On the universality of the third-order phase transition

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joint work with Paolo Facchi[2], Marilena Ligabò[2] & Pierpaolo Vivo[3]

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General framework

$$H = \frac{1}{2}\sum_{i\neq j}^{N} \underbrace{\varphi(|x_i - x_j|)}_{\text{repulsion}} + N\sum_{i=1}^{N} \underbrace{V(x_i)}_{\text{confinement}}, \quad x_i \in \mathbb{R}^d$$

Large-N limit: balance between repulsive interaction and external confinement $\Rightarrow$ the particles concentrate in a bounded region $\Gamma \subset \mathbb{R}^d$

$$F = -\lim_{N\to\infty} \frac{1}{\beta N^2} \log \int_{x_i \in \mathbb{R}^d} e^{-\beta H} \mathrm{d}x$$

Pushed-to-pulled transition: The same particles confined in a ball B_R.
Pulled: for R large; Critical: for $R = R_\star$; Pushed: for R small

$$F(B_R) = -\lim_{N\to\infty} \frac{1}{\beta N^2} \log \int_{x_i \in B_R} e^{-\beta H} \mathrm{d}x \neq F(\mathbb{R}^d)$$

Loss of analyticity of a free energy $\to$ occurrence of a phase transition.

Random Matrix Theory

GUE Random Hermitian matrix $M_{ij} = \overline{M_{ji}}$ with Gaussian entries;

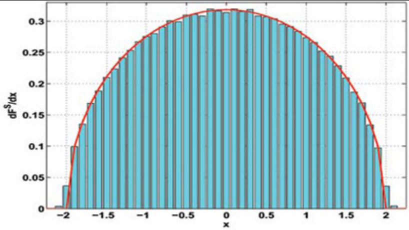
GinUE Complex matrix with independent Gaussian entries $M_{ij} \overset{i.i.d.}{\sim} N(0,1)$.

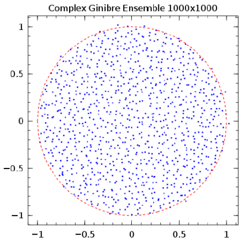
Eigenvalues: $P(x_1, \ldots, x_N) = \frac{1}{\mathcal{Z}_N} e^{-H}$ $\quad \begin{cases} x_i \in \mathbb{R} & \text{if} \quad M \in \text{GUE}, \\ x_i \in \mathbb{R}^2 & \text{if} \quad M \in \text{GinUE}, \end{cases}$

$$H(x_1, \ldots, x_N) = -\sum_{i \neq j} \log|x_i - x_j| + N \sum_k |x_k|^2 \quad \text{(log-gas)}$$

$\varrho_{\text{GUE}}(x) \propto \sqrt{R_\star - x^2}\, 1_{|x| \leq R_\star}$

$\varrho_{\text{GinUE}}(x) \propto 1_{|x| \leq R_\star}$





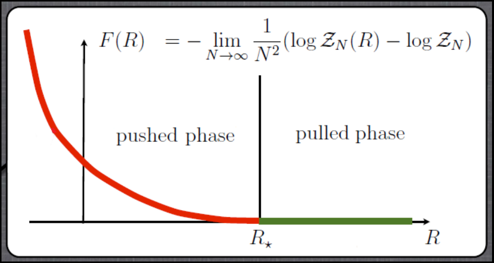
Random Matrix Theory

Concentration: $$\Pr(\max|x_i| \leq R) \to \begin{cases} 0 & \text{if} \quad R \leq R_\star, \\ 1 & \text{if} \quad R > R_\star. \end{cases}$$

Large-N scaling: $\Pr(\max|x_i| \leq R) \approx e^{-N^2 F(R)}$.

The exponential decay is decorated by a ‘large deviation function’ $F(R)$ (the excess free energy of the constrained log-gas)

$$F(R) = -\lim_{N\to\infty} \frac{\log \Pr(\max|x_i| \leq R)}{N^2} = \begin{cases} \psi(R) \geq 0 & \text{if} \quad R \leq R_\star, \\ 0 & \text{if} \quad R > R_\star, \end{cases}$$



Random matrices and third-order phase transitions

$$F_{\mathrm{GUE}}(R) = \begin{cases} \dfrac{1}{16}\left(8R^2 - R^4 - 16\log R - 12 + 8\log 2\right) & \text{if} \quad R \le R_\star, \\ 0 & \text{if} \quad R > R_\star, \end{cases}$$

▷ **D. S. Dean**, **S. N. Majumdar**, Phys. Rev. E 77, 041108 (2008).

$$F_{\mathrm{GinUE}}(R) = \begin{cases} \dfrac{1}{4}(4R^2 - R^4 - 4\log R - 3) & \text{if} \quad R \le R_\star, \\ 0 & \text{if} \quad R > R_\star. \end{cases}$$

▷ **F. D. Cunden**, **F. Mezzadri**, **P. Vivo**, J. Stat. Phys. **164**, 1062-1081 (2016).

The large deviation function is specific of the model ($F_{\mathrm{GUE}}(R) \neq F_{\mathrm{GinUE}}(R)$). Nevertheless, in both models the pulled-to-pushed transition is a third-order phase transition:

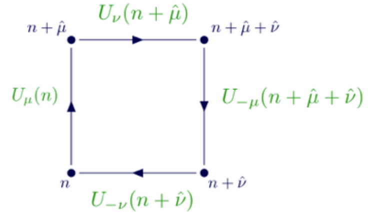
$$F_{\mathrm{GUE}}(R) \sim \frac{\sqrt{2}}{3}(R_\star - R)^3, \quad F_{\mathrm{GinUE}}(R) \sim \frac{4}{3}(R_\star - R)^3, \quad \text{as } R \to R_\star^- .$$

There is a long list of matrix models exhibiting this 3rd-order singularity.

2d Lattice gauge theories

Even the simplest integrals involving the unitary group are non-analytic in the coupling constant:

$$Z(g) = \int_{U(N)} \mathrm{d}U \exp\left\{ \frac{N}{g} \mathrm{Tr}(U + U^{\dagger}) \right\}$$

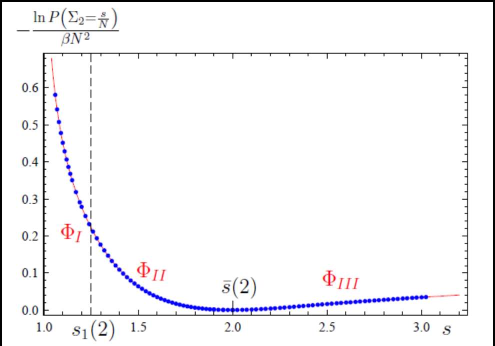


At the critical value g_c the free energy has a jump in the third derivative.

- **D. J. Gross**, **E. Witten**, Phys. Rev. D **21**, 446 (1980).
- **S. R. Wadia**, Phys. Lett. B **93**, 403 (1980).
- **M. R. Douglas**, **V. A. Kazakov**, Phys. Lett. B **319**, 219 (1993).

Entanglement distribution

The distribution of standard measures of bipartite entanglement for a random pure state of large quantum system is non-analytic.
Third-order transition between the ‘typical’ and ‘atypical’ states.



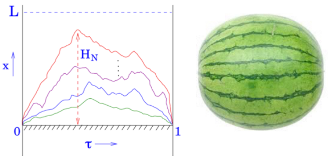
▷ **P. Facchi**, **U.Marzolino**, **G.Parisi**, **S.Pascazio**, **A.Scardicchio**, *Phys.Rev.Lett.* **101**, 050502(2008).
▷ **C. Nadal**, **S. N. Majumdar**, **M. Vergassola**, Phys. Rev. Lett. **104**, 110501 (2010).
▷ **A. De Pasquale**, **P. Facchi**, **G. Parisi**, **S. Pascazio**, **A. Scardicchio**, Phys. Rev. A **81**, 052324 (2010).

Vicious random walks and quantum chaos

Distribution of height in vicious watermelos:

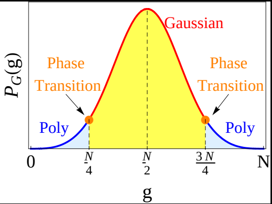
$$-\frac{\log \Pr(H_N \leq R)}{N^2} \xrightarrow{N\to\infty} \begin{cases} \psi(R) \geq 0 & R \leq R_\star, \\ 0 & R > R_\star, \end{cases}$$

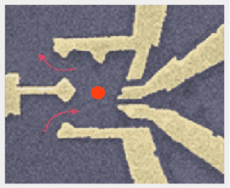
with $\psi(R \to R_\star) \sim \frac{16}{3}(R_\star - R)^3$.



- **G. Schehr**, **S. N. Majumdar**, **A. Comtet**, **J. Randon-Furling**, Phys. Rev. Lett. **101**, 150601 (2008).
- **N. Kobayashi**, **M. Izumi**, **M. Katori**, Phys. Rev. E **78**, 051102 (2008).

Distribution of conductance in chaotic cavities:





- **P. Vivo**, **S. N. Majumdar**, **O. Bohigas**, Phys. Rev. Lett. **101**(21), 216809 (2008);
- **F. D. Cunden**, **P. Facchi**, **P. Vivo**, EPL **110**, 50002 (2015).

Q: Is the 3rd-order phase transition universal?

It is tempting to suspect that *non-universal large deviation functions* of generic statistical models share the same *universal critical exponent* in presence of volume constraints.

$$H = \frac{1}{2}\sum_{i\neq j} \underbrace{\varphi_d(|x_i - x_j|)}_{d\text{-dim Coulomb}} + N\sum_k V(x_k)\,, \qquad (x_i \in \mathbb{R}^d)$$

$$\varphi_d(x) = \begin{cases} -x & \text{if} \quad d = 1, \\ -\log x & \text{if} \quad d = 2, \\ \dfrac{1}{(d-2)}\dfrac{1}{x^{d-2}} & \text{if} \quad d \geq 3. \end{cases}$$

Pr(the particles are confined in B_R) $\approx e^{-\beta N^2 F_d(R)}$:

$$F_d(R) = -\lim_{N\to\infty} \frac{1}{\beta N^2} \log \left(\frac{\displaystyle\int_{|x_1|\leq R} \cdots \int_{|x_N|\leq R} \mathrm{d}^N x\, e^{-\beta H}}{\displaystyle\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathrm{d}^N x\, e^{-\beta H}} \right).$$

Main Result

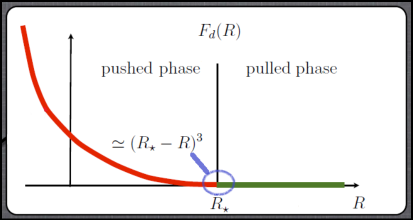
If $V(x) = v(|x|)$, with v smooth and convex, then

(i) the excess free energy is given by the explicit formula:

$$F_d(R) = \frac{1}{2} \int_{\min(R,R_\star)}^{R_\star} \left\{ r^{d-1} v'(r)^2 - \varphi_d'(r) - 2v'(r) \right\} \mathrm{d}r,$$

(ii) the system undergoes a third-order phase transition at $R = R_\star$, i.e.,

$$F_d(R) \simeq (R_\star - R)^3, \quad \text{as } R \to R_\star^- .$$



▷ **F. D. Cunden**, **P. Facchi**, **M. Ligabò**, **P. Vivo**, in preparation.

Take-home message

- Problem: characterisation of the order of a phase transition (without explicitly computing the thermodynamic potential!);
- Third-order phase transitions are ubiquitous in RMT and related models;
- Evidence that the critical exponent 3 is not specific of the log interaction;
- The third-order order transition might be a general phenomenon shared by rather generic systems with unstable interaction (mean field, zero temperature).

For more details see:

- **S. N. Majumdar**, and **G. Schehr**, *Top eigenvalue of a random matrix: large deviations and third order phase transition*, J. Stat. Mech. **P01012** (2014).
- **F. D. Cunden**, **P. Facchi**, and **P. Vivo**, *A shortcut through the Coulomb gas method for spectral linear statistics on random matrices*, J. Phys. A: Math. Theor. **49**, 135202 (2016).

Thank you and Merry Xmas!