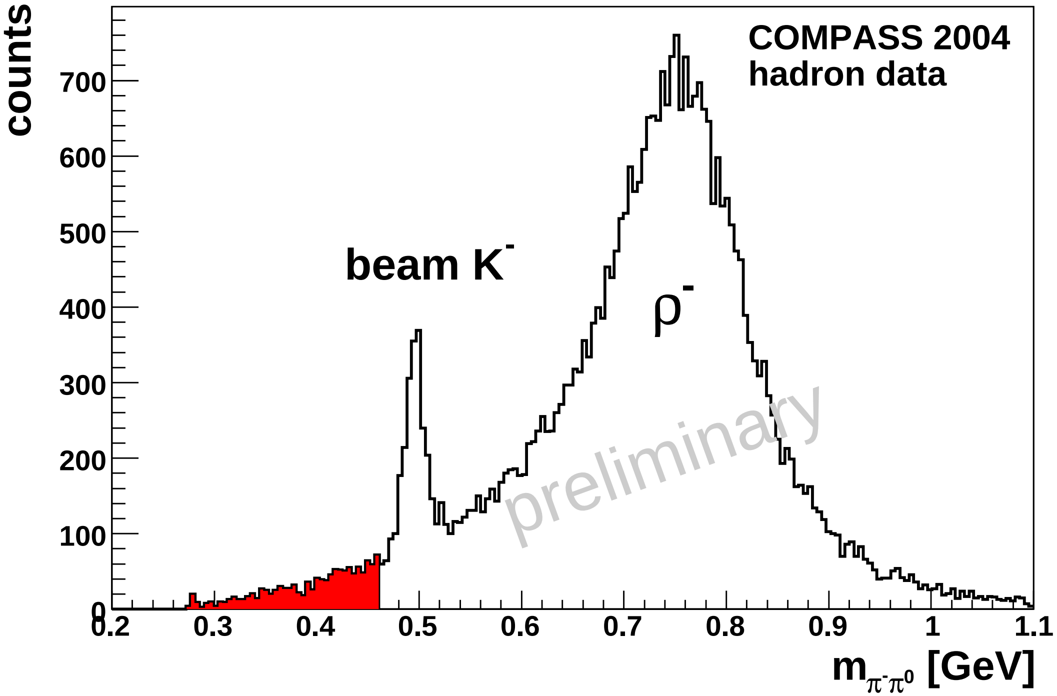

figure courtesy of T. Nagel 2009

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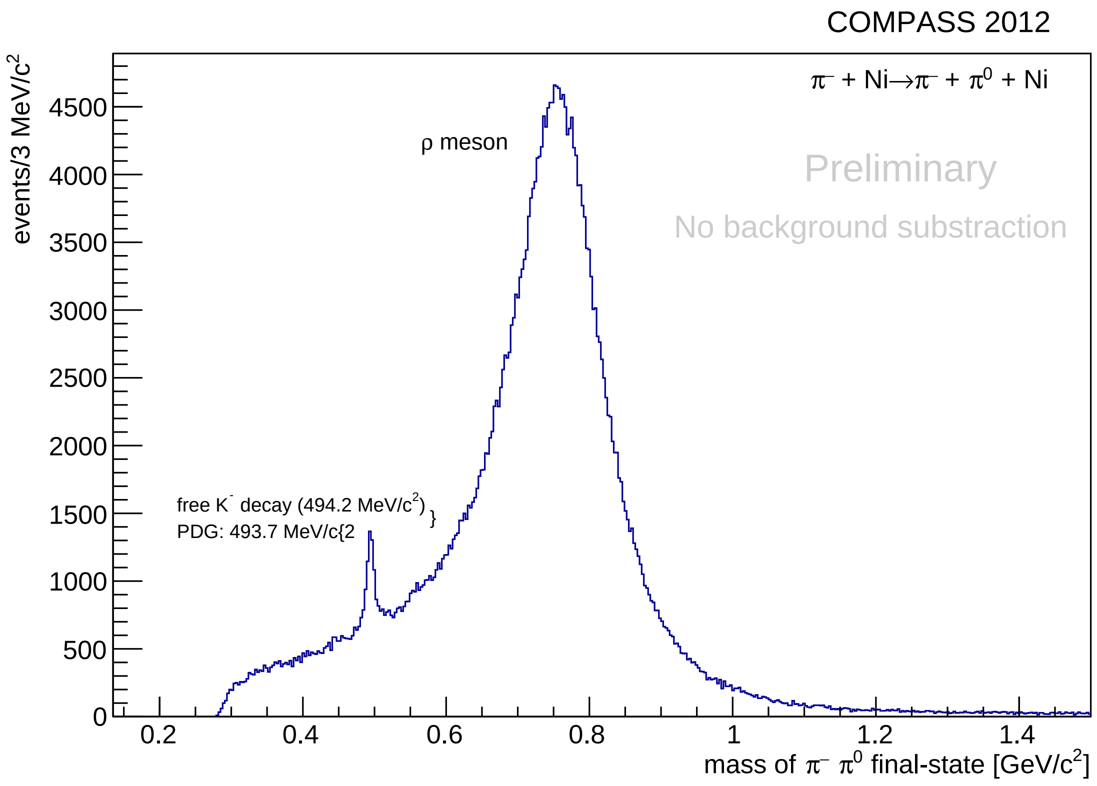
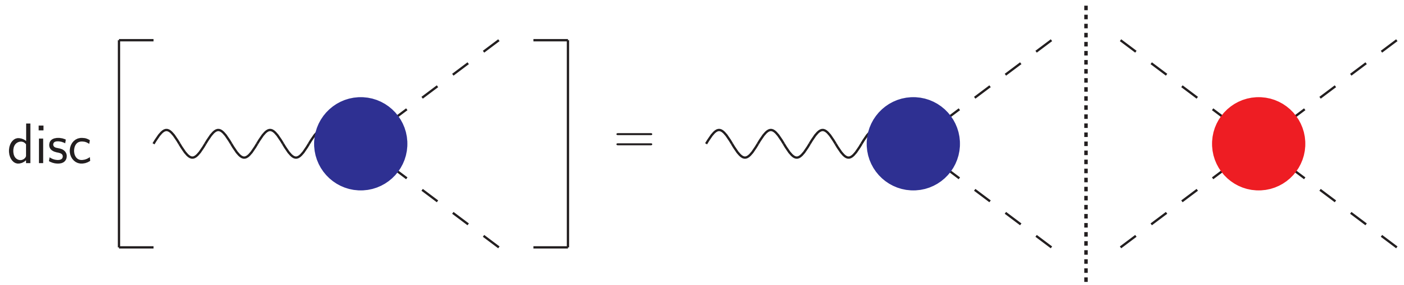


J. Seyfried, MSc thesis 2017

Warm-up: pion form factor from dispersion relations

- just two particles in final state: form factor; from unitarity:



$$\frac{1}{2i}\,\text{disc}\,F_I(s) \;=\; \text{Im}\,F_I(s) \;=\; F_I(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_I(s)\,e^{-i\delta_I(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_I(s)$ is just $\delta_I(s)$ Watson 1954

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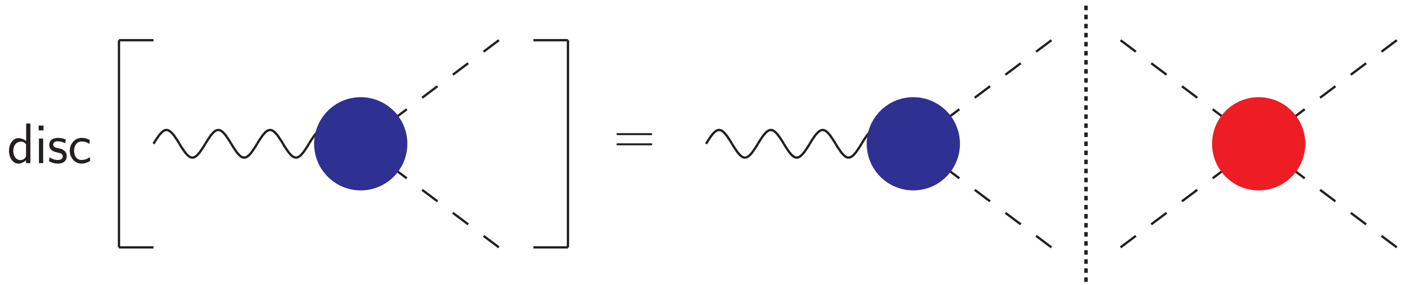

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- solution to this homogeneous integral equation known:

$$F_I(s) = P_I(s)\Omega_I(s)\;, \quad \Omega_I(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_I(s')}{s'(s'-s)}\right\}$$

$P_I(s)$ polynomial, $\Omega_I(s)$ Omnès function Omnès 1958

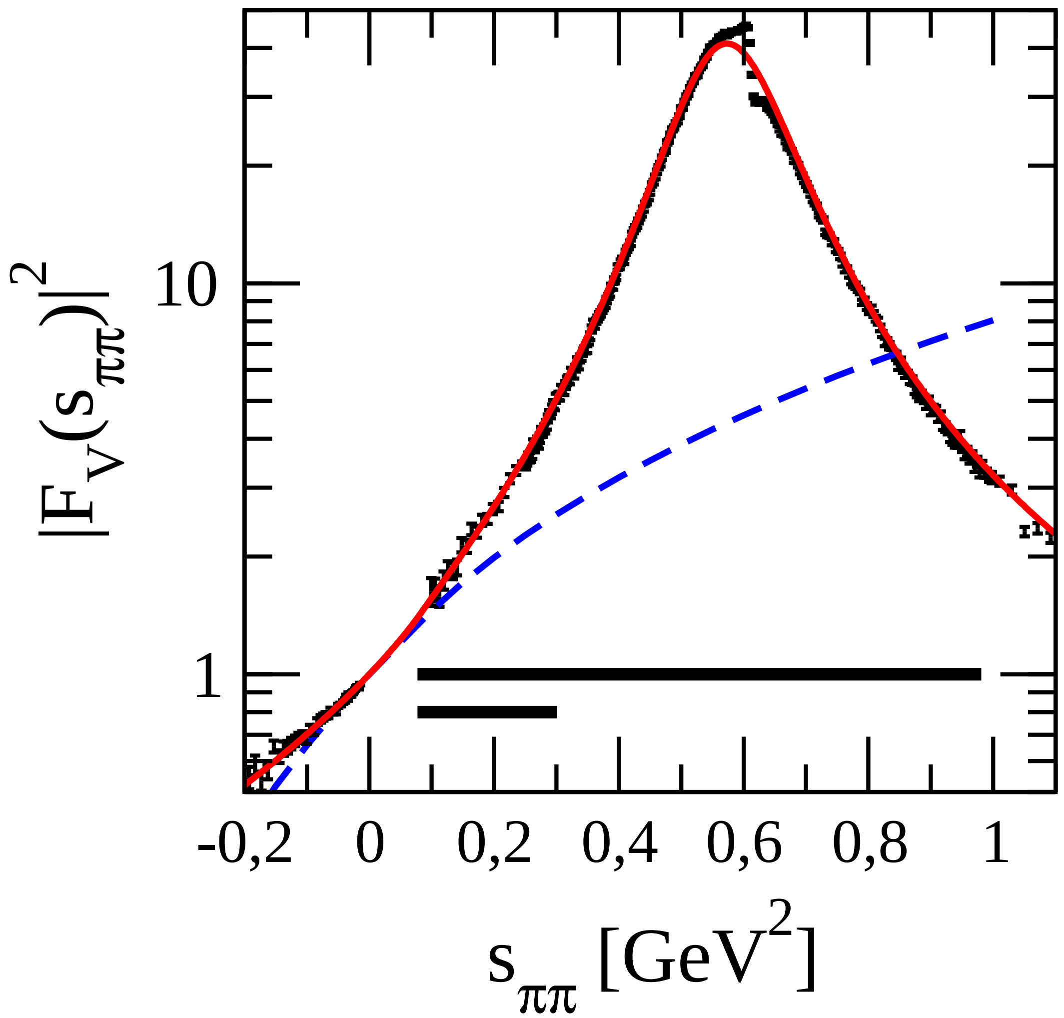
- today: high-accuracy $\pi\pi$ phase shifts available

Ananthanarayan et al. 2001, García-Martín et al. 2011

- constrain $P_I(s)$ using symmetries (normalisation at $s=0$ etc.)

Pion vector form factor from dispersion relations

- pion vector form factor clearly non-perturbative: ρ resonance



ChPT at one loop

data on $e^+e^- \rightarrow \pi^+\pi^-$

Omnès representation

Stollenwerk et al. 2012

$\longrightarrow$ Omnès representation vastly extends range of applicability

Dispersion relations for 3 pions

- $\gamma\pi \to \pi\pi$ particularly simple system: odd partial waves
  $\longrightarrow$ P-wave interactions only (neglecting F- and higher)
- amplitude decomposed into single-variable functions

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta} n^\mu p^\nu_{\pi^+} p^\alpha_{\pi^-} p^\beta_{\pi^0} \, \mathcal{F}(s,t,u)$$

$$\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$$

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\, \mathcal{F}(s) = 2i \{ \underbrace{\mathcal{F}(s)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{left-hand cut}} \} \times \theta(s - 4M_\pi^2) \times \sin \delta_1^1(s)\, e^{-i\delta_1^1(s)}$$

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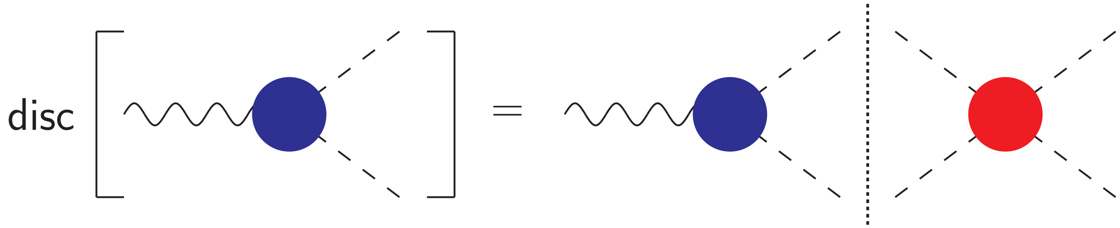
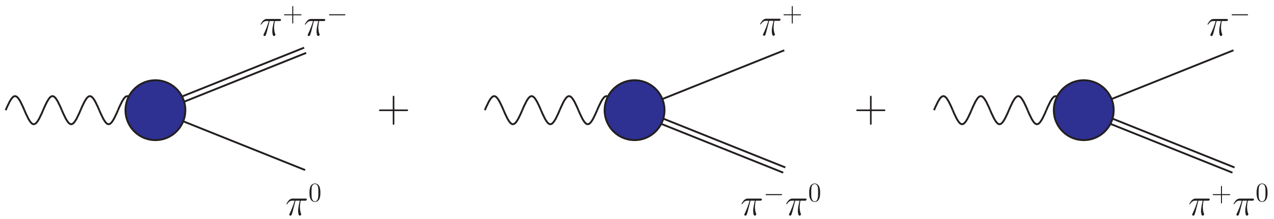


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s) = P(s)\,\Omega(s)\;, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

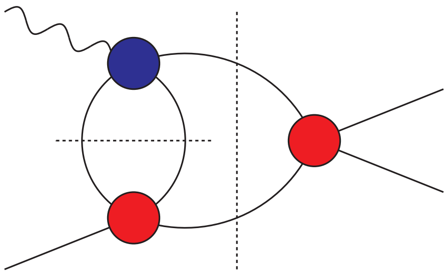
$\mathcal{F}(s,t,u) =$



Dispersion relations for 3 pions

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\,\mathcal{F}(s) = 2i\{\underbrace{\mathcal{F}(s)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{left-hand cut}}\} \times \theta(s-4M_\pi^2) \times \sin\delta_1^1(s)\, e^{-i\delta_1^1(s)}$$

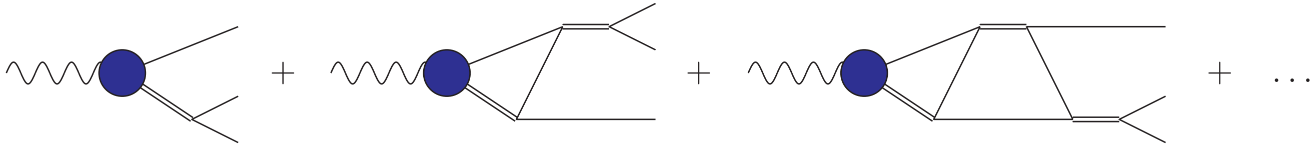


- inhomogeneities $\hat{\mathcal{F}}(s)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s) = \Omega(s)\left\{\frac{C_2^{(1)}}{3}\big(1-\dot{\Omega}(0)s\big) + \frac{C_2^{(2)}}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z)\big)$$

$$\mathcal{F}(s) =$$



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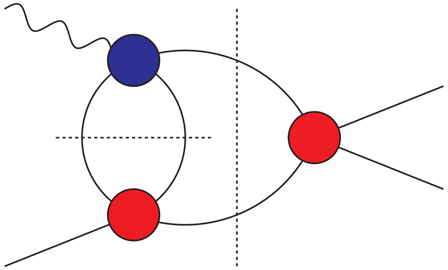

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$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1}dz\,(1-z^2)\mathcal{F}\big(t(s,z)\big)$$

- admits crossed-channel scattering between s-, t-, and u-channel (left-hand cuts)

Omnès solution for $\gamma\pi \to \pi\pi$

$$\mathcal{F}(s) = \Omega(s)\left\{\frac{C_2^{(1)}}{3}\big(1-\dot{\Omega}(0)s\big) + \frac{C_2^{(2)}}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

- important observation: $\mathcal{F}(s)$ linear in $C_2^{(i)}$

$$\mathcal{F}(s) = C_2^{(1)}\mathcal{F}^{(1)}(s) + C_2^{(2)}\mathcal{F}^{(2)}(s)$$

$\longrightarrow$ basis functions $\mathcal{F}^{(i)}(s)$ calculated independently of $C_2^{(i)}$

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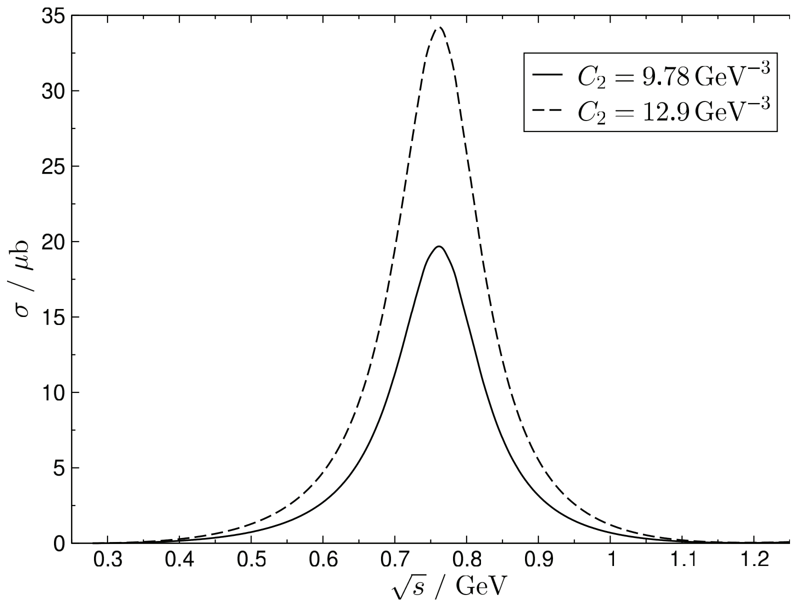
- representation of cross section in terms of two parameters

$\longrightarrow$ fit to data, extract

$$F_{3\pi} \simeq C_2 = C_2^{(1)} + C_2^{(2)}M_\pi^2$$

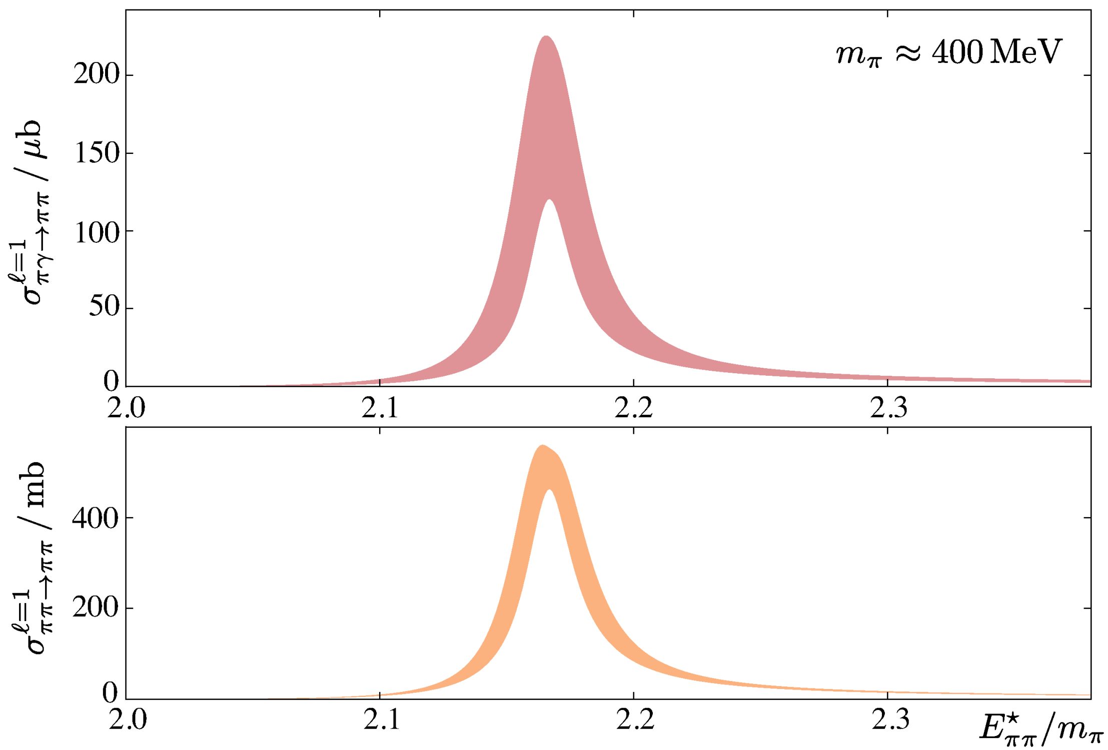
$\longrightarrow \sigma \propto (C_2)^2$ also in ρ region



Hoferichter, BK, Sakkas 2012

$\gamma\pi \to \pi\pi$: plans & extensions

- $\gamma\pi \to \pi\pi$ on the lattice



Briceño et al. (HadSpec Coll.) 2015

- study quark-mass extrapolation

Niehus, MSc thesis 2017

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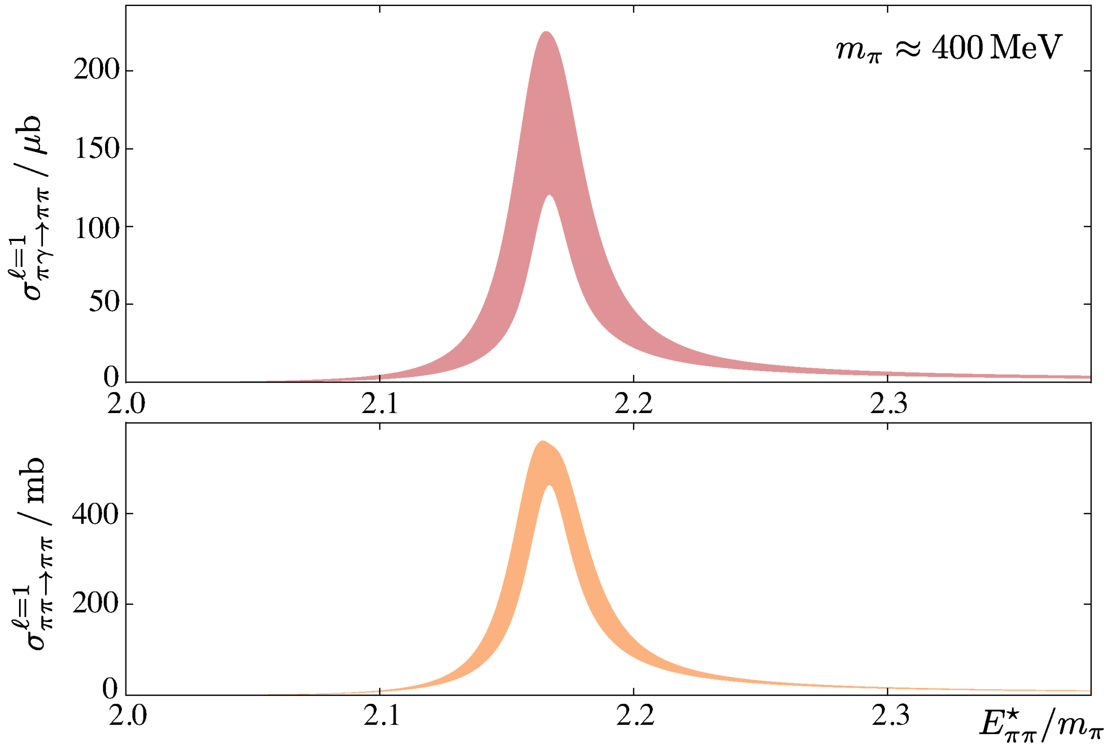


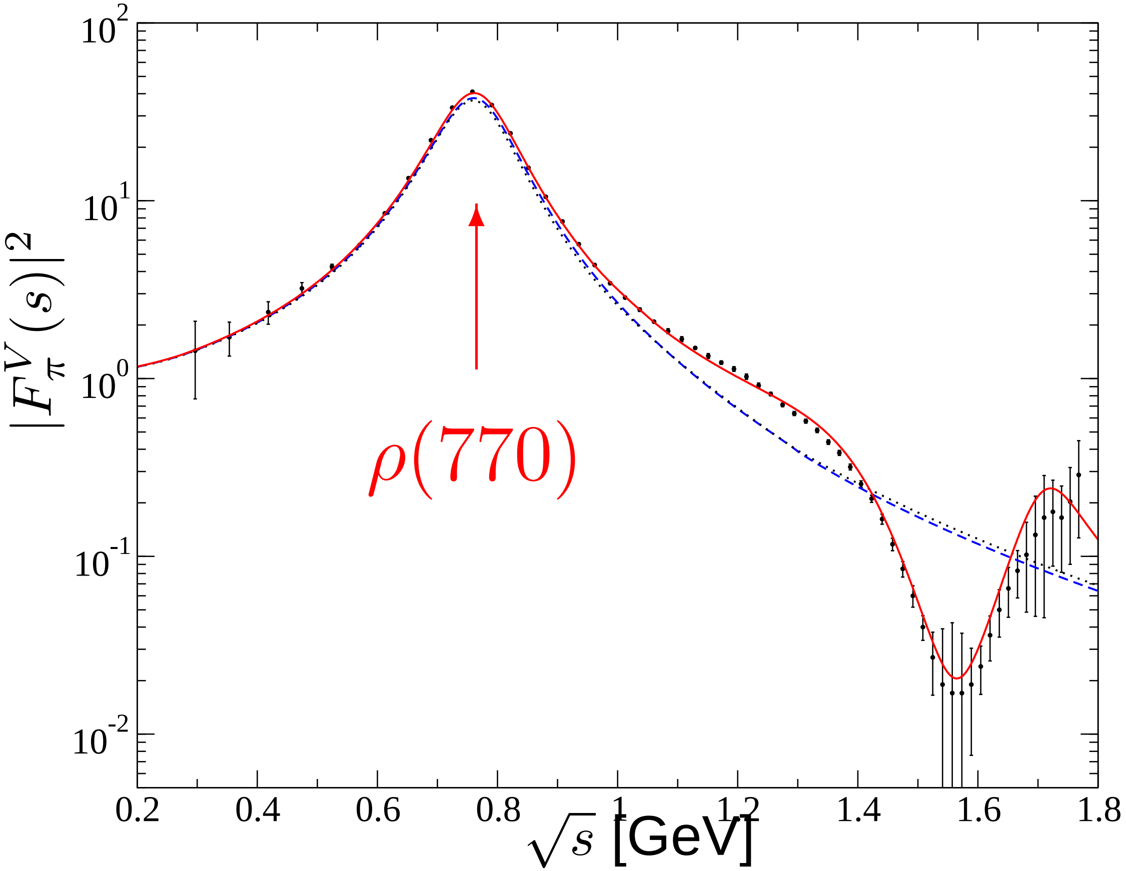
Briceño et al. (HadSpec Coll.) 2015

- study quark-mass extrapolation — Niehus, MSc thesis 2017
- only odd partial waves allowed $\longrightarrow$ estimate F-wave? $\rho_3(1690)$?
  comparison to $\rho'(1450)$, $\rho''(1700)$ effects? — Zanke, BSc thesis 2017

The simplest of all resonances: ρ(770)

Data on pion form factor in $\tau^- \to \pi^-\pi^0\nu_\tau$

Belle 2008

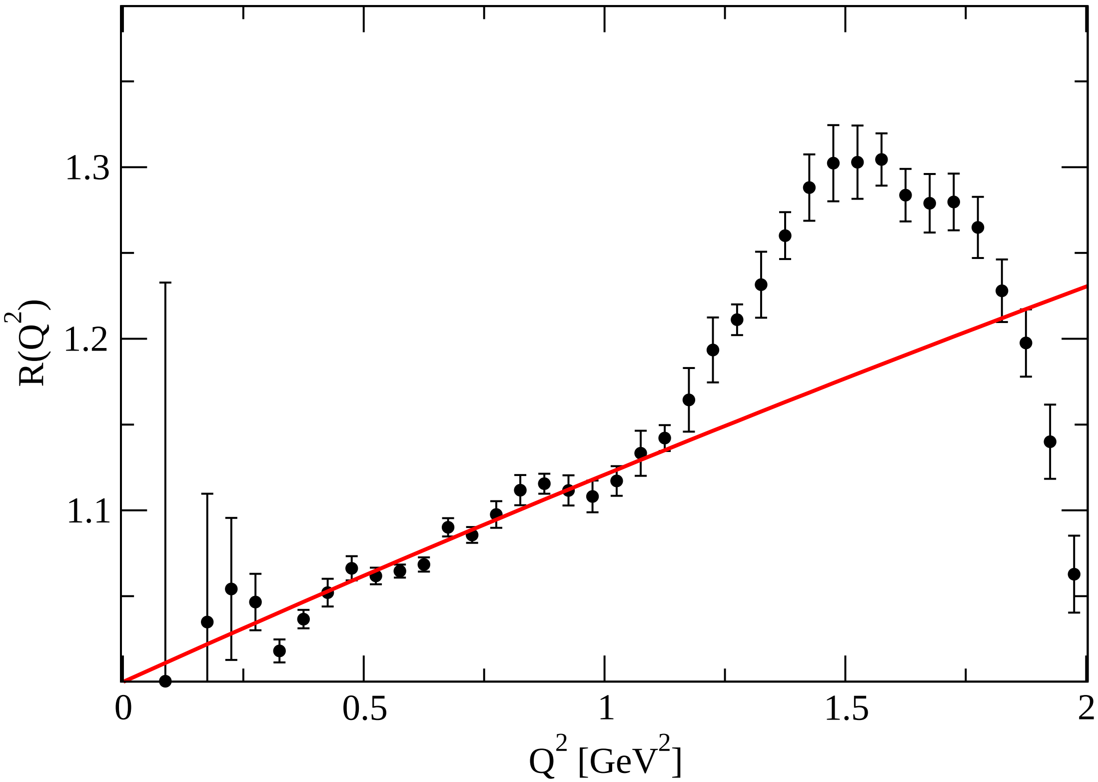


Schneider et al. 2012

- can we understand what's there "below the peak"?
- how is the $\rho(770)$ line shape modified in different reactions?

Pion vector form factor vs. Omnès representation

- divide $\tau^- \to \pi^-\pi^0\nu_\tau$ form factor by Omnès function:



Hanhart et al. 2013

$\longrightarrow$ linear below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\mathrm{GeV}^{-2} s)\Omega(s)$

$\longrightarrow$ above: inelastic resonances ρ', $\rho''\ldots$

Final-state universality: η, $\eta' \to \pi^+\pi^-\gamma$

- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{F}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(t) \times \Omega(t)$, $P(t) = 1 + \alpha^{(\prime)} t$, $t = M^2_{\pi\pi}$

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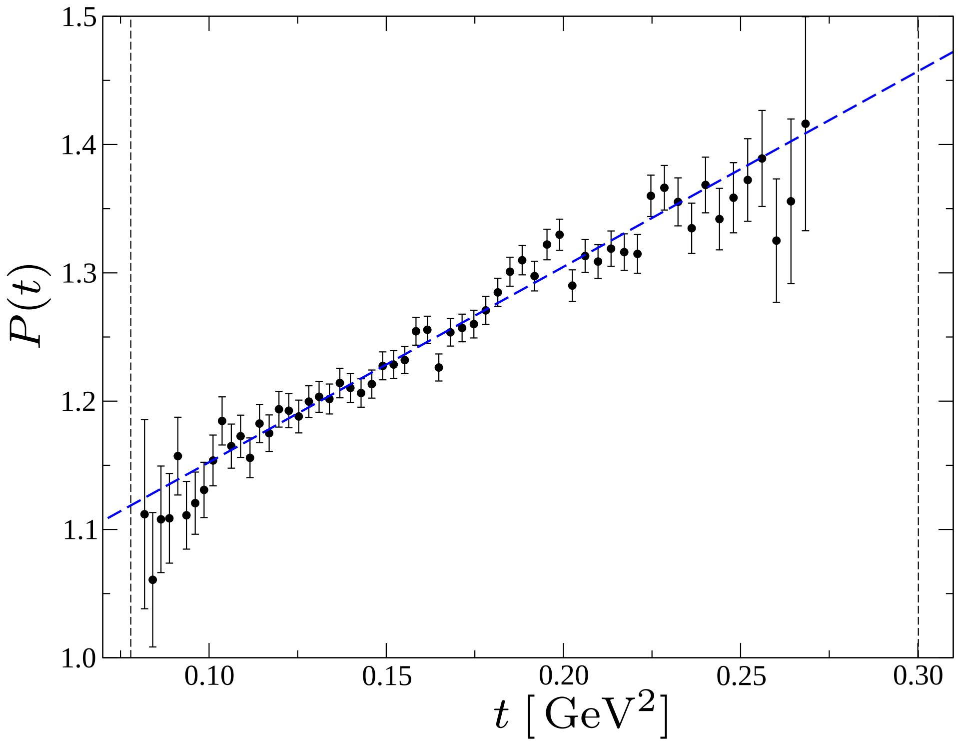
- divide data by pion form factor $\longrightarrow P(t)$ Stollenwerk et al. 2012

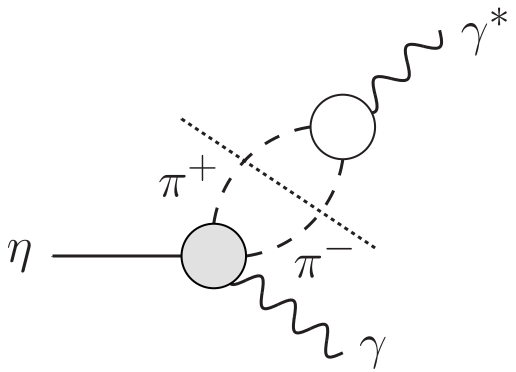


$\longrightarrow$ exp.: $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)$ GeV^{-2} cf. KLOE 2013

Anomalous decay $\eta \to \pi^+\pi^-\gamma$

- $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)\,\text{GeV}^{-2}$ large
  $\longrightarrow$ implausible to explain through ρ', ρ''...
- for large t, expect $P(t) \to$ const. rather
- important input for
  $\eta \to \gamma^*\gamma$ transition form factor:
  $\longrightarrow$ dispersion integral covers
  larger energy range

Hanhart et al. 2013



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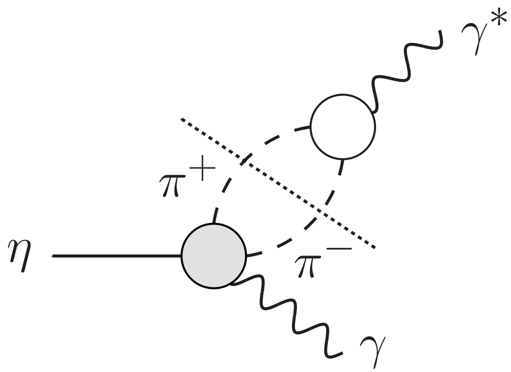


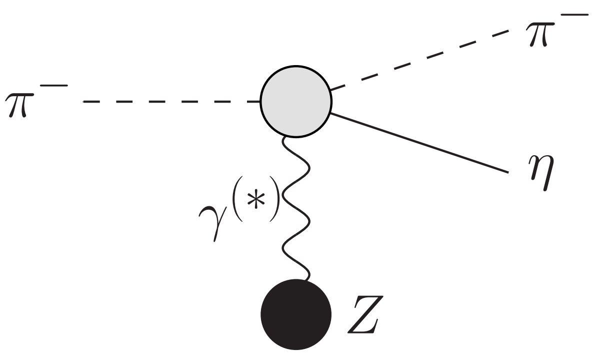
Intriguing observation:

- naive continuation of $\mathcal{F}^{\eta}_{\pi\pi\gamma} = A(1 + \alpha t)\Omega(t)$ has zero
  at $t = -1/\alpha \approx -0.66\,\text{GeV}^2$
  $\longrightarrow$ test this in crossed process $\gamma\pi^- \to \pi^-\eta$
  $\longrightarrow$ "left-hand cuts" in $\pi\eta$ system?

BK, Plenter 2015

Primakoff reaction $\gamma\pi \to \pi\eta$

- can be measured in
  Primakoff reaction COMPASS
- S-wave forbidden
  P-wave exotic: $J^{PC} = 1^{-+}$
  D-wave $a_2(1320)$ first resonance



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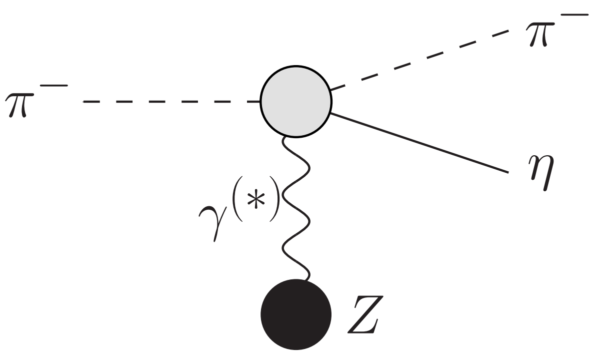

- S-wave forbidden
  P-wave exotic: $J^{PC} = 1^{-+}$
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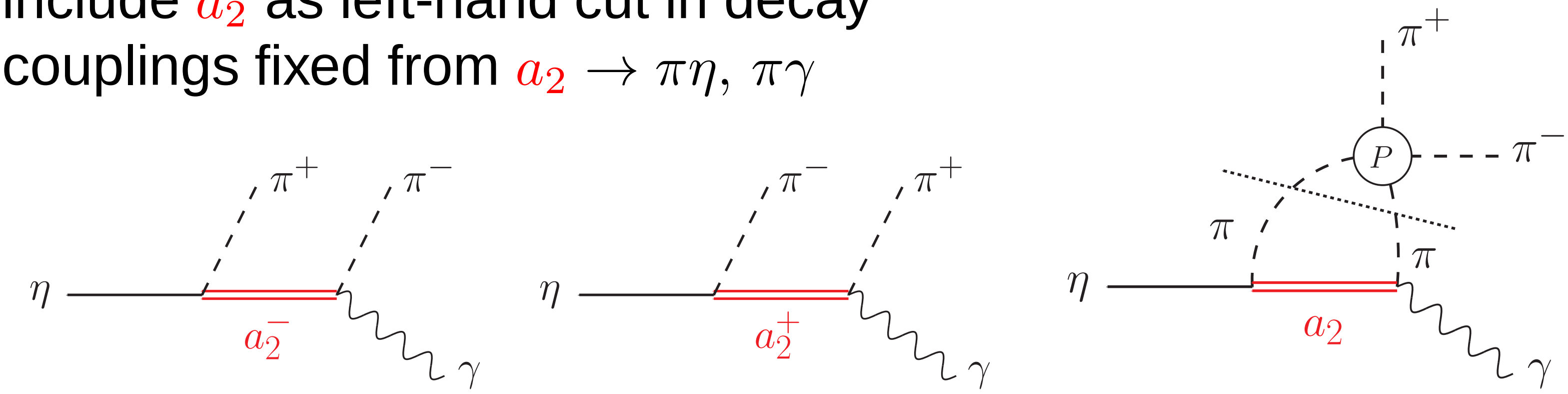
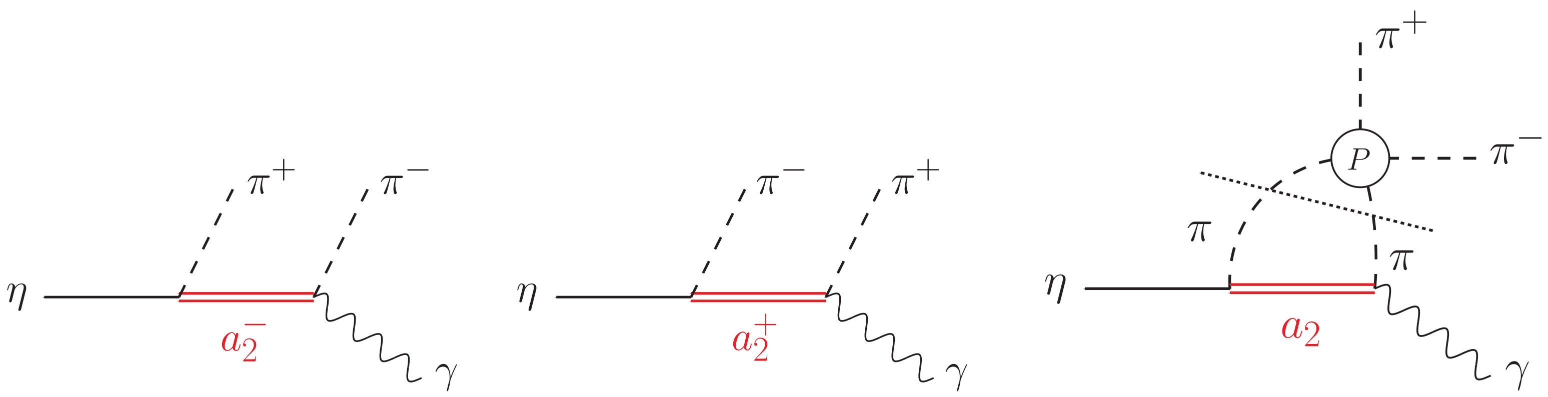
- include a_2 as left-hand cut in decay
  couplings fixed from $a_2 \to \pi\eta,\ \pi\gamma$



  - ▷ compatible with decay data?
  - ▷ first s-channel resonance
    $\longrightarrow$ breakdown scale for t-channel dominance
  - ▷ does the amplitude zero survive?

Formalism including left-hand cuts



- a_2 + rescattering essential to preserve Watson's theorem
- formally:

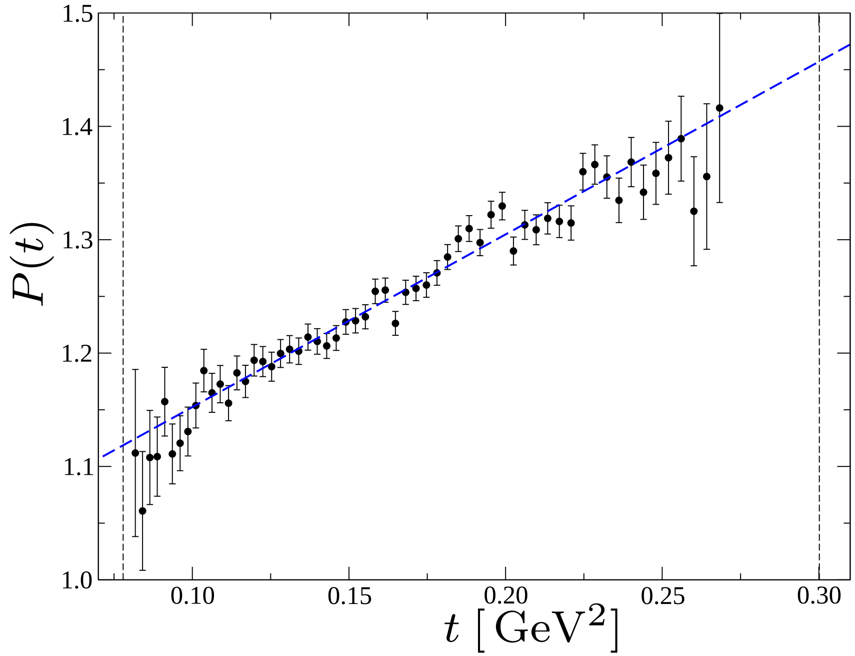
$$\mathcal{F}^{\eta}_{\pi\pi\gamma}(s,t,u) = \mathcal{F}(t) + \mathcal{G}_{a_2}(s,t,u) + \mathcal{G}_{a_2}(u,t,s)$$

$$\mathcal{F}(t) = \Omega(t)\left\{ A(1+\alpha t) + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dx}{x^2}\frac{\sin\delta(x)\hat{\mathcal{G}}(x)}{|\Omega(x)|(x-t)}\right\}$$

$\hat{\mathcal{G}}$: t-channel P-wave projection of a_2 exchange graphs

- re-fit subtraction constants A, α

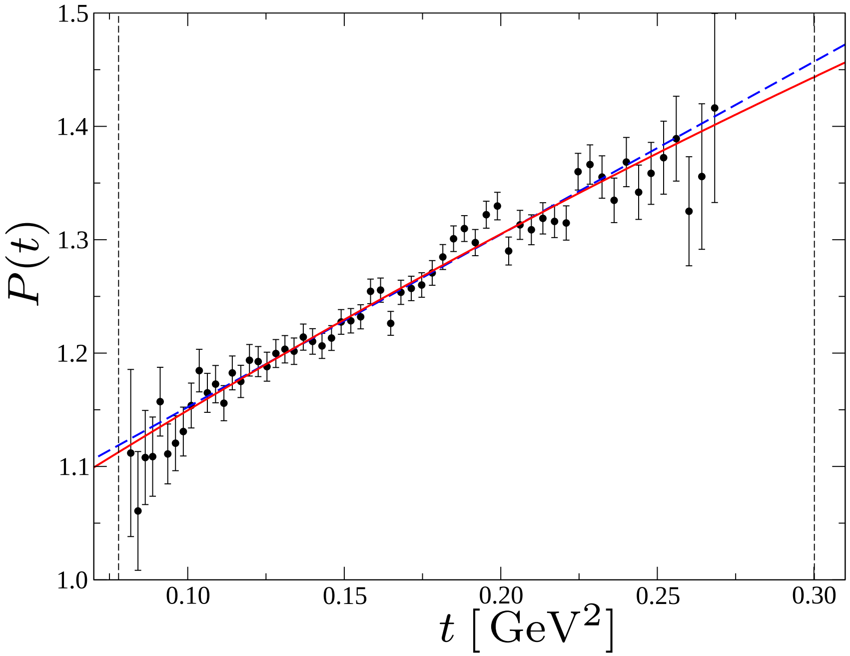
$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with a_2



KLOE 2013

$\alpha = 1.52 \pm 0.06$, χ^2/ndof $= 0.94$

$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with a_2



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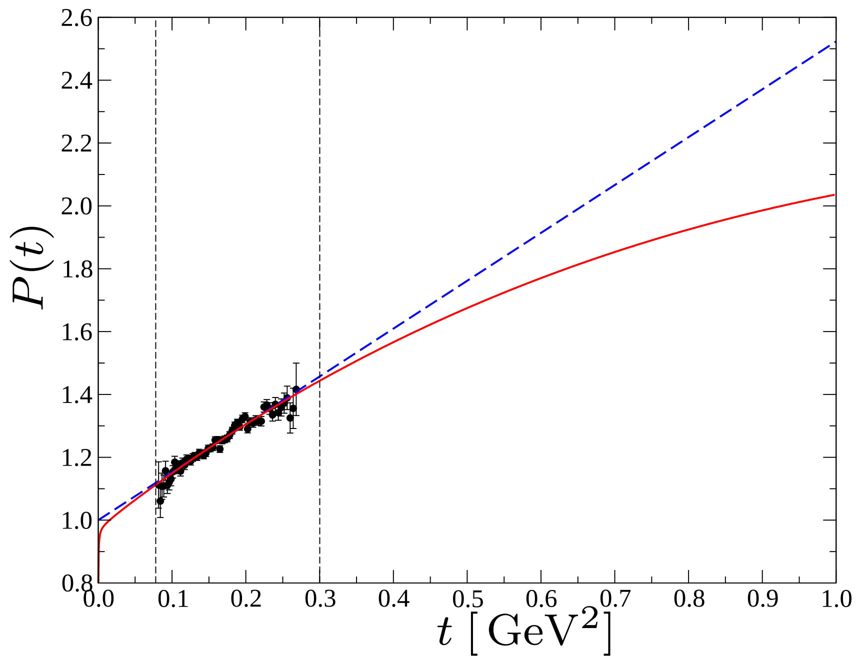
$\alpha = 1.52 \pm 0.06$, $\chi^2/\text{ndof} = 0.94$

$\longrightarrow \alpha = 1.42 \pm 0.06$, $\chi^2/\text{ndof} = 0.90$

$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with a_2



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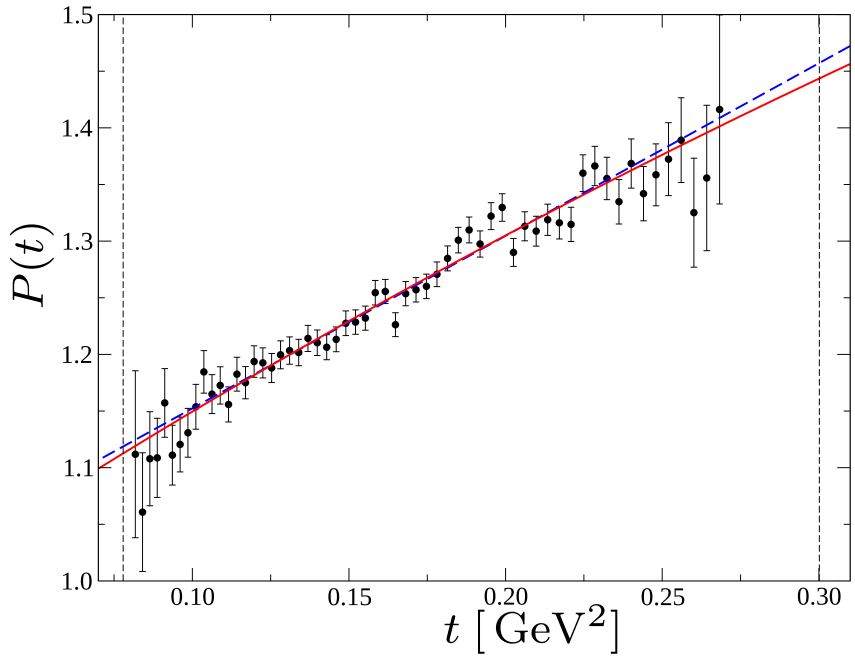
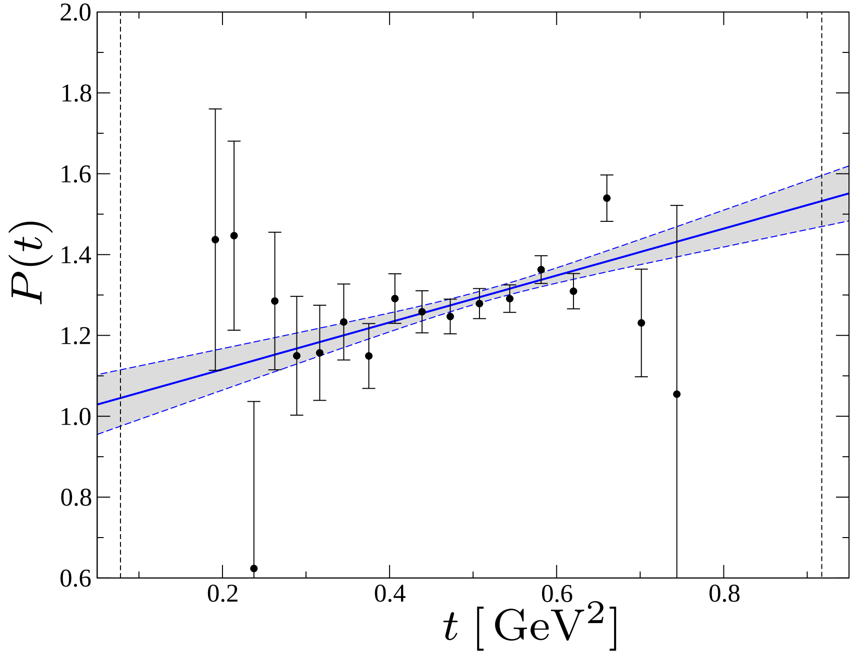
- equally good—why care? sum rule for $\eta \to \gamma^*\gamma$ transition form factor slope reduced by $7 - 8\%$ cf. Hanhart et al. 2013

$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with a_2





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Crystal Barrel 1997

$\alpha' = 0.6 \pm 0.2$, $\chi^2/\text{ndof} = 1.2$

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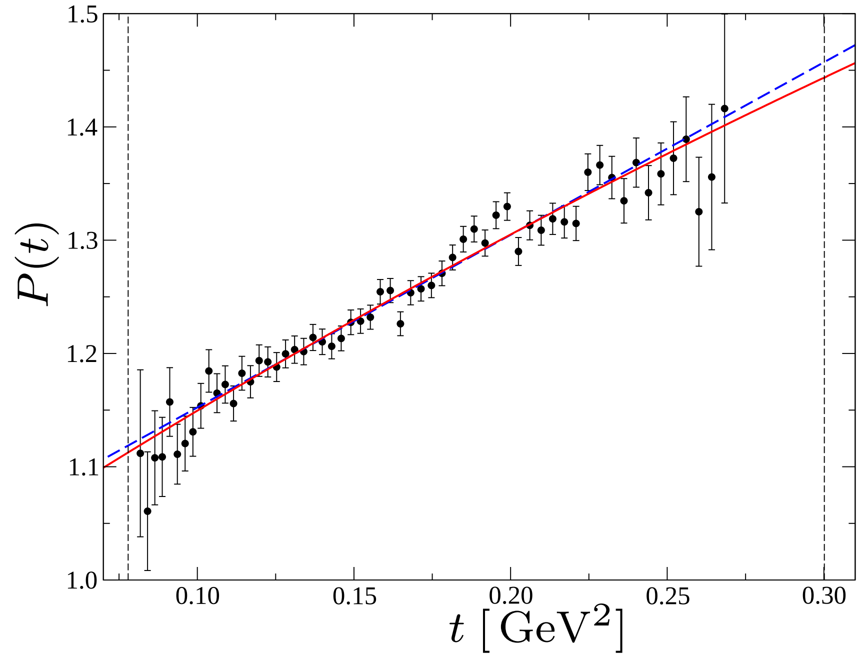
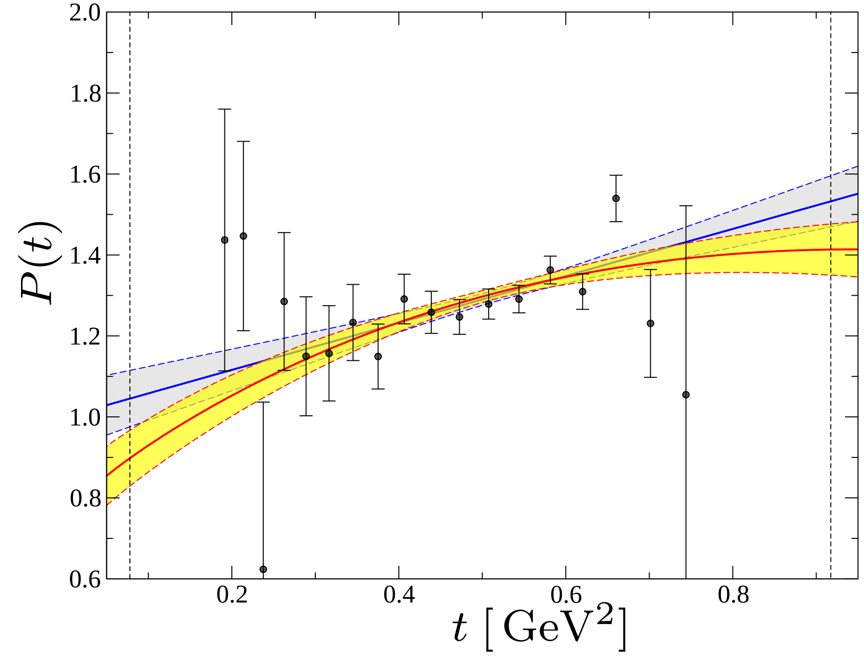
$\eta, \eta' \to \pi^+\pi^-\gamma$ with a_2





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$\longrightarrow \alpha = 1.42 \pm 0.06$, $\chi^2/\text{ndof} = 0.90$

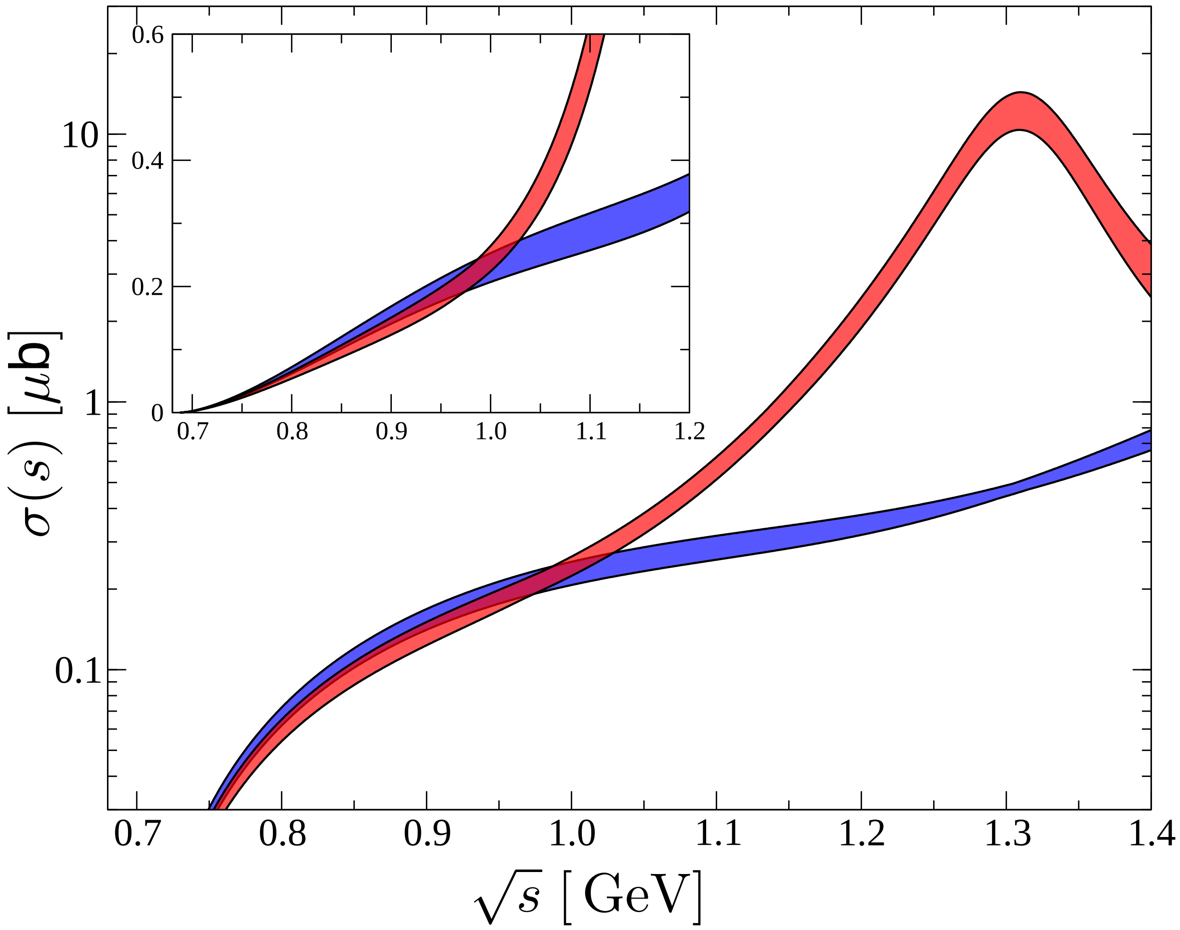
Crystal Barrel 1997

$\alpha' = 0.6 \pm 0.2$, $\chi^2/\text{ndof} = 1.2$

$\longrightarrow \alpha' = 1.4 \pm 0.4$, $\chi^2/\text{ndof} = 1.4$

- equally good—why care? sum rule for $\eta \to \gamma^*\gamma$ transition form factor slope reduced by $7-8\%$ cf. Hanhart et al. 2013
- $\alpha \approx \alpha'$ (large-N_c) better fulfilled including a_2 BK, Plenter 2015

Total cross section $\gamma\pi \to \pi\eta$

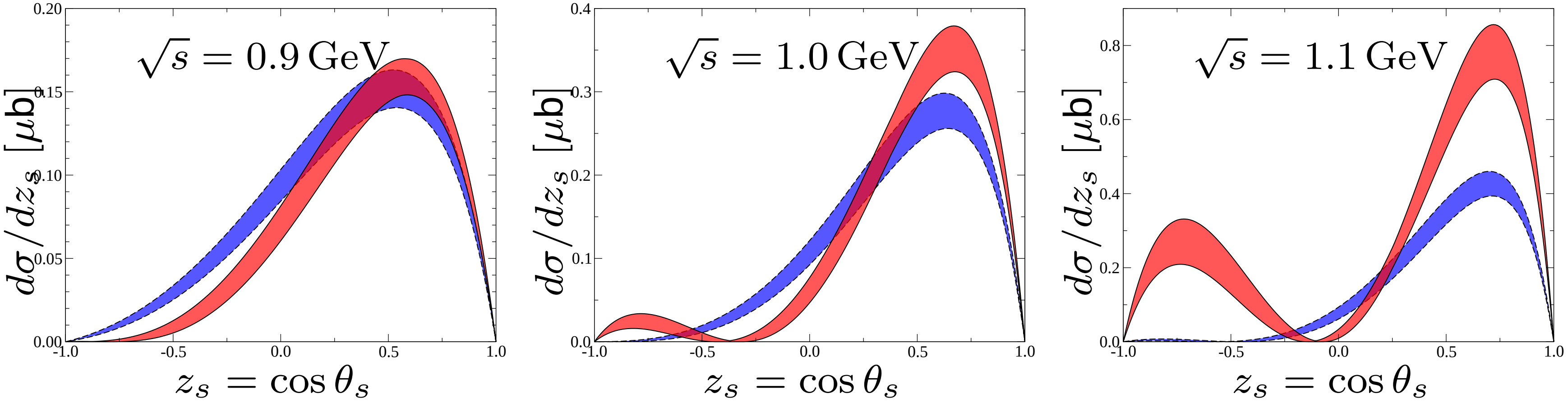


blue: t-channel dynamics / "ρ" only red: full amplitude

- t-channel dynamics dominate below $\sqrt{s} \approx 1\,\mathrm{GeV}$
- uncertainty bands: $\Gamma(\eta \to \pi^+\pi^-\gamma)$, α, a_2 couplings BK, Plenter 2015

Differential cross sections $\gamma\pi \to \pi\eta$

- amplitude zero visible in differential cross sections:



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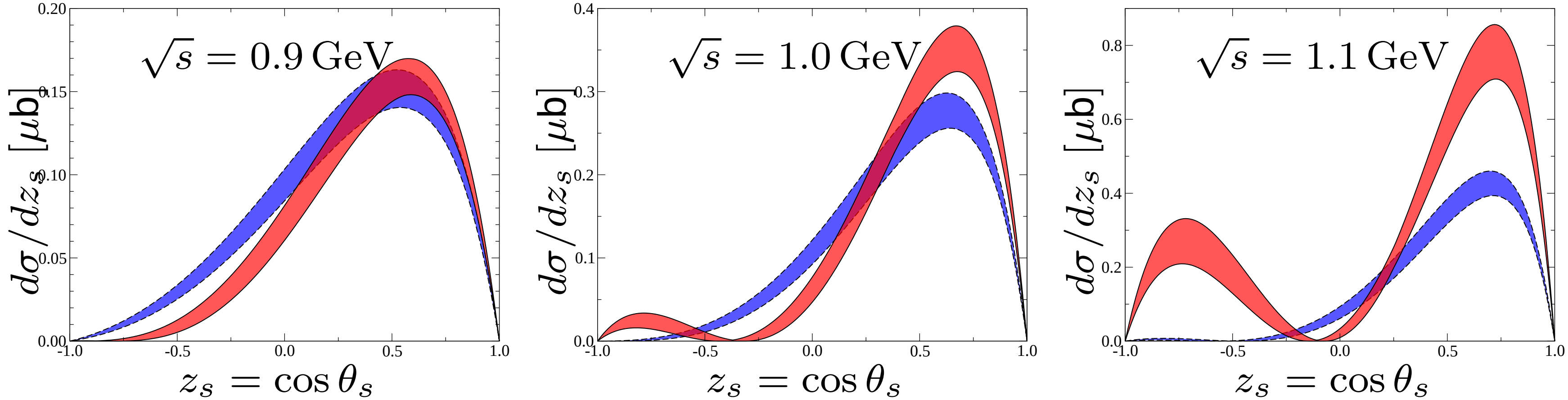
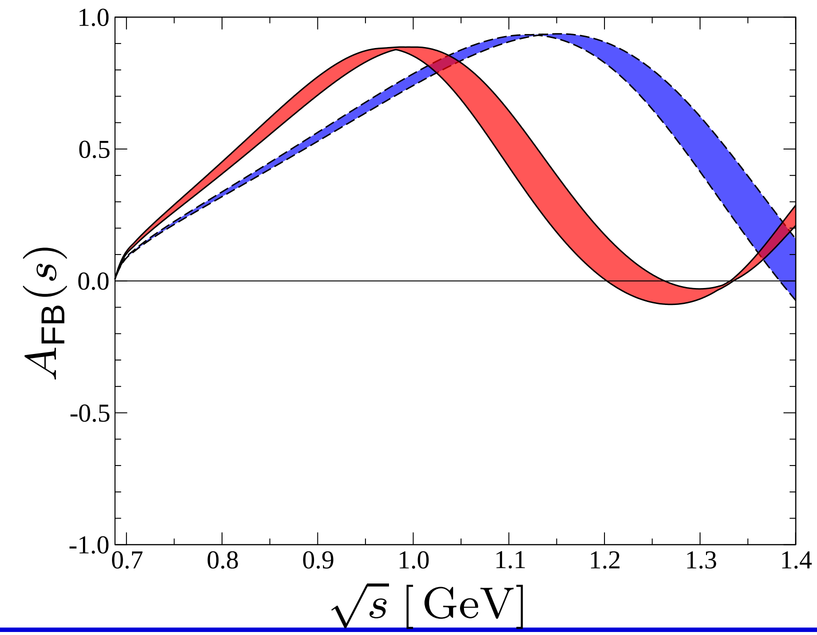


blue: t-channel dynamics / "ρ" only red: full amplitude

- strong P-D-wave interference
- can be expressed as forward-backward asymmetry

$$A_{\rm FB} = \frac{\sigma(\cos\theta > 0) - \sigma(\cos\theta < 0)}{\sigma_{\rm total}}$$



Summary / Outlook

Dispersion relations for light-meson processes

- based on unitarity, analyticity, crossing symmetry
- extends range of applicability (at least) to full elastic regime
- matching to ChPT where it works best

$\boldsymbol{\gamma\pi^- \to \pi^-\pi^0}$

- improved extraction of $F_{3\pi}$ from COMPASS data up to 1 GeV

$\boldsymbol{\gamma\pi^- \to \pi^-\eta}$

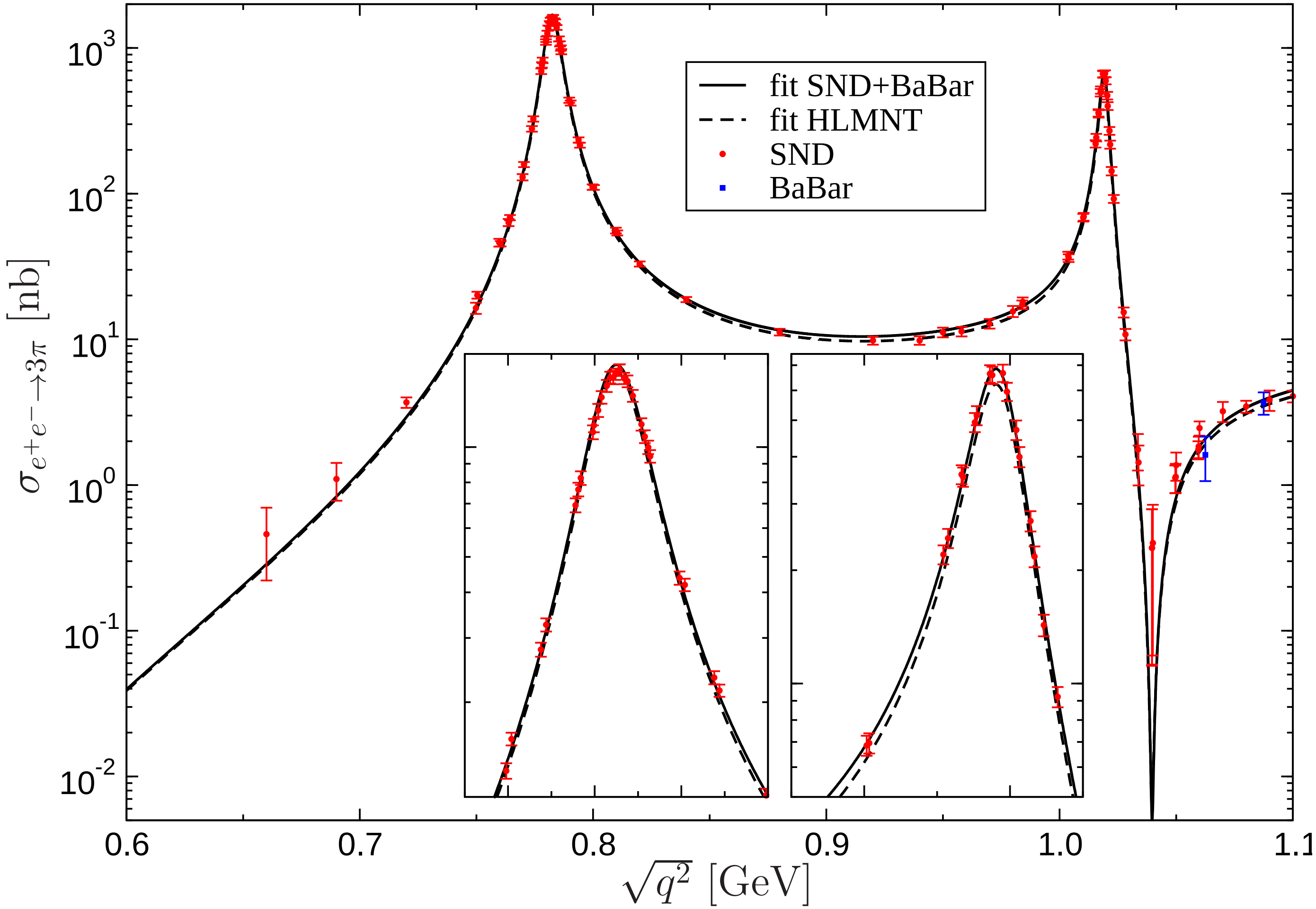
- cross section & forward-backward asymmetry below $a_2(1320)$: extends $\eta \to \pi^+\pi^-\gamma$ amplitude
- first COMPASS feasibility studies — Altenbach, Diploma thesis 2016

Impact:

- π^0 and η transition form factors: $\longrightarrow$ hadron physics in $(g-2)_\mu$
- study resonance line shapes affected by crossed-channel effects

Spares

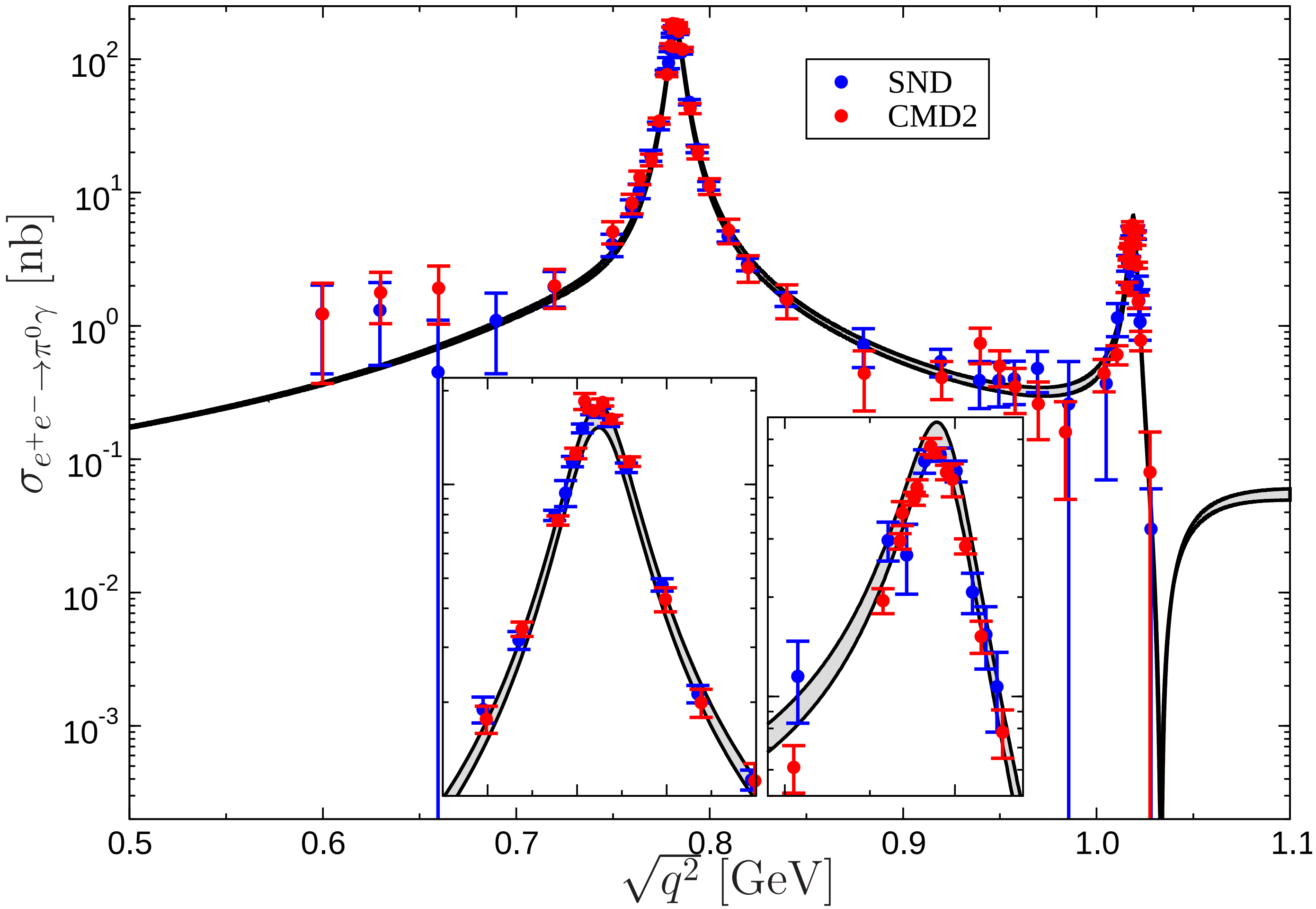
Fit to $e^+e^- \to 3\pi$ data



Hoferichter, BK, Leupold, Niecknig, Schneider 2014

- one subtraction/normalisation at $q^2 = 0$ fixed by $\gamma \to 3\pi$
- fitted: ω, ϕ residues, linear subtraction β

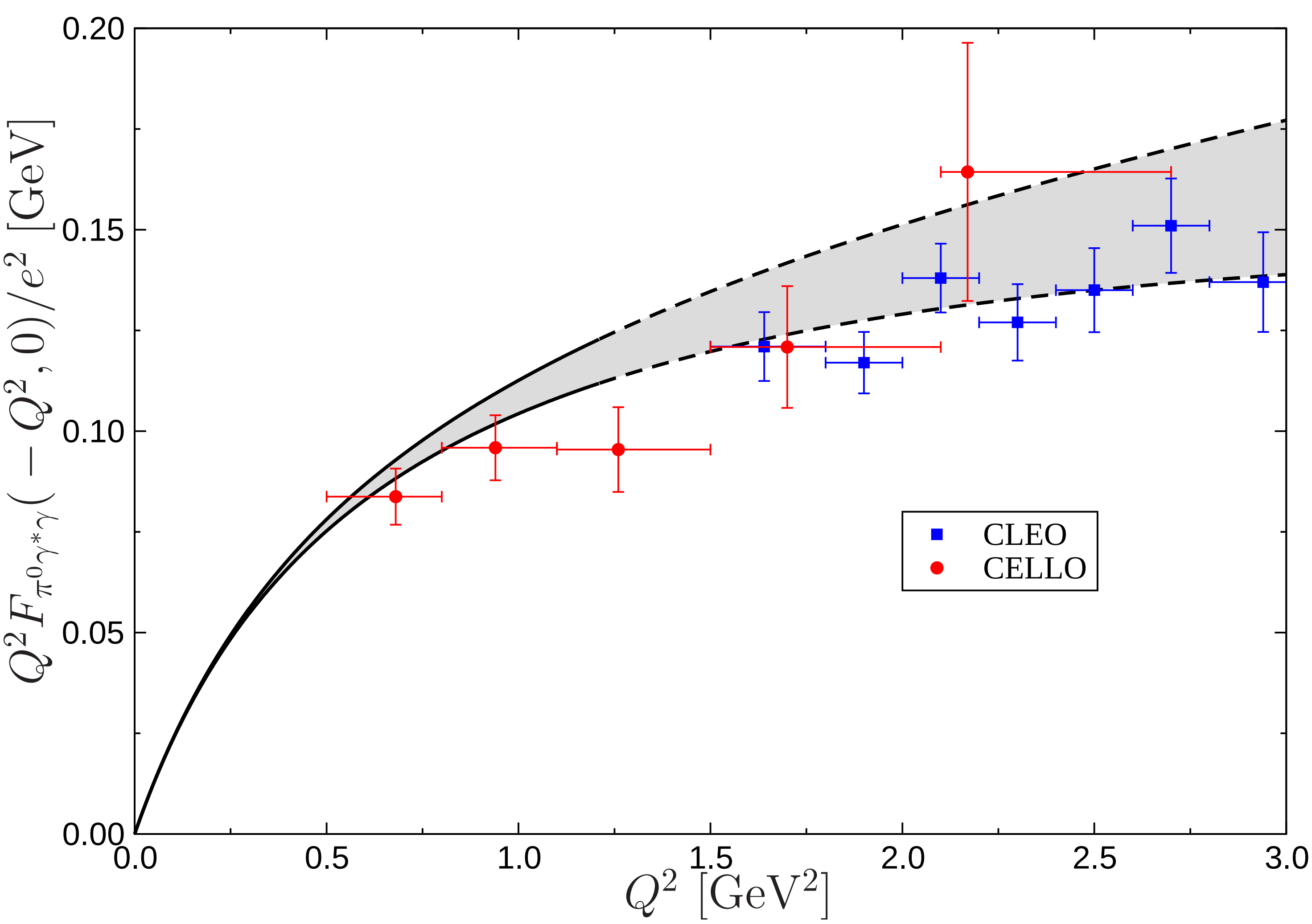
Comparison to $e^+e^- \to \pi^0\gamma$ data



Hoferichter, BK, Leupold, Niecknig, Schneider 2014

- "prediction"—no further parameters adjusted
- data very well reproduced

Prediction spacelike form factor



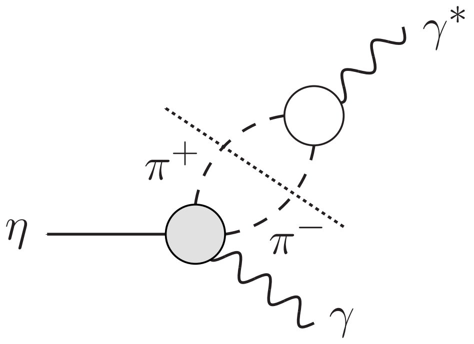
Hoferichter, BK, Leupold, Niecknig, Schneider 2014

$\longrightarrow$ more precise low-energy spacelike data to come BESIII

Transition form factor $\eta \to \gamma^*\gamma$

Hanhart et al. 2013

$$\bar{F}_{\eta\gamma^*\gamma}(q^2,0) = 1 + \frac{\kappa_\eta q^2}{96\pi^2 F_\pi^2}\int_{4M_\pi^2}^\infty ds\sigma(s)^3 P(s)\frac{|F_\pi^V(s)|^2}{s-q^2}$$
$$+ \Delta F_{\eta\gamma^*\gamma}^{I=0}(q^2,0) \; [\longrightarrow \text{VMD}]$$



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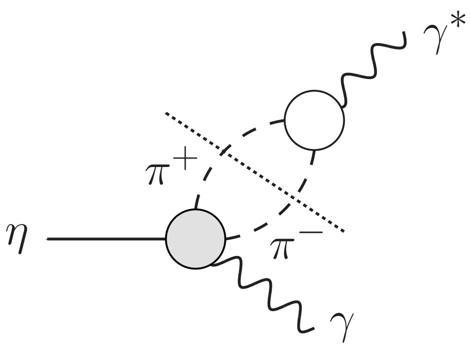
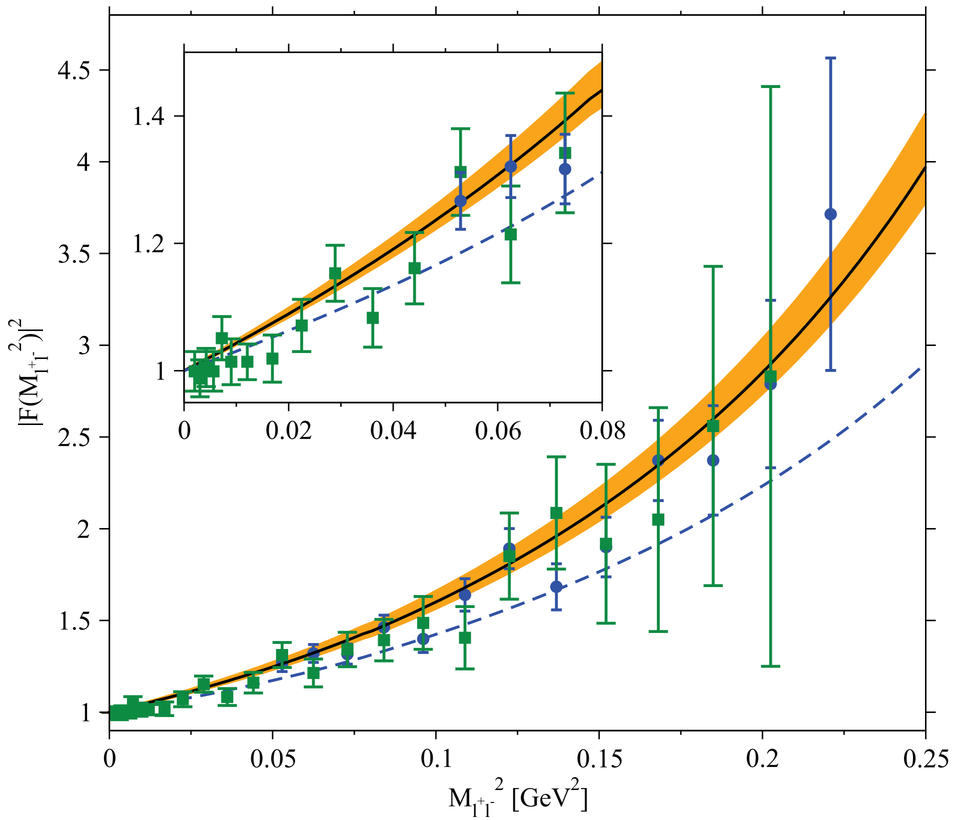


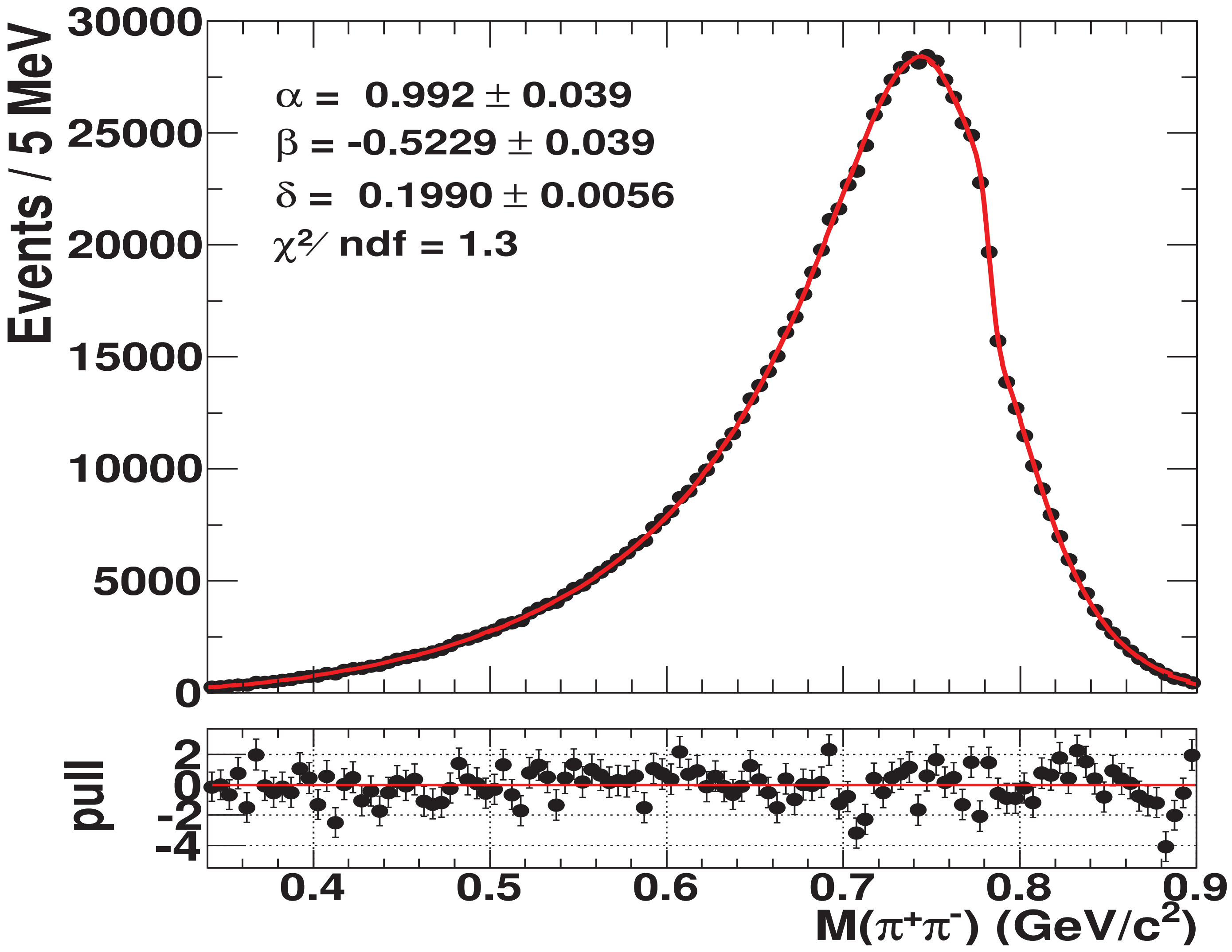


$\longrightarrow$ huge statistical advantage of using hadronic input for $\eta \to \pi^+\pi^-\gamma$ over direct measurement of $\eta \to e^+e^-\gamma$ (rate suppressed by α^2_{QED})

figure courtesy of C. Hanhart

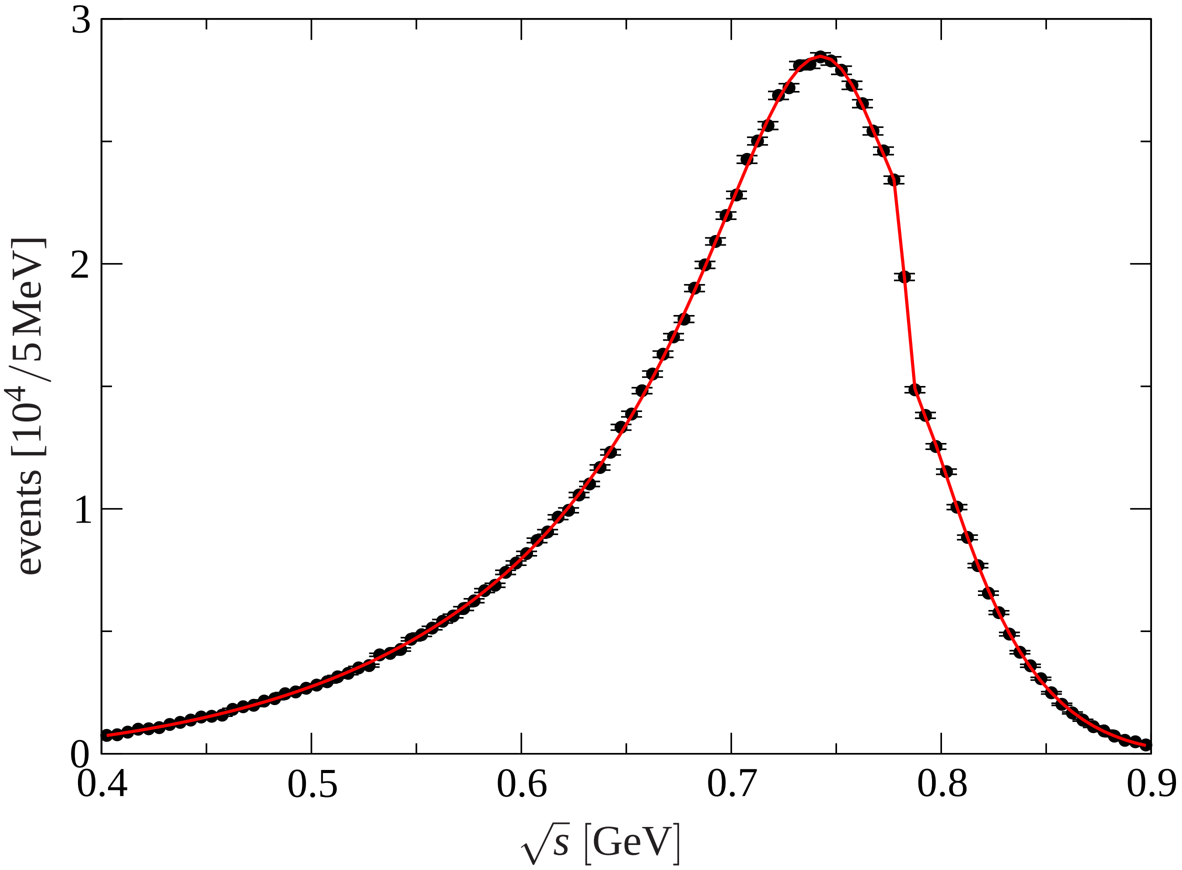
data: NA60 2011, A2 2014

New data on $\eta' \to \pi^+\pi^-\gamma$



BESIII preliminary, Fang 2015

New data on $\eta' \to \pi^+\pi^-\gamma$



fit to pseudodata after BESIII preliminary

- fit form $$\left[A(1+\alpha t+\beta t^2)+\frac{\kappa}{m_\omega^2-t-im_\omega\Gamma_\omega}\right]\times\Omega(t)$$

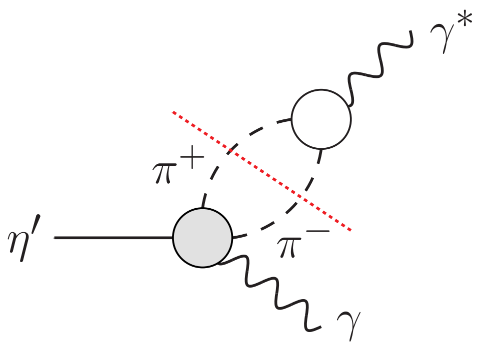
$\longrightarrow$ curvature $\propto \beta t^2$ essential (smaller than a_2 prediction)

$\longrightarrow$ even ρ–ω mixing clearly visible

Hanhart et al. 2017

Prediction for η' transition form factor

- isovector: combine high-precision data on $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- isoscalar: VMD, couplings fixed from $\eta' \to \omega\gamma$ and $\phi \to \eta'\gamma$



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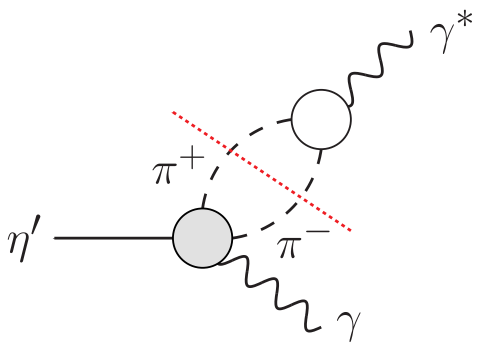
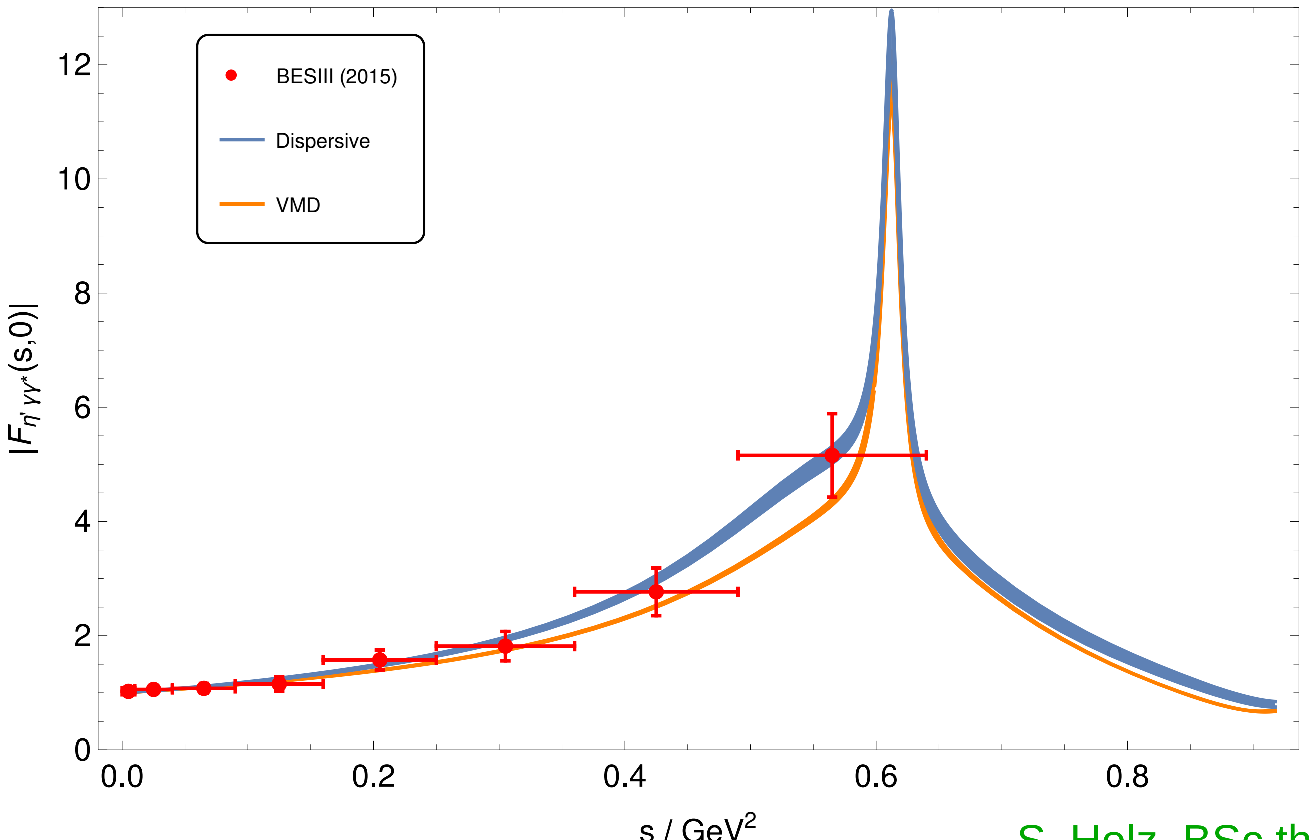
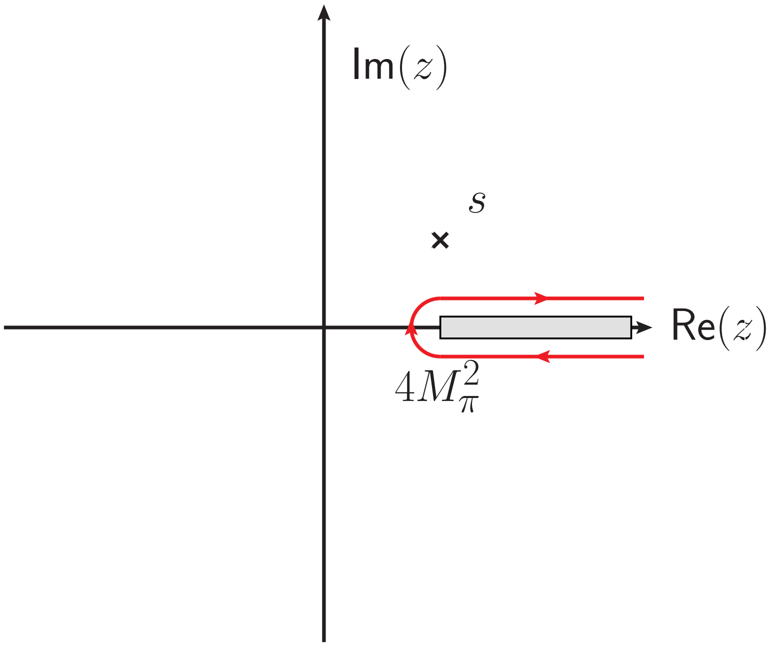


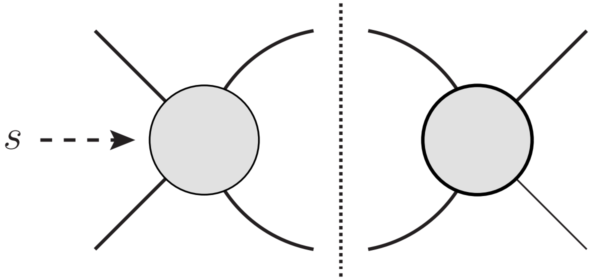


S. Holz, BSc thesis 2016

What are left-hand cuts?

Example: pion–pion scattering





- right-hand cut due to unitarity: $s \geq 4M_\pi^2$

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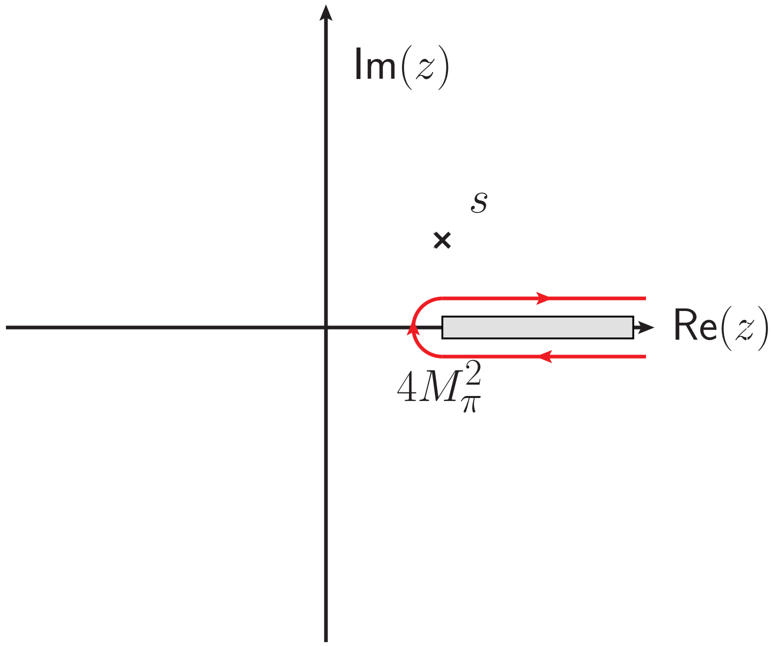
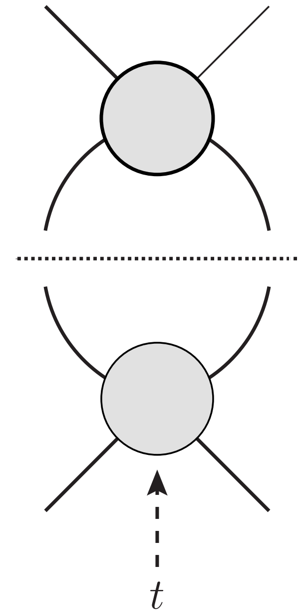
Example: pion–pion scattering





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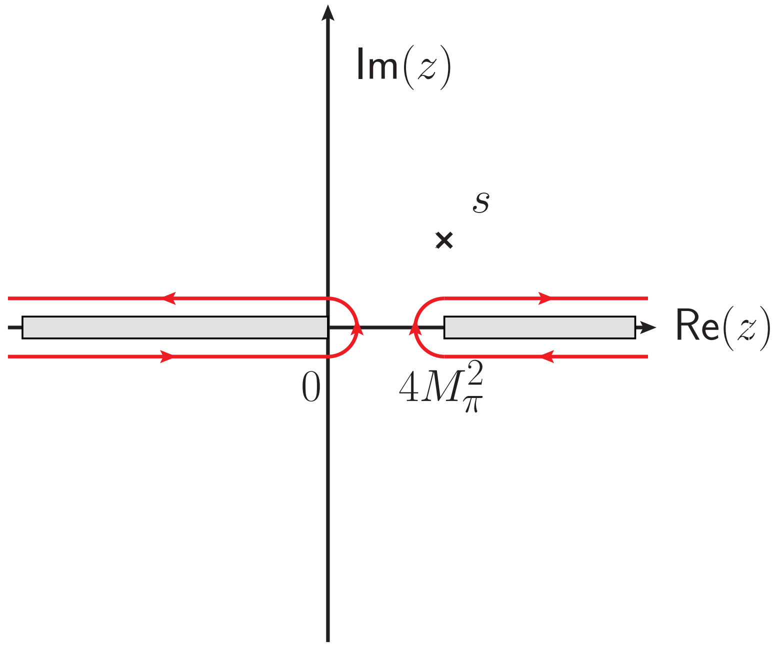

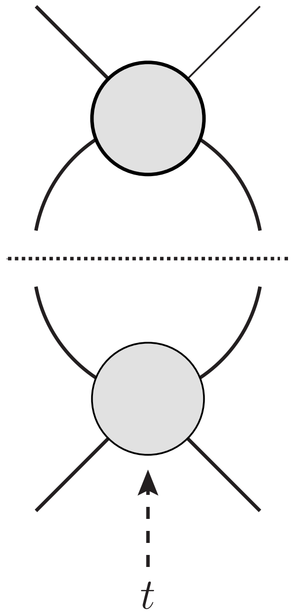

- right-hand cut due to unitarity: $s \geq 4M_\pi^2$
- crossing symmetry: cuts also for $t,\ u \geq 4M_\pi^2$
- partial-wave projection: $T(s,t) = 32\pi \sum_i T_i(s) P_i(\cos\theta)$

$$t(s,\cos\theta) = \frac{1-\cos\theta}{2}(4M_\pi^2 - s)$$

$\longrightarrow$ cut for $t \geq 4M_\pi^2$ becomes cut for $s \leq 0$ in partial wave

$\pi\pi$ scattering constrained by analyticity and unitarity

Roy equations = coupled system of partial-wave dispersion relations
+ crossing symmetry + unitarity

- twice-subtracted fixed-t dispersion relation:

$$T(s,t) = c(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \left\{ \frac{s^2}{s'^2(s'-s)} + \frac{u^2}{s'^2(s'-u)} \right\} \mathrm{Im}\,T(s',t)$$

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- subtraction function $c(t)$ determined from crossing symmetry
- project onto partial waves $t_J^I(s)$ (angular momentum J, isospin I) $\longrightarrow$ coupled system of partial-wave integral equations

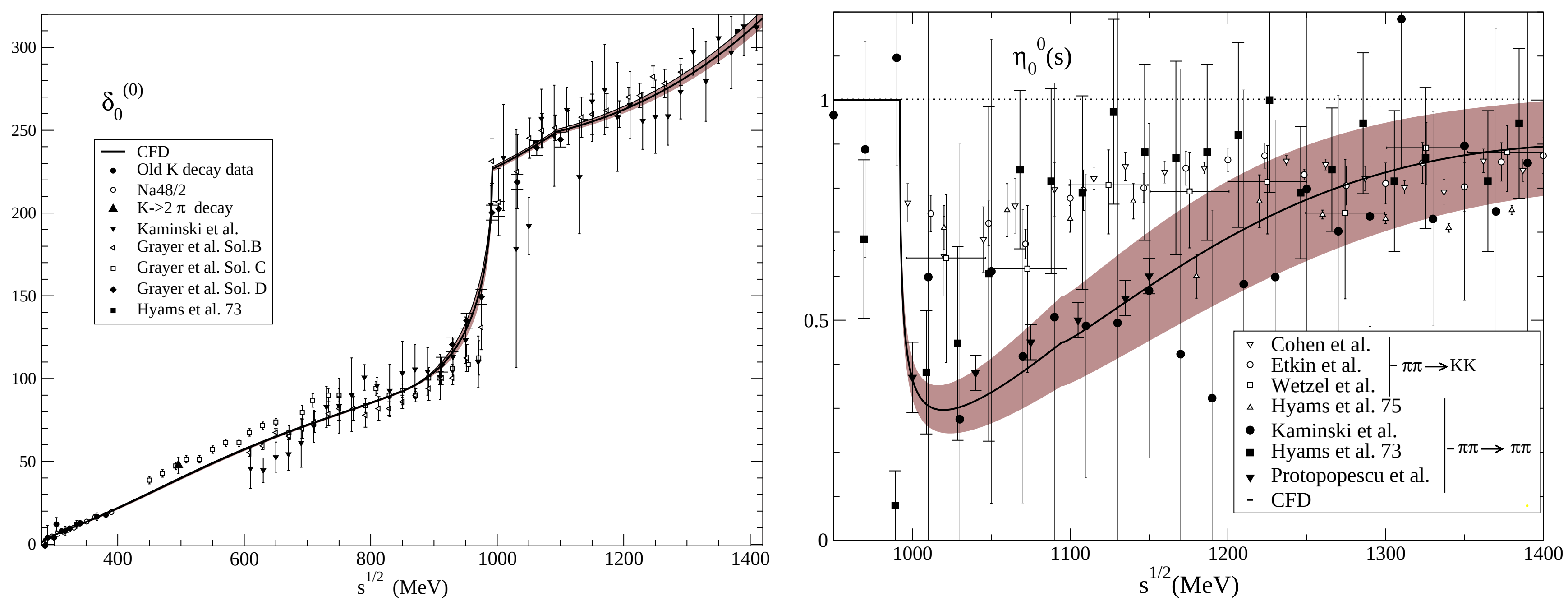
$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s')\,\mathrm{Im}t_{J'}^{I'}(s')$$

Roy 1971

- subtraction polynomial $k_J^I(s)$: $\pi\pi$ scattering lengths can be matched to chiral perturbation theory   Colangelo et al. 2001
- kernel functions $K_{JJ'}^{II'}(s,s')$ known analytically

$\pi\pi$ scattering constrained by analyticity and unitarity

- elastic unitarity $\longrightarrow$ coupled integral equations for phase shifts
- modern precision analyses:
  - ▷ $\pi\pi$ scattering — Ananthanarayan et al. 2001, García-Martín et al. 2011
  - ▷ πK scattering — Büttiker et al. 2004
- example: $\pi\pi$ $I = 0$ S-wave phase shift & inelasticity



García-Martín et al. 2011

- strong constraints on data from analyticity and unitarity!