Leading isospin breaking effects on the lattice

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OUTLINE

- Motivations
- Isospin breaking effects on the lattice
- Results: • $M_{\pi^+} - M_{\pi^0}$ • $M_{K^+} - M_{K^0}$ • $(m_d - m_u)$ • $M_{D^+} - M_{D^0}$

In collaboration with:
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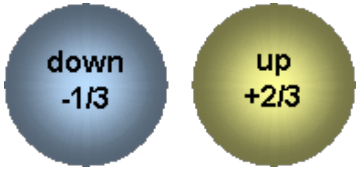
ISOSPIN BREAKING EFFECTS

Isospin symmetry is an almost exact property of the strong interactions



Isospin breaking effects are induced by:

$m_u \neq m_d$: $O[(m_d - m_u)/\Lambda_{QCD}] \approx 1/100$ "Strong"

$Q_u \neq Q_d$: $O(\alpha_{e.m.}) \approx 1/100$ "Electromagnetic"

Since electromagnetic interactions renormalize quark masses the two corrections are intrinsically related

Though small, IB effects can play a very important role

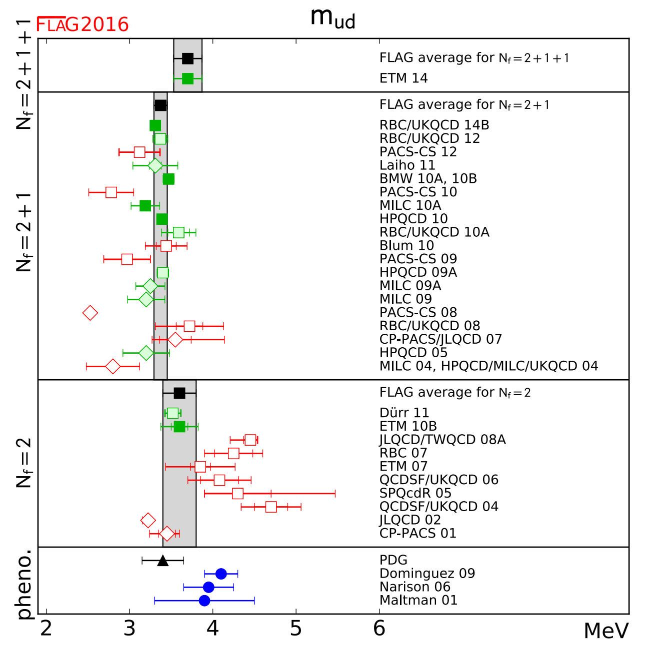
MOTIVATIONS

Theoretical motivations

Quark masses are fundamental parameters of the Standard Model

Lattice QCD in the isosymmetric limit (m_u=m_d, Q_u=Q_d) provide a determination of the average up-down quark mass:

$$m_{ud}=\left(m_u+m_d\right)/2$$



$$m_{ud} = 3.70(17)\ \text{MeV}\ (\text{Nf}=2+1+1)$$

But the knowledge of m_u and m_d separately is important

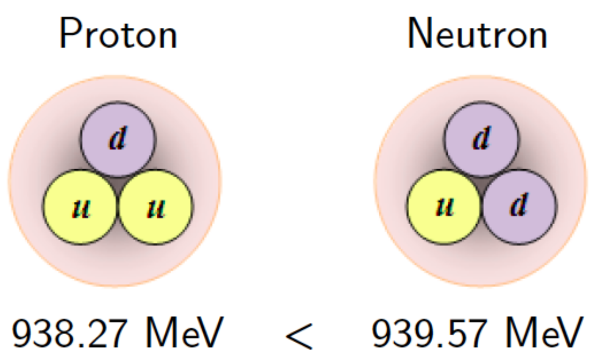
- Accurate knowledge of quark masses is important for our understanding of flavor physics at the fundamental level

$m_u \simeq 2.5$ MeV	$m_d \simeq 5$ MeV
$m_c \simeq 1.2$ GeV	$m_s \simeq 100$ MeV
$m_t \simeq 175$ GeV	$m_b \simeq 4.3$ GeV

A remarkable relation:

$$\left(\frac{m_d}{m_s}\right)^{1/2} \simeq \left(\frac{m_u}{m_c}\right)^{1/4} \simeq V_{us} \simeq 0.22$$

- The actual values of the mass difference $m_d - m_u$ and quark charges Q_d, Q_u implies $M_n > M_p$ and guarantees the stability of matter

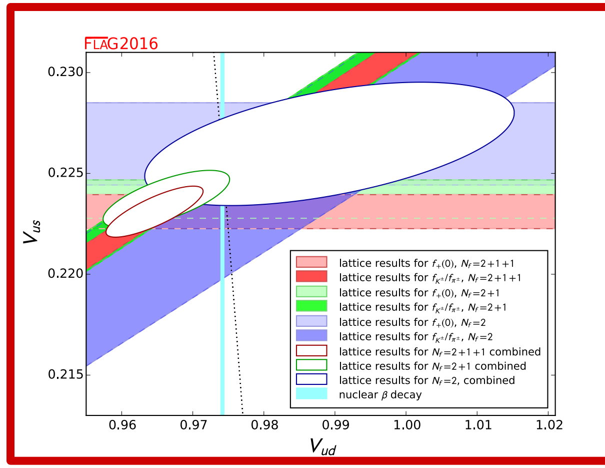


$M(n) - M(p) = 1.3$ MeV $= 0.14\%$

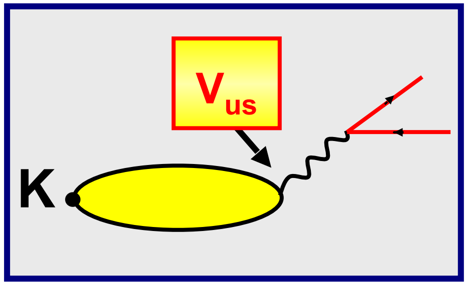
IB effects cannot be neglected at present in flavor physics phenomenology

Example: determination of Vus and the CKM first row unitarity test

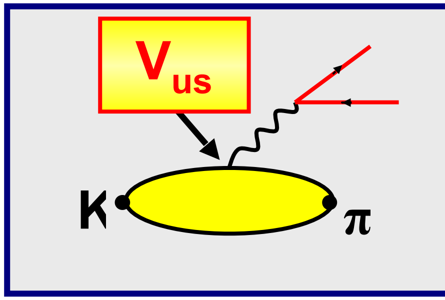
$$|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2=1$$



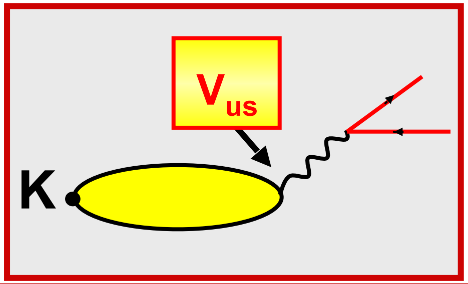
$$\frac{\Gamma(K\to\mu\bar{\nu}_\mu(\gamma))}{\Gamma(\pi\to\mu\bar{\nu}_\mu(\gamma))}=\frac{|V_{us}|^2}{|V_{ud}|^2}\left(\frac{f_K}{f_\pi}\right)^2\frac{m_K\left(1-\frac{m_\mu^2}{m_K^2}\right)}{m_\pi\left(1-\frac{m_\mu^2}{m_\pi^2}\right)}\times 0.9930(35)$$



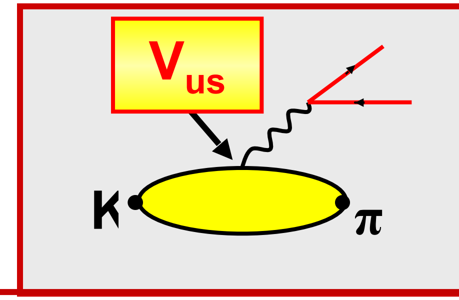
$$\Gamma_{K\to\pi l\nu}=C_K^2\,\frac{G_F^2 m_K^5}{192\pi^2}\,S_{EW}[1+\Delta_{SU(2)}+2\Delta_{EM}]\times|V_{us}|^2|f_+^{K\pi}(0)|^2$$



EXPERIMENTS



FlaviA net Kaon WG



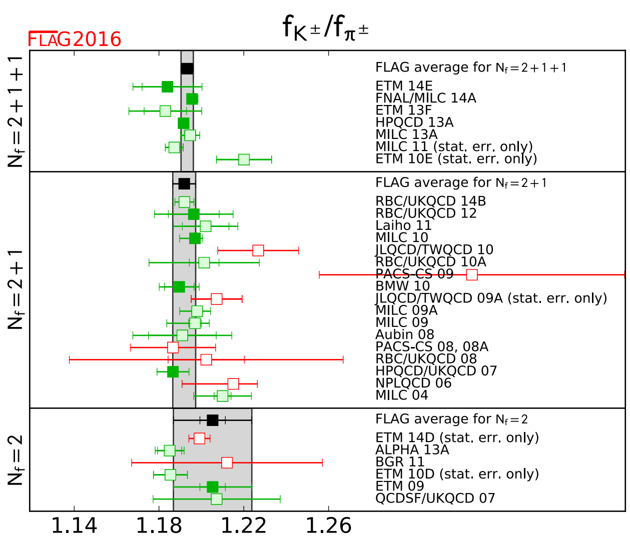
$$\left|\frac{V_{us} f_K}{V_{ud} f_\pi}\right| = 0.2758(5)$$

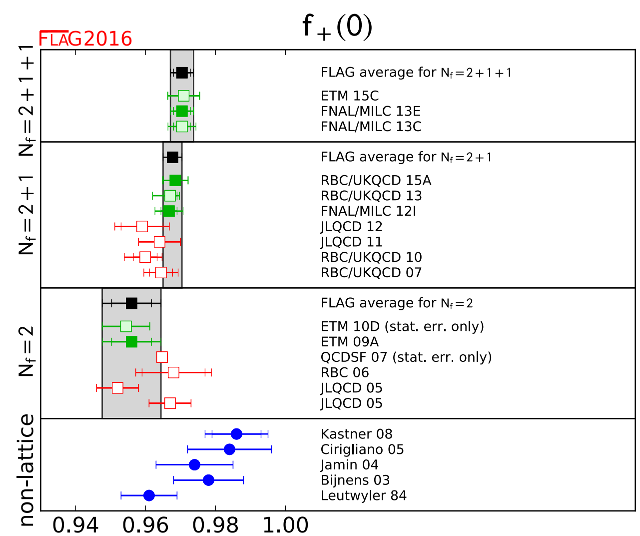
0.2%

$$|V_{us}| f_+(0) = 0.2163(5)$$

THEORY: LATTICE QCD

FLAG





$f_K / f_\pi = 1.193(3)$

0.3%

$f_+(0) = 0.9704(33)$

ISOSPIN BREAKING EFFECTS ON THE LATTICE

A strategy for Lattice QCD:

The isospin breaking part of the Lagrangian is treated as a perturbation

Expand in: $m_d - m_u$, α_{em}

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Isospin breaking effects due to the up-down mass difference in lattice QCD

RM123 collaboration
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Leading isospin breaking effects on the lattice

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(RM123 Collaboration) arXiv:1303.4896

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The $(m_d - m_u)$ expansion

- Identify the isospin breaking term in the QCD action

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}\left(m_u + m_d\right)\left(\bar{u}u + \bar{d}d\right) - \frac{1}{2}\left(m_d - m_u\right)\left(\bar{u}u - \bar{d}d\right) \right] =$$

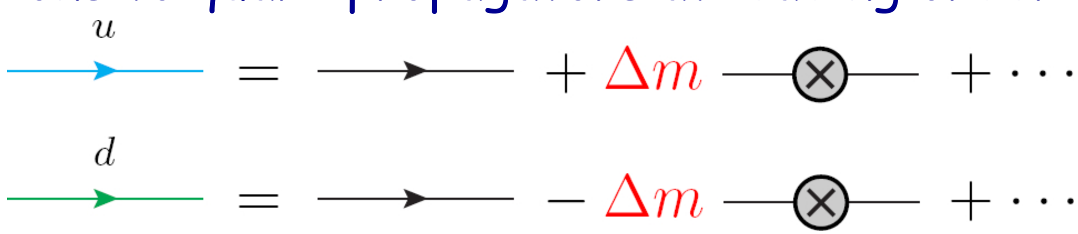
$$= \sum_x \left[m_{ud}\left(\bar{u}u + \bar{d}d\right) - \Delta m\left(\bar{u}u - \bar{d}d\right) \right] = S_0 - \Delta m \hat{S}$$

$$\boxed{\hat{S} = \sum_x \left(\bar{u}u - \bar{d}d\right)}$$

- Expand the functional integral in powers of Δm

$$\langle O \rangle = \frac{\int D\phi\, O\, e^{-S_0 + \Delta m \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O\, e^{-S_0}\left(1 + \Delta m \hat{S}\right)}{\int D\phi\, e^{-S_0}\left(1 + \Delta m \hat{S}\right)} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0}{1 + \cancel{\Delta m \langle \hat{S} \rangle_0}}$$

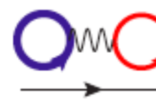
Corrections to quark propagators at leading order in Δm:

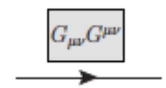


The expansion for the quark propagator

$\Delta \longrightarrow^{\pm} =$

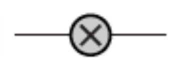
$(e_f e)^2$  $+ (e_f e)^2$  $- [m_f - m_f^0]$  $\mp [m_f^{cr} - m_0^{cr}]$ 

$-e^2 e_f \sum_{f_1} e_{f_1}$  $- e^2 \sum_{f_1} e_{f_1}^2$  $- e^2 \sum_{f_1} e_{f_1}^2$  $+ e^2 \sum_{f_1 f_2} e_{f_1} e_{f_2}$ 

$+ \sum_{f_1} \pm [m_{f_1}^{cr} - m_0^{cr}]$  $+ \sum_{f_1} [m_{f_1} - m_{f_1}^0]$  $+ [g_s^2 - (g_s^0)^2]$ 

.

In the electro-quenched approximation:

$\Delta \longrightarrow^{\pm} = (e_f e)^2 \Big[$  $+$  $\Big] - [m_f - m_f^0]$  $\mp [m_f^{cr} - m_0^{cr}]$  .

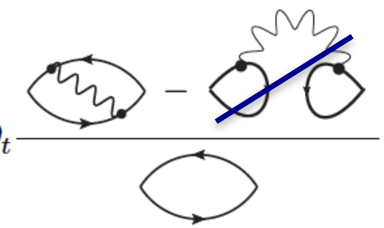
preview

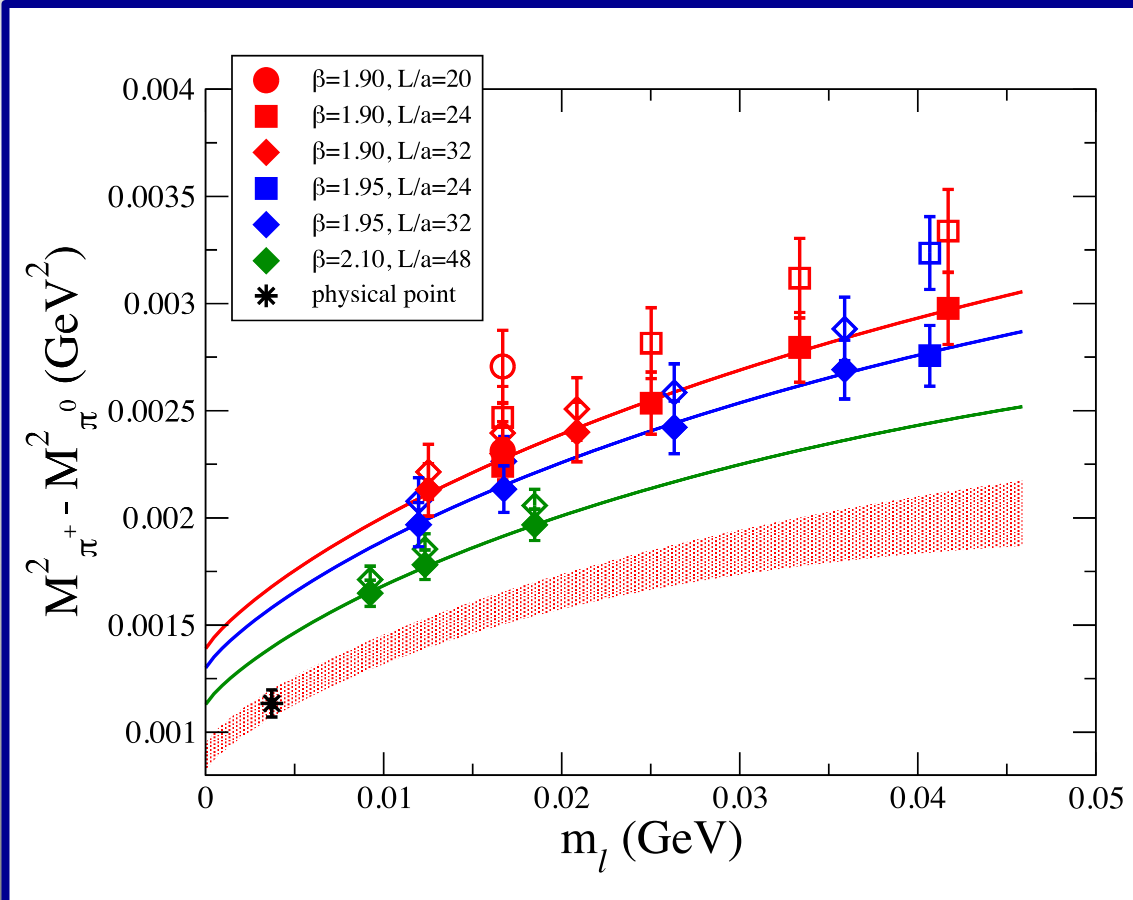
RESULTS

The charged-neutral pion mass splitting

The charged and neutral pion masses

$$M_{\pi^+} - M_{\pi^0} = \frac{(e_u - e_d)^2}{2} e^2 \partial_t$$





The charged-neutral pion mass splitting

We obtain:

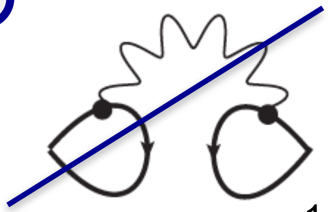
$$\boxed{\begin{aligned} M_{\pi^+} - M_{\pi^0} &= 4.21(23)_{stat}(13)_{syst} \ \text{MeV} \\ &= 4.21(26) \ \text{MeV} \end{aligned}}$$

where the errors are statistical and systematic (estimated from chiral and continuum extrapolations and FSE).

The result is in good agreement with the experimental determination

$$\left[M_{\pi^+} - M_{\pi^0}\right]^{\exp} = 4.5936(5) \ \text{MeV}$$

It suggests, a posteriori, that the effect of having neglected the disconnected contribution of $O(\alpha_{em} m_{ud})$ is small



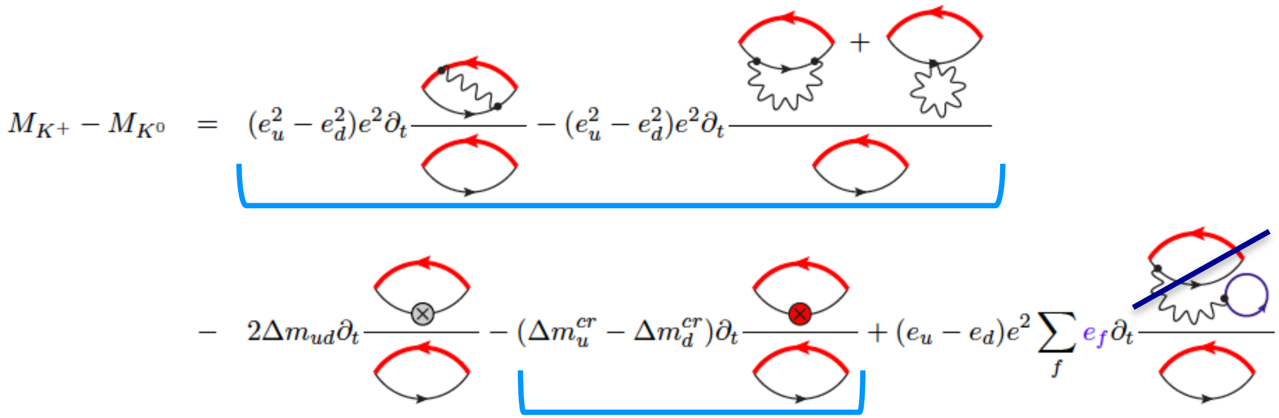
preview

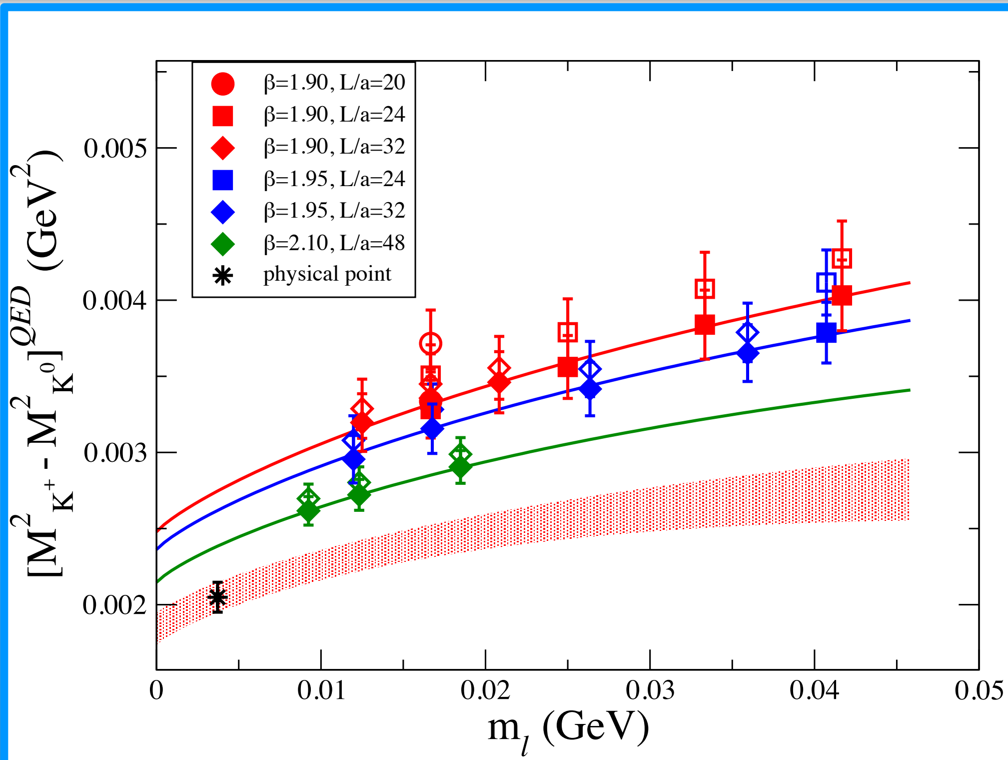
RESULTS

The charged-neutral

kaon mass splitting and $(m_d - m_u)$

The charged and neutral kaon masses





Kaon mass splitting: results

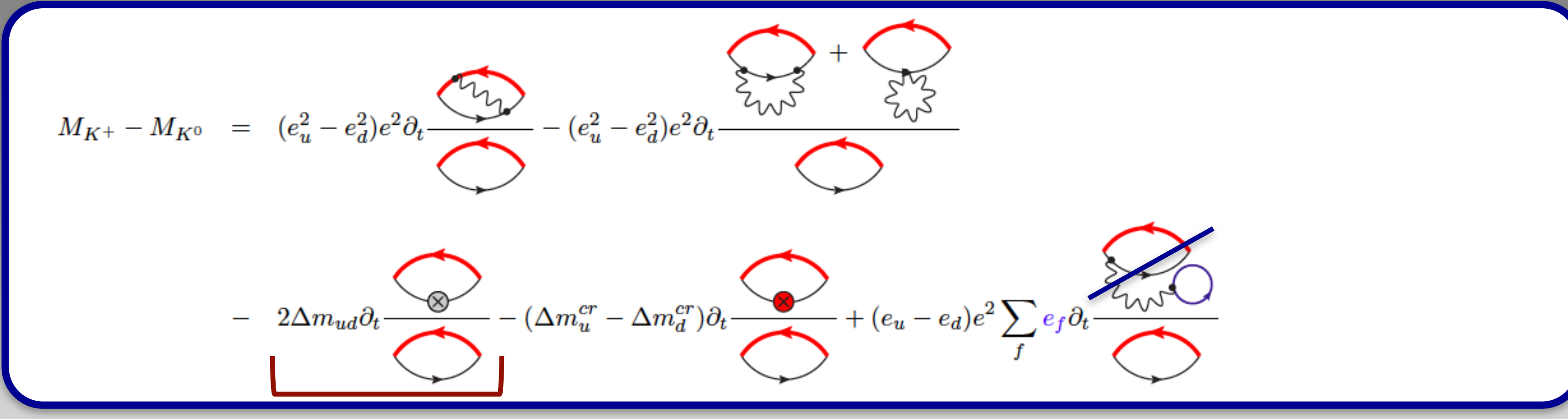
$$\boxed{\left[\mathrm{M}_{K^+} - \mathrm{M}_{K^0}\right]^{QED} = 2.07(15)\ \mathrm{MeV}}$$

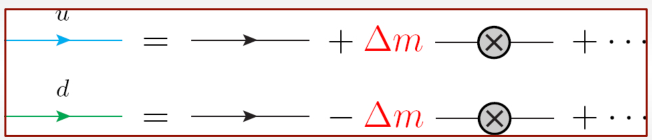
$$\downarrow$$

$$\boxed{\left[\mathrm{M}_{K^+} - \mathrm{M}_{K^0}\right]^{\exp} = -3.934(20)\ \mathrm{MeV}}$$

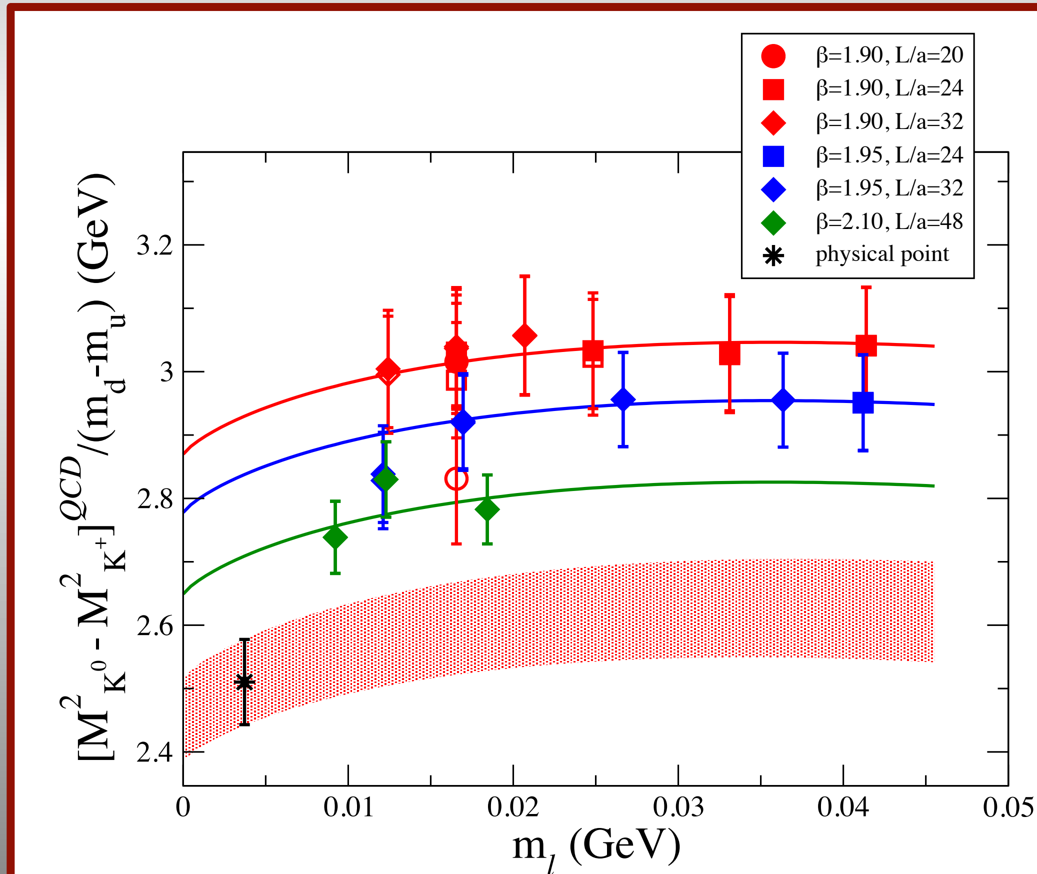
$$\downarrow$$

$$\boxed{\left[\mathrm{M}_{K^+} - \mathrm{M}_{K^0}\right]^{QCD} = -6.00(15)\ \mathrm{MeV}}$$





$$\boxed{\hat{S} = \sum_x \left(\bar{u}u - \bar{d}d\right)}$$

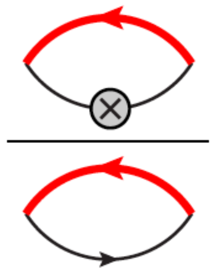


The up and down quark masses and related…

From:

$$\boxed{\frac{\left[M^2_{K^0} - M^2_{K^+}\right]^{QCD}}{m_d - m_u} = 2.51(18)\ \mathrm{GeV}}$$

$2\Delta m_{ud}\partial_t$



(all masses in the MSbar scheme at 2 GeV), one obtains:

$$(m_d - m_u) = 2.38(9)_{stat}(16)_{syst}\ \mathrm{MeV}$$
$$= 2.38(18)\ \mathrm{MeV}$$

$$m_u / m_d = 0.513(18)_{stat}(24)_{syst}$$
$$= 0.513(30)$$

$$\mathrm{m}_{ud} = 3.70(17)\ \mathrm{MeV}$$

$$m_u = 2.50(17)\ \mathrm{MeV}$$
$$m_d = 4.88(20)\ \mathrm{MeV}$$

BMW (2017) $\mathrm{m}_u = 2.27(9)$ MeV

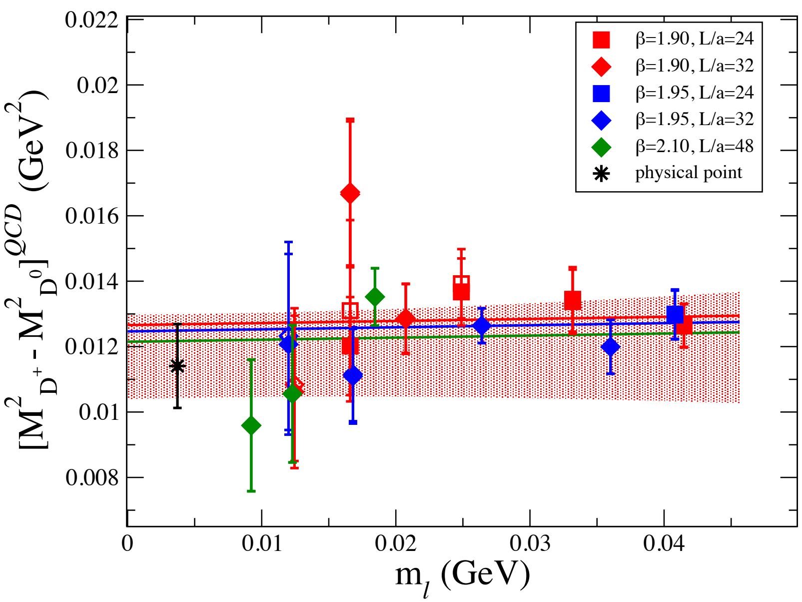
BMW (2017) $\mathrm{m}_d = 4.67(9)$ MeV

1st lattice results

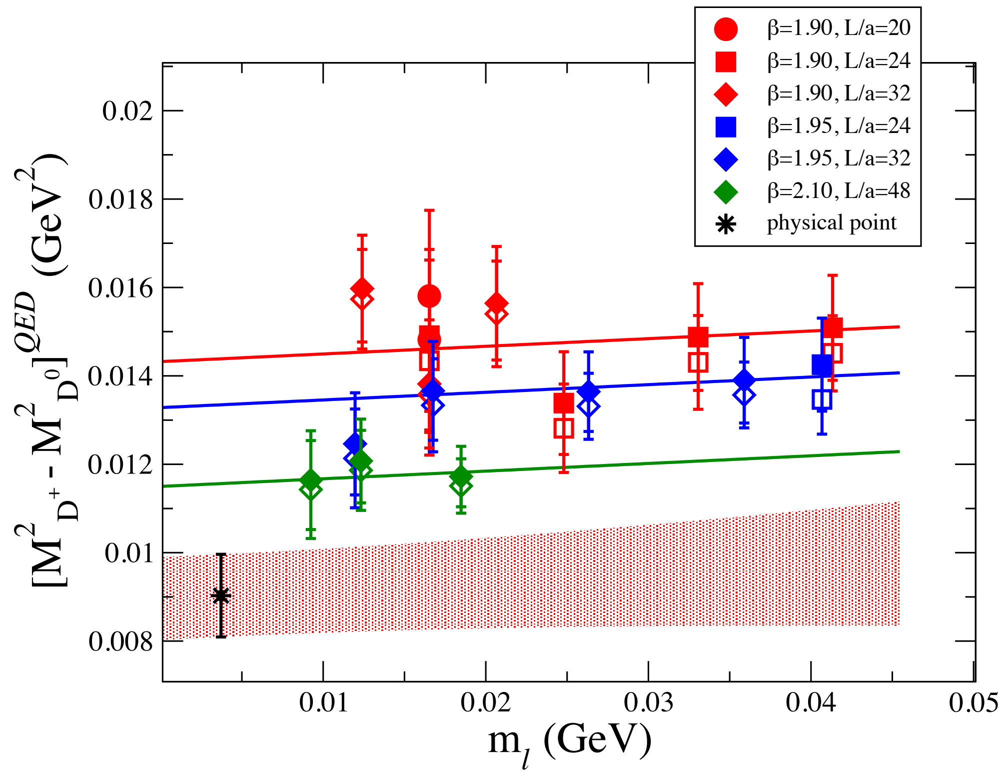
RESULTS

The charged-neutral

D meson mass splitting



$$\left[M_{D^+} - M_{D^0}\right]^{QCD} = 3.06(27)_{stat}(7)_{syst}\ \text{MeV}$$



$$\left[M_{D^+} - M_{D^0}\right]^{QED} = 2.42(22)_{stat}(46)_{syst}\ \text{MeV}$$

$$M_{D^+} - M_{D^0} = 5.47(53)\ \text{MeV}$$

$$\left[M_{D^+} - M_{D^0}\right]^{exp} = 4.75(8)\ \text{MeV}$$

CONCLUSIONS

Several results

$$M_{\pi^+} - M_{\pi^0} = 4.21(26)\ \mathrm{MeV}$$

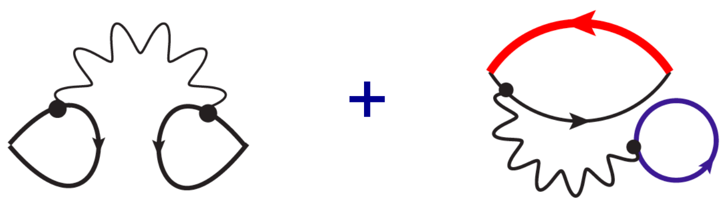
$$\left[\mathrm{M}_{K^+} - \mathrm{M}_{K^0}\right]^{QED} = 2.07(15)\ \mathrm{MeV}$$

$$\left[\mathrm{M}_{K^+} - \mathrm{M}_{K^0}\right]^{QCD} = -6.00(15)\ \mathrm{MeV}$$

$$m_u = 2.50(17)\ \mathrm{MeV}$$
$$m_d = 4.88(20)\ \mathrm{MeV}$$

$$M_{D^+} - M_{D^0} = 5.47(53)\ \mathrm{MeV}$$

Future perspectives



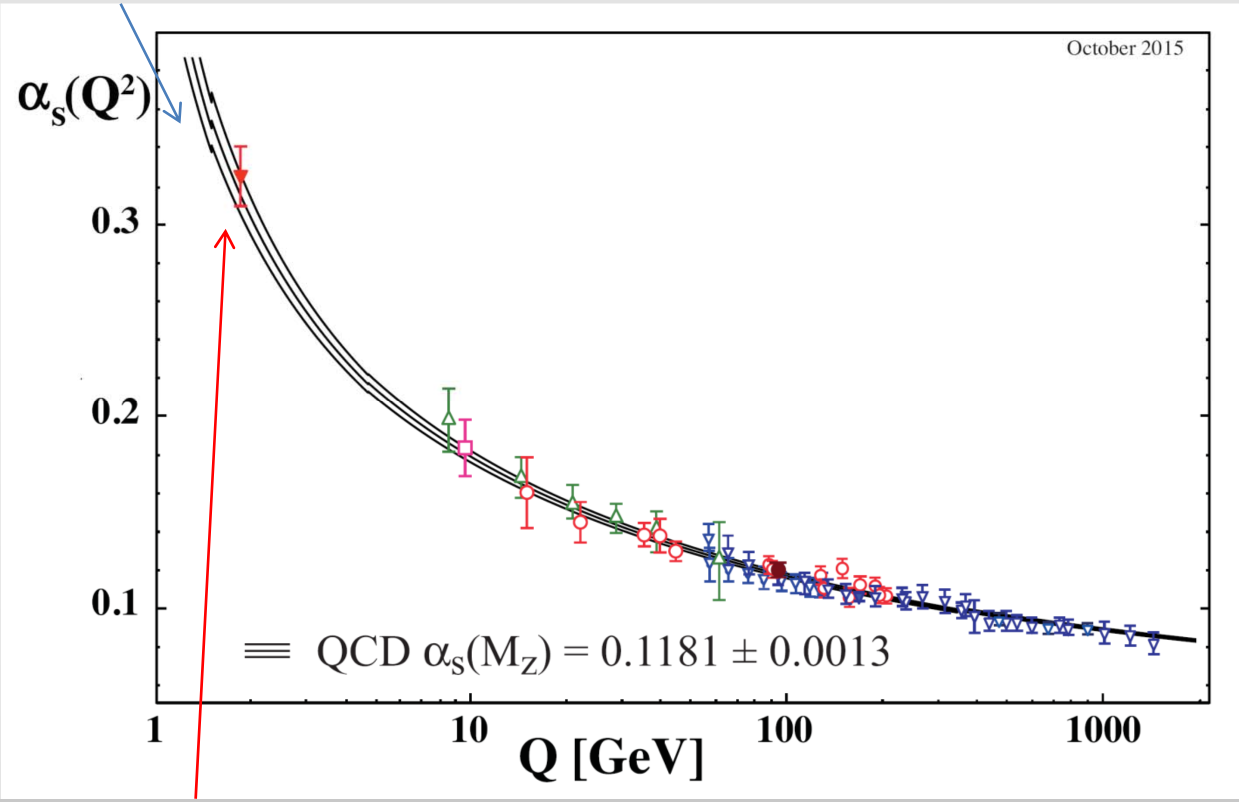
ADDITIONAL

SLIDES

Quantum Chromodynamics (QCD)

describes strong interactions

Quark confinement



low energies (hadronic scale)

non-perturbative methods ⟶ Lattice QCD (LQCD)

LQCD

• non-perturbative approach based only on first principles

• free parameters of the fundamental theory [10 = 6 (quark masses) + 4 (CKM)]

LQCD provides a UV and IR regularization of the theory

Lattice spacing a

Finite volume

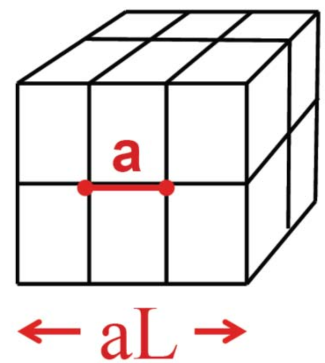
$$|p| \leq \frac{\pi}{a}$$

$$p_{\min} = \frac{2\pi}{aL}$$

QCD is obtained in the limits:

$a \to 0$ Continuum limit

$L \to \infty$ Thermodynamic limit



The fields are defined on a 4-dim hypercubic lattice of volume

$V = a^4 \, (L^3 \times T)$

$$\varphi(x) \to \varphi(an)$$

$$n = (n_t, n_x, n_y, n_z)$$

$$\partial_\mu \varphi(x) \to \frac{\varphi(x + a\hat{\mu}) - \varphi(x)}{a}$$

Numerical simulations in LQCD

$$\langle O \rangle = \frac{1}{Z} \int D\varphi \, O(\varphi) \, e^{-S(\varphi)}$$

Path-Integral formalism in Euclidean space-time

Monte Carlo methods

Importance Sampling

$$P(\varphi) = \frac{e^{-S(\varphi)}}{Z}$$

$$\boxed{\langle O \rangle \simeq \frac{1}{N_c} \sum_{i=1}^{N_c} O(\varphi_i)}$$

Metropolis and *Molecular Dynamics* algorithms

CORRELATION FUNCTIONS

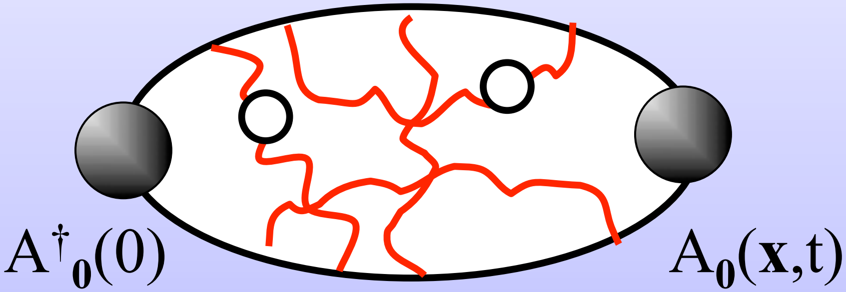
Hadron masses and matrix elements

$$G(t) = \sum_{\mathbf{x}} \langle A_0(\mathbf{x},t)\, A^\dagger{}_0(\mathbf{0},0) \rangle =$$

The operator $A_0 = \bar{u}\gamma_0\gamma_5 d$ can excite 1-π, 3-π etc. states

$$= \sum_{\mathbf{x}} \sum_n \frac{\langle 0 | e^{iPx} A_0(0)\, e^{-iPx} | n \rangle \langle n | A^\dagger{}_0(0) | 0 \rangle}{2 E_n}$$

$$= \sum_n \frac{|\langle 0 | A_0 | n \rangle|^2}{2 m_n} \exp[-m_n t]$$

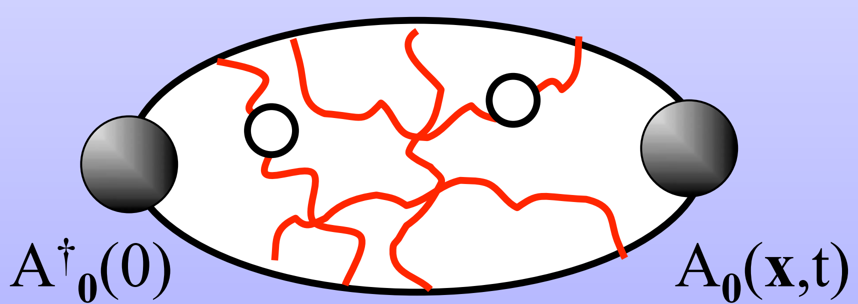


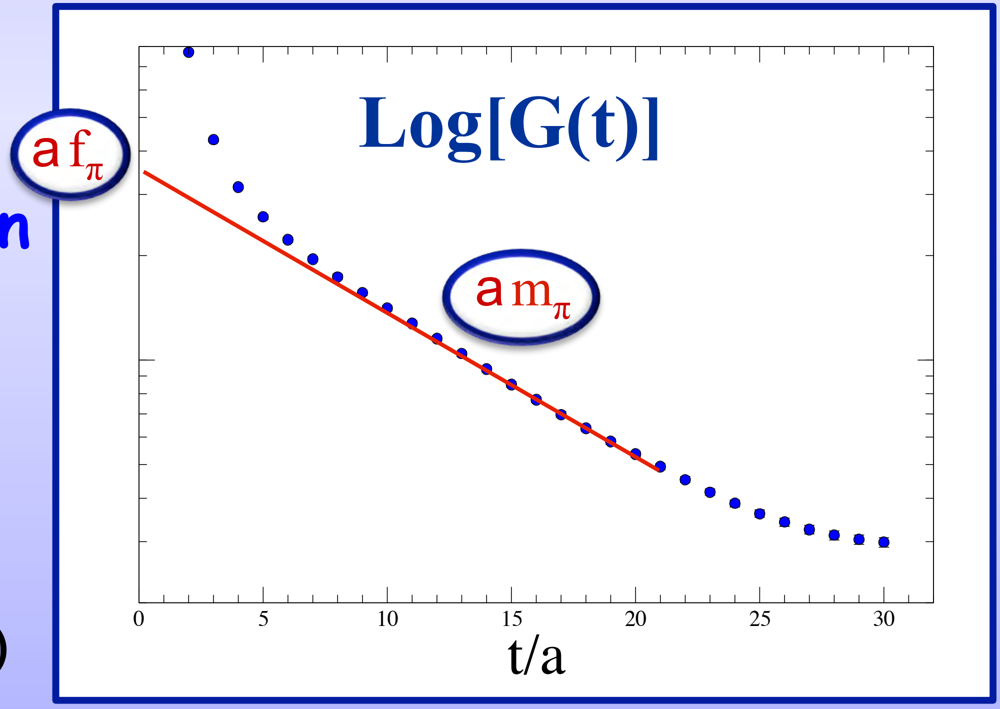
$$\xrightarrow{t \to \infty} \frac{|\langle 0 | A_0 | \pi \rangle|^2}{2 m_\pi} \exp[-m_\pi t] = \frac{f_\pi^2 m_\pi}{2} \exp[-m_\pi t]$$

$$G(t) = \sum_{\mathbf{x}} \langle A_0(\mathbf{x},t)\, A^{\dagger}{}_0(\mathbf{0},0) \rangle \rightarrow$$

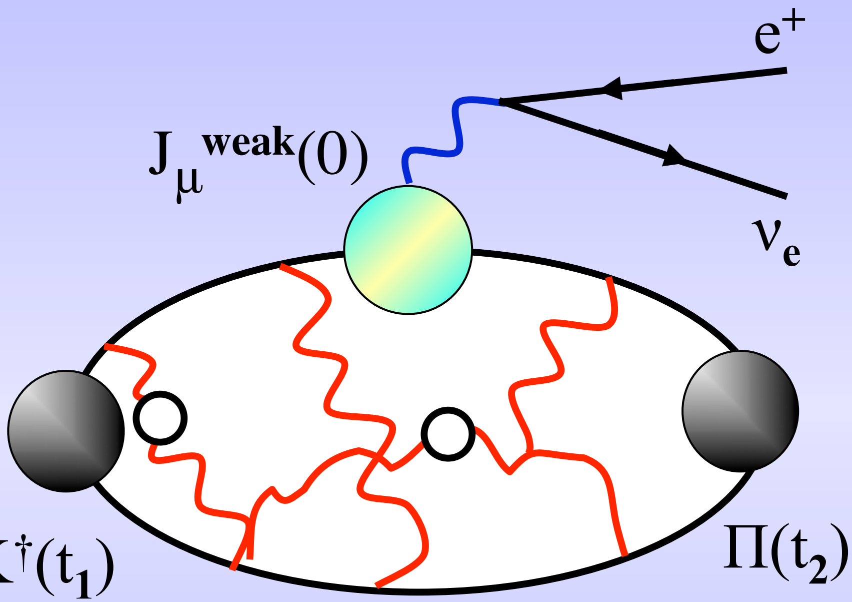
$$\rightarrow \frac{|\langle 0|A_0|\pi\rangle|^2}{2\,m_\pi} \exp[-m_\pi t] = \frac{f_\pi^2\, m_\pi}{2} \exp[-m_\pi t]$$

Hadron mass and **⟨0|A|h⟩ matrix elements** from the **2-point correlation function**





3-point functions



$$K^\dagger(t_1) = \sum_{\mathbf{x}} K^\dagger(\mathbf{x}, t_1) \exp[-i\mathbf{p_K}\mathbf{x}]$$
$$\Pi(t_2) = \sum_{\mathbf{x}} \Pi(\mathbf{x}, t_2) \exp[+i\mathbf{p_\pi}\mathbf{x}]$$

$$\langle \Pi(t_2)\, J_\mu^{\mathbf{weak}}(0)\, K^\dagger(t_1) \rangle \longrightarrow$$

$$\frac{\langle 0|\Pi|\pi\rangle\langle K|\, K^\dagger|0\rangle \exp[-\, E_K t_1 -\, E_\pi t_2]}{(2E_K)\,(2E_\pi)} \times \langle \pi(\mathbf{p_\pi}) \mid J_\mu^{\mathbf{weak}}(0) \mid K(\mathbf{p_K}) \rangle$$

Also e.m. form factors, structure functions, etc

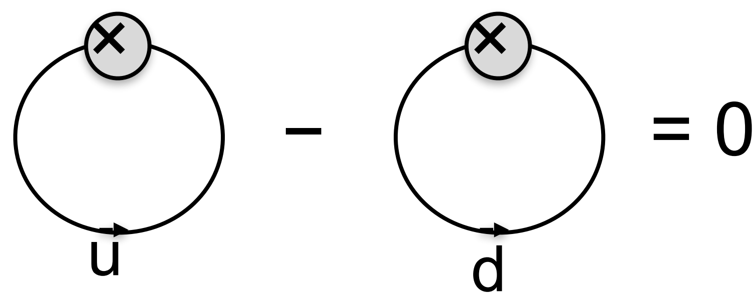
ISOSPIN BREAKING EFFECTS ON THE LATTICE

IB effects due to the m_d-m_u mass difference

$$\langle O \rangle \simeq \langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0 \longrightarrow \frac{\Delta O}{O} = \frac{\langle O \rangle - \langle O \rangle_0}{\langle O \rangle_0} = \Delta m \frac{\langle O \hat{S} \rangle_0}{\langle O \rangle_0}$$

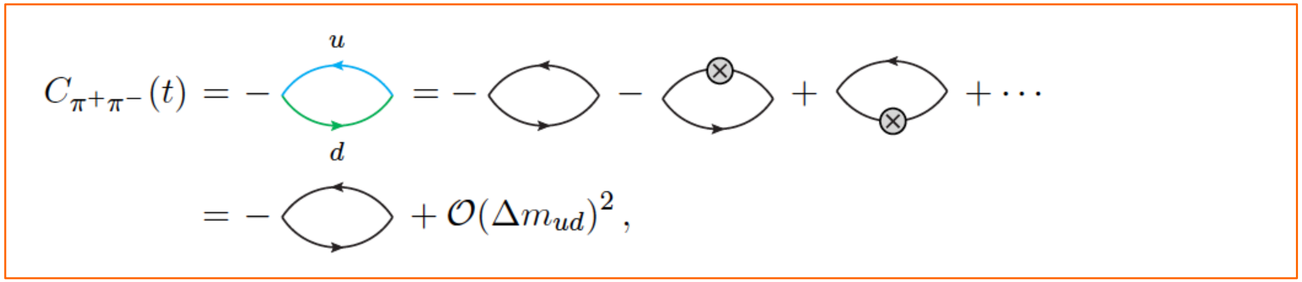
Very precise on the lattice because of the statistical correlation

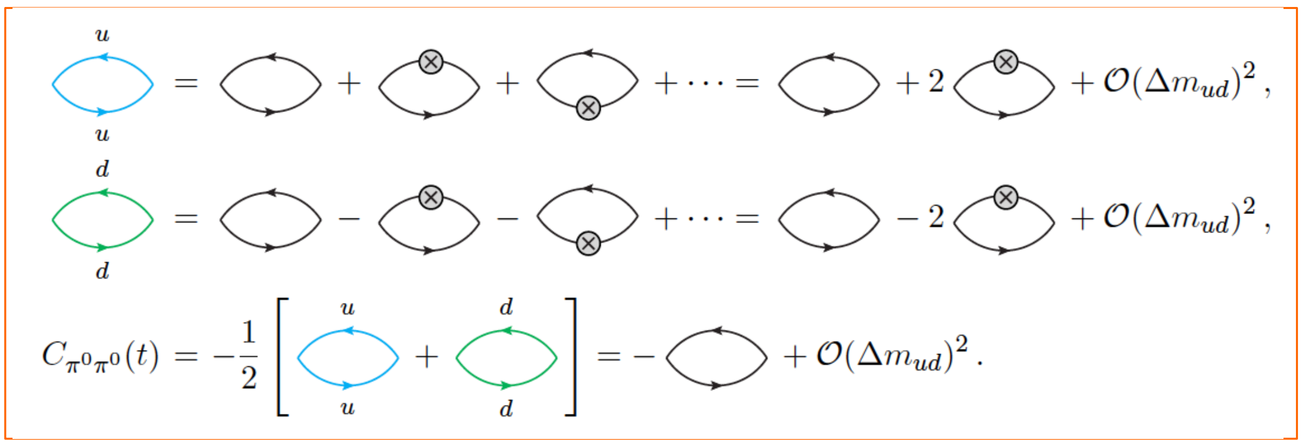
Disconnected Wick contractions generated by the insertion of $\hat{S} = \Sigma_x(\bar{u}u-\bar{d}d)$ vanish due to isospin symmetry:



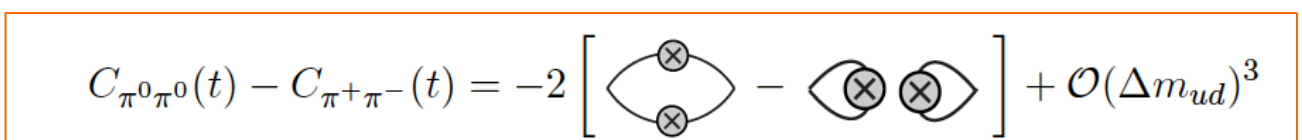
An example: the charged and neutral pions

Because of the u $\longleftrightarrow$ d symmetry, the corrections cancel at 1st order





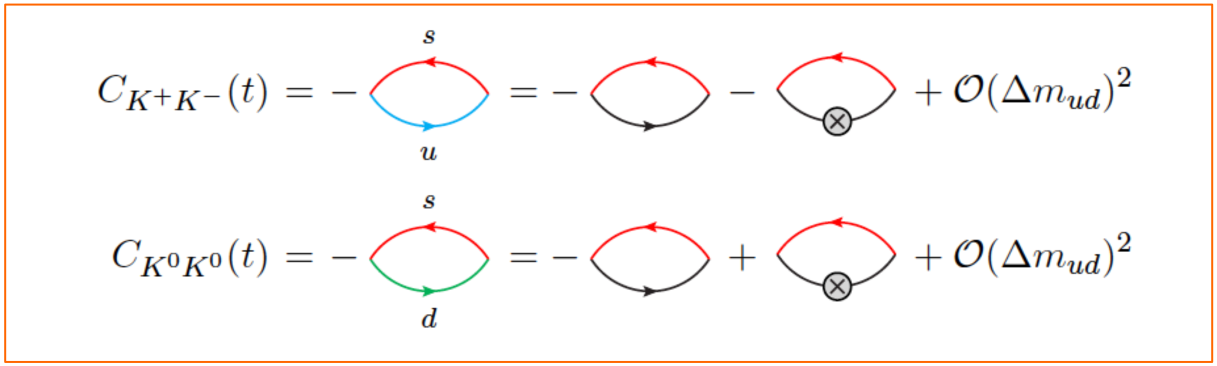
This is certainly not the case at 2nd order:



The charged and neutral kaons

Corrections to the charged and neutral kaons are equal and opposite

at 1st order:



ISOSPIN BREAKING EFFECTS ON THE LATTICE

Electromagnetic corrections

LATTICE QED

- Non-compact QED: the dynamical variable is the gauge potential $A_\mu(x)$ in a fixed (covariant) gauge

- The covariant derivatives are defined by introducing the QED links:

$$A_\mu(x) \to E_\mu(x) = e^{-iaeA_\mu(x)}$$

$$aD^+_\mu q_f(x) = \left[E_\mu(x)\right]^{e_f} q_f(x+a\hat{\mu}) - q_f(x)$$

- Gauge transformations for the quarks and photon fields are:

$$q_f(x) \to e^{iee_f\lambda(x)} q_f(x)$$
$$\bar{q}_f(x) \to e^{-iee_f\lambda(x)} \bar{q}_f(x)$$

$$A_\mu(x) \to A_\mu(x) + \nabla^+_\mu \lambda(x) = $$
$$= A_\mu(x) + \big(\lambda(x+a\hat{\mu}) - \lambda(x)\big)/a$$

- The QED link then transforms as: $E_\mu(x) \to e^{-iae\left[A_\mu(x)+\nabla_\mu\lambda(x)\right]} = e^{ie\lambda(x)} E_\mu(x) e^{-ie\lambda(x+a\hat{\mu})}$

and $D^+_\mu q_f(x) = \big(\partial_\mu - iee_f A_\mu(x)\big) q_f(x) + O(a)$ is manifestly covariant.

LATTICE QED

- For non-compact QED the pure gauge action is:

$$S_{QED} = \frac{1}{4}\sum_{x} F_{\mu\nu}(x)F_{\mu\nu}(x) = \frac{1}{4}\sum_{x;\mu\nu}\left(\nabla^{+}_{\mu}A_{\nu}(x) - \nabla^{+}_{\nu}A_{\mu}(x)\right)^{2} =$$

$$= -\frac{1}{4}\sum_{x;\mu\nu}\left[A_{\nu}(x)\nabla^{-}_{\mu}\left(\nabla^{+}_{\mu}A_{\nu}(x) - \nabla^{+}_{\nu}A_{\mu}(x)\right) - A_{\mu}(x)\nabla^{-}_{\nu}\left(\nabla^{+}_{\mu}A_{\nu}(x) - \nabla^{+}_{\nu}A_{\mu}(x)\right)\right]$$

- By using a covariant gauge fixing one gets:

$$\boxed{\nabla^{-}_{\mu}A_{\mu}(x) = 0} \longrightarrow \boxed{S_{QED} = \frac{1}{2}\sum_{x;\mu\nu} A_{\nu}(x)\left(-\nabla^{-}_{\mu}\nabla^{+}_{\mu}\right)A_{\nu}(x)}$$

- Imposing periodic b.c. and looking at the action in momentum space reveals a problem with the zero momentum mode:

$$\boxed{S_{QED} = \frac{1}{2}\sum_{k;\mu\nu} \tilde{A}^{*}_{\nu}(k)\left(2\sin(k_{\mu}/2)\right)^{2}\tilde{A}_{\nu}(k)}$$

The photon propagator is infrared divergent

LATTICE QED

- The infrared problem is not specific of the lattice regularization but it is general for QED in a finite volume with periodic b.c. Already at the classical level, the Gauss' law for a charged particle is inconsistent for the zero mode:

$$\nabla^-_\mu F_{\mu\nu}(x) = j_\nu(x) \longrightarrow \nabla^-_i E_i(x) = \rho(x) \longrightarrow \boxed{0 = \sum_{\vec{x}} \nabla^-_i E_i(x) = e \sum_{\vec{x}} \delta^3(t,x) = e}$$

- A solution to the infrared problem consists in removing the zero mode:

$$D^\perp_{\mu\nu}(x-y) = \sum_{k\neq 0} \frac{\delta_{\mu\nu}\, e^{ik(x-y)}}{\left[2\sin(k_\rho/2)\right]^2}$$

- We subtracted the zero mode in x-space and applied a stochastic technique

$$P^\perp\phi(x) \equiv \phi(x) - \frac{1}{V}\sum_y \phi(y)$$

$$\left[-\nabla^-_\rho \nabla^+_\rho\right]\phi_\mu(x) = P^\perp \eta_\mu(x) \longleftarrow \boxed{\text{Real } Z_2 \text{ noise}}$$

$$\phi_\mu(x) = \left[\frac{\delta_{\mu\nu}}{-\nabla^-_\rho \nabla^+_\rho} P^\perp\right]\eta_\nu(x) = \sum_y D^\perp_{\mu\nu}(x-y)\eta_\nu(y)$$

The leading isospin breaking expansion

- The QCD + QED action is written in terms of the full covariant derivative:

$$D^{+}_{\mu} q_f(x) = \left[E_\mu(x) \right]^{e_f} U_\mu(x)\, q_f(x+\hat{\mu}) - q_f(x)$$

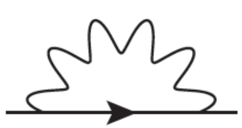
QED QCD

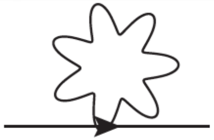
- Since $E_\mu(x) = e^{-i e A_\mu(x)} = 1 - i e A_\mu(x) - 1/2\, e^2 A_\mu^2(x) + \ldots$ the expansion of the lattice action up to $O(e^2)$ contains 2 contributions:

$$S_f = \sum_x \bar{q}_f(x) D_f[U, A]\, q_f(x) =$$

$$= S_f(e=0) + \sum_{x,\mu} \left[e_f e A_\mu(x) V_\mu^f(x) + \frac{(e_f e)^2}{2} A_\mu(x) A_\mu(x) T_\mu^f(x) + \ldots \right]$$

Both contributions are required for gauge invariance

$(e_f e)^2$ 

$(e_f e)^2$ 

The leading isospin breaking expansion

- Switching on the e.m. interactions requires the introduction of new counterterms which renormalize the couplings of the theory:

$$\vec{g}^0 = \left(0, g_s^0, m_u^0, m_d^0, m_s^0, \ldots\right) \to \vec{g} = \left(e^2, g_s, m_u, m_d, m_s, \ldots\right)$$

- For any observable, the leading isospin breaking expansion reads,

$$O(\vec{g}) = O(\vec{g}^0) + \left[e^2 \frac{\partial}{\partial e^2} + \left(g_s^2 - (g_s^0)^2\right) \frac{\partial}{\partial g_s^2} + \left(m_f - m_f^0\right) \frac{\partial}{\partial m_f} + \ldots \right] O(\vec{g}) \Big|_{\vec{g} = \vec{g}^0}$$

or, in terms of renormalized couplings,

$$O(\vec{\hat{g}}) = O(\vec{\hat{g}}^0) + \left[\hat{e}^2 \frac{\partial}{\partial \hat{e}^2} + \left(\hat{g}_s^2 - \left(\frac{Z_{g_s}}{Z_{g_s}^0} g_s^0 \right)^2 \right) \frac{\partial}{\partial \hat{g}_s^2} + \left(m_f - \frac{Z_{m_f}}{Z_{m_f}^0} m_f^0 \right) \frac{\partial}{\partial \hat{m}_f} + \ldots \right] O(\vec{g}) \Big|_{\vec{g} = \vec{g}^0}$$

ETMC gauge ensembles
Nf=2+1+1

ensemble	β	V/a^4	M_π(MeV)	M_K(MeV)	M_D(MeV)
$A30.32$	1.90	$32^3 \times 64$	275 (10)	568 (22)	2012 (77)
$A40.32$			316 (12)	578 (22)	2008 (77)
$A50.32$			350 (13)	586 (22)	2014 (77)
$A40.24$		$24^3 \times 48$	322 (13)	582 (23)	2017 (77)
$A60.24$			386 (15)	599 (23)	2018 (77)
$A80.24$			442 (17)	618 (24)	2032 (78)
$A100.24$			495 (19)	639 (24)	2044 (78)
$A40.20$		$20^3 \times 48$	330 (13)	586 (23)	2029 (79)
$B25.32$	1.95	$32^3 \times 64$	259 (9)	546 (19)	1942 (67)
$B35.32$			302 (10)	555 (19)	1945 (67)
$B55.32$			375 (13)	578 (20)	1957 (68)
$B75.32$			436 (15)	599 (21)	1970 (68)
$B85.24$		$24^3 \times 48$	468 (16)	613 (21)	1972 (68)
$D15.48$	2.10	$48^3 \times 96$	223 (6)	529 (14)	1929 (49)
$D20.48$			255 (7)	535 (14)	1933 (50)
$D30.48$			318 (8)	550 (14)	1937 (49)

European Twisted Mass Collaboration

Finite size effects (FSE)

In pure QCD, the existence of a mass gap renders FSE exponentially small ~ $e^{-M_\pi \cdot L}$ (in most of the cases)

The QED photon is massless, the e.m. interactions are long ranged and FSE are only power suppressed.

With our regularization of the zero mode, FSE are expressed by:

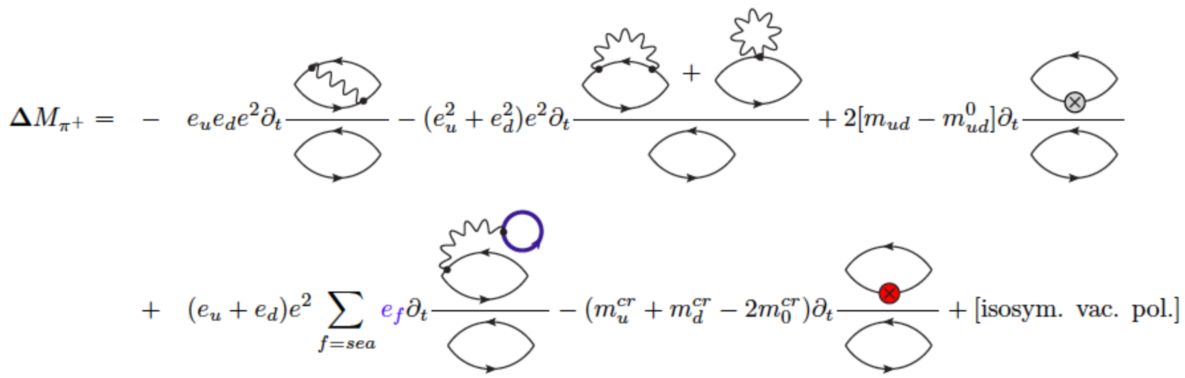
$$M_{PS}^2(T,L) \underset{T,L\to+\infty}{\sim} M_{PS}^2 \left\{1 - q^2\alpha_{em}\left[\frac{\kappa}{M_{PS}L}\left(1+\frac{2}{M_{PS}L}\right)\right]\right\}$$

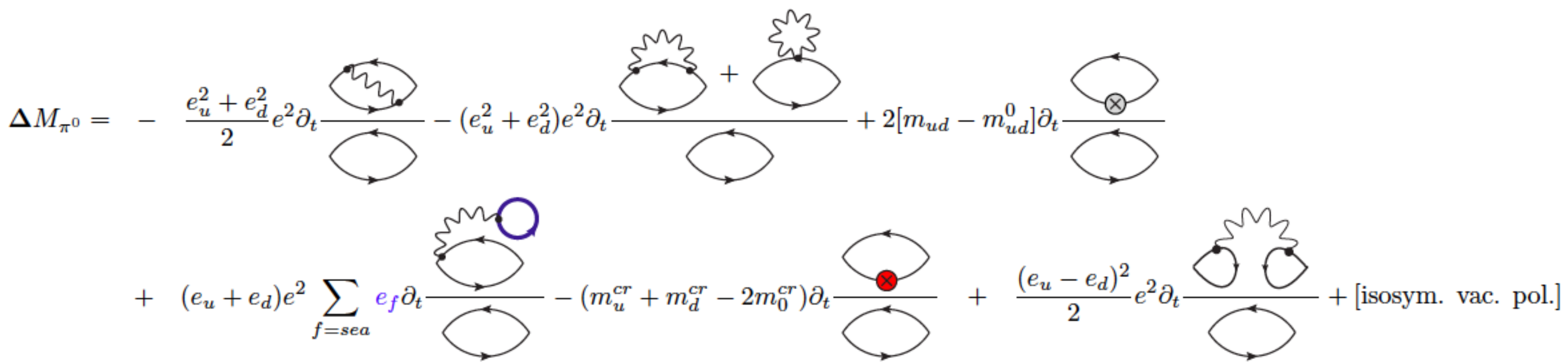
$$\boxed{\kappa = 2.837297(1)}$$

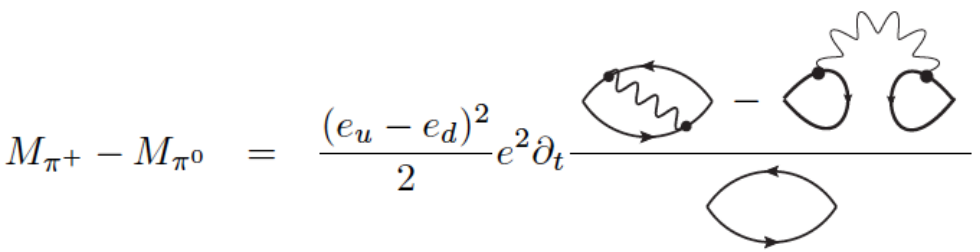
$\boxed{\mathrm{QED}_L \ \text{(i.e. } A_\mu(k_0, \vec{k}=\vec{0}) \equiv 0 \text{ for all } k_0)}$

Sz. Borsanyi *et al.*
arXiv:1406.4088 [hep-ph]

The charged and neutral pion masses



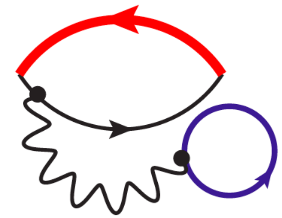




Only 2 diagrams contribute to the pion mass splitting.
The disconnected diagram, of $O(\alpha_{em} m_{ud})$, has been neglected in the present calculation

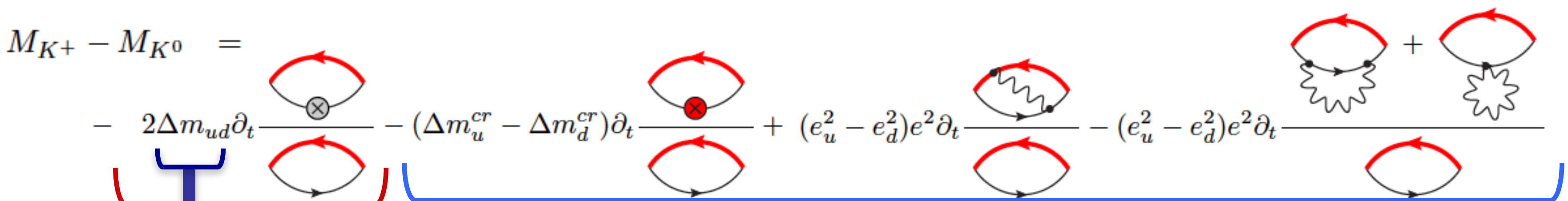
The charged/neutral kaon mass splitting

- We adopt the electro-quenched approximation:
  - To be addressed in a future calculation



= 0

- By choosing a specific renormalization scheme and scale for the quark masses, one can separate the QCD and QED contributions:



$\propto e^2$, separately divergent

$$\Delta m_{ud} = \frac{1}{2}\left(\frac{\hat{m}_d}{Z_{m_d}} - \frac{\hat{m}_u}{Z_{m_u}}\right) = \frac{1}{2}\left(\frac{1}{Z_{m_d}} + \frac{1}{Z_{m_u}}\right)\Delta\hat{m}_{ud} + \frac{1}{2}\left(\frac{1}{Z_{m_d}} - \frac{1}{Z_{m_u}}\right)\hat{m}_{ud}$$

S. Aoki *et al.*
arXiv:9802034 [hep-ph]

$$\boxed{\frac{1}{Z_{ud}}(\overline{\mathrm{MS}},\mu) = \frac{e^2(e_d^2 - e_u^2)}{32\pi^2}\left[6\log(a\mu) - 22.596...\right] Z_P^0}$$

Comparison with other approaches/results

- Other lattice studies of QCD + QED have been /are being performed.

- They are based on the "standard" approach: QED is introduced directly in the Monte Carlo simulation, like QCD.

- Advantages of our approach:
  - The small parameters Δm and e are factorized in the expansion
  - No need to generate new gauge configurations
  - IB is introduced only where needed (no large overall FSE due to e.m.)
  - Specific diagrammatic contributions can be easily isolated.
  E.g. separation between strong and e.m. IB effects

- Disadvantages:
  - More vertices and correlations functions to be computed