Anomalous Diffusion

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Diffusion Equation

White Noise

Langevin Equation:

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = -\dfrac{1}{m}\dfrac{dV(x)}{dx} - \gamma v + \xi(t) \end{cases}$$

$$\langle \xi(t)\xi(s)\rangle = \frac{2 k_B T \gamma}{M}\delta(t-s)$$

Fluctuation Dissipation

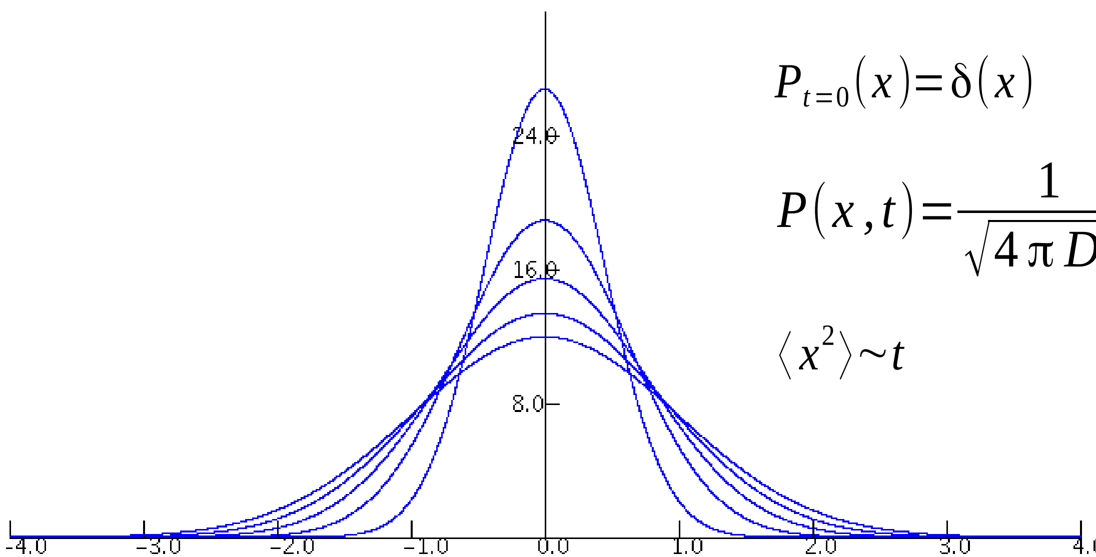
$$\gamma \dot{x} = \xi$$

- *Free*
- *Overdamped*

Fokker-Planck Equation:

$$\frac{\partial P(x,t)}{\partial t} = D_0 \frac{\partial^2 P}{\partial x^2}$$

$$D_0 = \frac{k_B T}{m\gamma}$$



$$P_{t=0}(x) = \delta(x)$$

$$P(x,t) = \frac{1}{\sqrt{4\pi D_0 t}} e^{-\frac{x^2}{4 D_0 t}}$$

$$\langle x^2 \rangle \sim t$$

Central Limit Theorem

X_i i.i.d., $average(X_i)=\mu$, $variance(X_i)=\sigma^2$ $\quad S_n=\sum_{i=1}^{n} X_i \quad S_n \overset{n\to\infty}{\sim} N(n\mu, n\sigma^2)$

$$\varphi(u)=L\{f(t)\}:=\int_0^\infty e^{-ut} f(t)\,dt$$

$$f(t)*g(t):=\int_0^t f(t-\tau)g(\tau)\,d\tau=g(t)*f(t) \quad \longleftrightarrow \quad \begin{array}{l} X_1\sim f \quad X_2\sim g \\ X_1+X_2\sim f*g \\ \varphi_{f*g}(u)=\varphi_f(u)\varphi_g(u) \end{array}$$

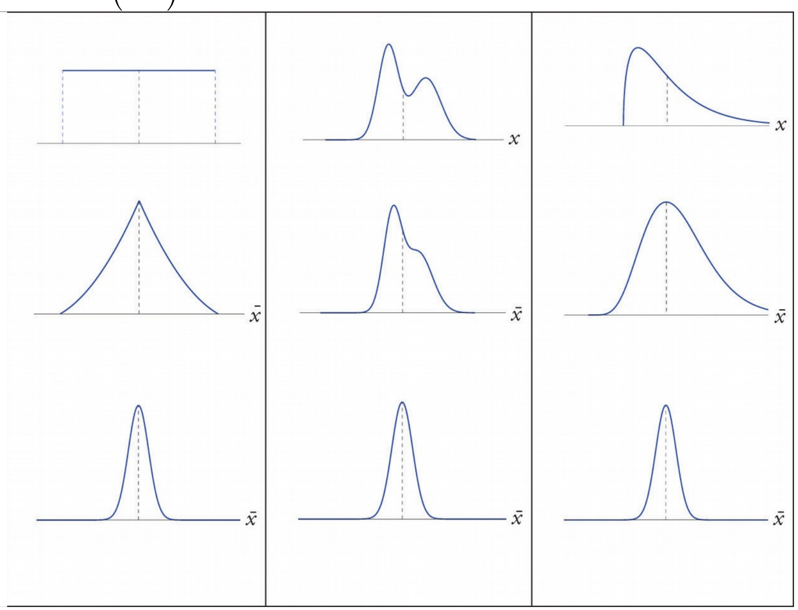
$$N(a_1^2,b_1)*N(a_2^2,b_2)=N(a_1^2+a_2^2,b_1+b_2)$$

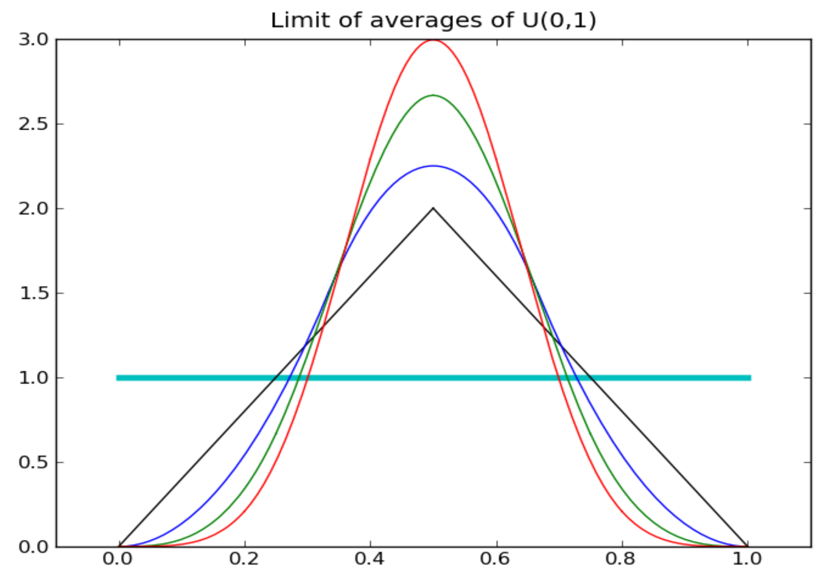
$P(X)^{*n}$

n=1

n=5

n=30





Fractional Brownian Motion

Continuous Time Gaussian Process: $\begin{cases} E[B_H(t)]=0 \\ E[B_H(t)B_H(s)]=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}) \end{cases}$

$H>1/2$: *Increments positively correlated*

$H\in(0,1)$ $H=1/2$: *Brownian Motion*

Hurst Index

$H<1/2$: *Increments negatively correlated*

The following integral satisfies the definition

$$B_H(t)=\frac{1}{C(H)}\int_{-\infty}^{0}[(t-s)^{H-1/2}-(-s)^{H-1/2}]dB(s)+\int_{0}^{t}[(t-s)^{H-1/2}]dB(s)$$

with $B(t)$ *Brownian Motion*

$$B_H(t)=\frac{\Gamma(H+1/2)}{C(H)}(I^{H-1/2})B(t)$$

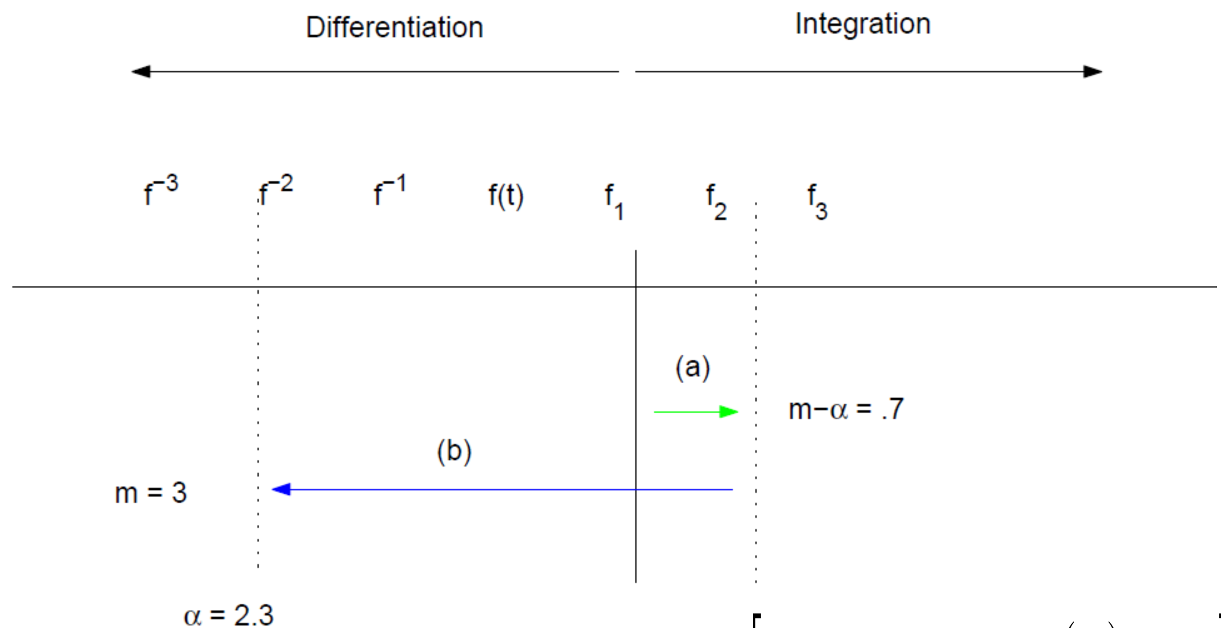
I : *differintegral operator*

B.B.Mandelbrot and J. Van Ness, SIAM Rev. ***10****, 422-437 (1968)*

Fractional Integral & Fractional Derivative (1)

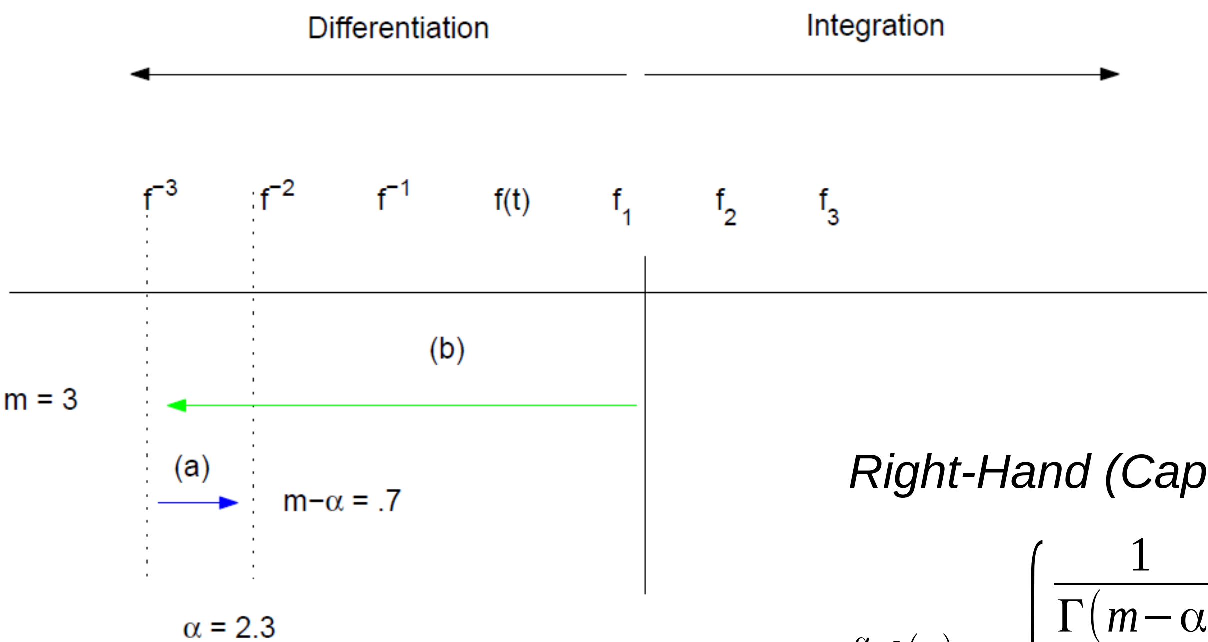
$$\int_a^x d\sigma_1 \int_a^{\sigma_1} d\sigma_2 \ldots \int_a^{\sigma_{n-1}} d\sigma_n f(\sigma_n) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau$$

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$



Left-Hand fractional Derivative: $D^\alpha f(t) := \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}}\right], m-1 < \alpha < m \\ \dfrac{d^m}{dt^m} f(t), \alpha = m \end{cases}$

Fractional Integral & Fractional Derivative (2)



Right-Hand (Caputo) fractional Derivative:

$$D_*^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1<\alpha<m \\ \dfrac{d^m}{dt^m} f(t), & \alpha=m \end{cases}$$

- *If C is a constant* $D_*^{\alpha} C = 0 \quad D^{\alpha} C \neq 0$
- *For RH Fractional Differential Equation we don't need fractional order initial conditions*

Mittag-Leffler function

$$E_{\alpha}(-\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k+1)}$$

Colored noise

$$M\ddot{x}(t)+M\int_0^t \gamma(t-\tau)\dot{x}(\tau)\,d\tau+U'(x)=\xi$$ *Langevin equation with friction kernel*

$$\langle \xi(t)\xi(s)\rangle = K(t-s)$$ *Colored noise*

$$M k_B T\,\gamma(t-s)=K(t-s)$$ *Generalized Fluctuation-Dissipation*

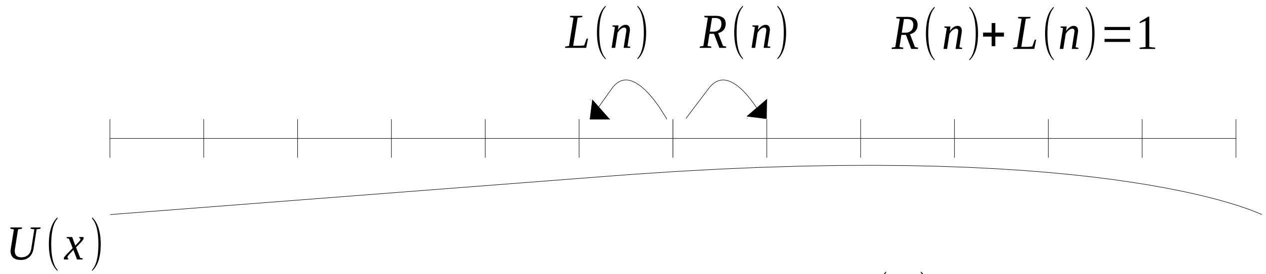
$$M\ddot{x}(t)+M\gamma_\alpha \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\dot{x}(t)+U'(x)=\xi$$ *Fractional form*

$$K(t-s)\sim|t-s|^{-\alpha} \xrightarrow[\alpha\neq 1]{0\leqslant\alpha\leqslant 2} C_v(t)=E_{2-\alpha}(-const\,t^{2-\alpha}) \longrightarrow MSD=\langle x^2\rangle\sim t^\alpha$$

$$\begin{cases} \dfrac{\partial P_{fBM}(x,t)}{\partial t}=\alpha D_0 t^{\alpha-1}\dfrac{\partial^2 P_{fBM}(x,t)}{\partial x^2} \\ P_{t=0}(x)=\delta(x) \end{cases} \longrightarrow \boxed{P(x,t)=\frac{1}{\sqrt{4\pi D_0 t^\alpha}}e^{-\frac{x^2}{4D_0 t^\alpha}}}$$

*E. Lutz, Phys. Rev. E, **64**, 051106 (2001)*

Continuum Limit of CTRWs (1)



$$R(n)+L(n)=1$$

$$\psi(\tau)=\textit{Waiting time distribution}$$

$$p_i(n)=\textit{Probability}\text{ to }\textit{be}\text{ in }\textit{site } n \textit{ after } i \textit{ jumps}$$

$$p_{i+1}(n)=R(n-1)\,p_i(n-1)+L(n+1)\,p_i(n+1)$$

$$P(n,t)=\sum_{i=0}^{\infty} p_i(n)\,Q_i(t)$$

$$Q_i(t)=\textit{Probability of } i \textit{ jumps until time } t$$

$$p_i(n) \longleftrightarrow p_i(x) \qquad \textit{Continuum limit}$$

Taylor Expansion

$$R(x-a)\,p_i(x-a)=R(x)\,p_i(x)+\frac{\partial}{\partial x}\left[R(x)\,p_i(x)\right]+\frac{\partial^2}{\partial x^2}\left[R(x)\,p_i(x)\right]\frac{a^2}{2}+ \quad \ldots\ldots\ldots\ldots$$

Detailed Balance

$$R(x)-L(x)\simeq\frac{a\,F(x)}{2k_B T} \qquad R(x)\simeq L(x)\simeq\frac{1}{2}: F(x)\,\textit{slowly varying}$$

*E. Barkai, R. Metzler and J. Klafter, Phys. Rev. E, **61**, 132 (2000)*

Continuum Limit of CTRWs (2)

If the first moment of $\psi(\tau)$ exists

$$\frac{p_{i+1}(x)-p_i(x)}{\langle\tau\rangle}=\frac{a^2}{2\langle\tau\rangle}\left[\frac{\partial^2}{\partial x^2}p_i(x)-\frac{\partial}{\partial x}\left(\frac{F(x)}{k_B T}p_i(x)\right)\right] \longrightarrow \textit{Normal Fokker-Planck}$$

$$\frac{\partial p(x,t)}{\partial t} \qquad K_1=\lim_{\substack{\langle\tau\rangle\to 0\\ a^2\to 0}}\frac{a^2}{2\langle\tau\rangle}=D_0$$

What about $\psi(\tau)\overset{\tau\, large}{\sim}\frac{\alpha A_\alpha}{\Gamma(1-\alpha)\tau^{1+\alpha}}$ $\qquad \alpha<1 \longrightarrow \langle\tau\rangle$ *diverges*

$$\hat{\psi}(u)=1-A_\alpha u^\alpha+c_1(A_\alpha u^\alpha)^2+\ \ldots\ldots\ldots\ldots \longrightarrow u^\alpha\hat{P}(x,u)=K_\alpha L_{fp}\hat{P}(x,u)$$

$$K_\alpha=\lim_{\substack{A_\alpha\to 0\\ a^2\to 0}}\frac{a^2}{2A_\alpha}$$

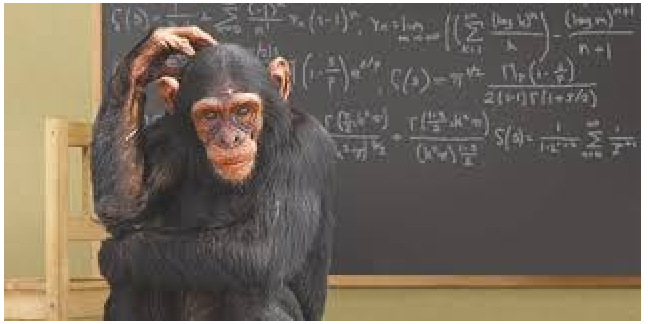
Generalized Diffusion Constant $\qquad [K_\alpha]=\left[\frac{m^2}{sec^\alpha}\right]$

Fractional Fokker-Planck Equation

$$D_*^\alpha P(x,t)=K_\alpha\left[\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\left(\frac{F(x)}{k_B T}\right)\right]P(x,t)$$

$MSD=\langle x^2\rangle\sim t^\alpha \qquad \alpha<1 \qquad$ *Subdiffusion*

Thank you & Merry Christmas !



....of the monkey year!

Backup slide 1: Central Limit Theorem (2)

$X_1^{(1)}$ $\qquad n=1,2,\ \ldots\ldots\ldots\ldots$

$X_{1,}^{(2)} X_2^{(2)}$ $\qquad k=1,2,\ \ldots\ldots\ldots\ldots\ ,n$

$X_{1,}^{(3)} X_{2,}^{(3)} X_3^{(3)}$

$\vdots \qquad \ddots$

$X_{1,}^{(n)} X_{2,}^{(n)} X_{3,}^{(n)} \ \ldots\ldots\ldots\ldots \ X_n^{(n)} \longrightarrow S_n = \sum_{k=1}^{n} X_k^{(n)}$

$\vdots \qquad \ddots$

- *In every row* $n \in \mathbb{N}$ *the random variables* $X_{1,}^{(n)} X_{2,}^{(n)} X_{3,}^{(n)} \ \ldots\ldots\ldots\ldots \ X_n^{(n)}$ *are independent*
- *The* $X_k^{(n)}$ *are uniformly, asymptotically negligible, namely* $max_k P\{|X_k^{(n)}| \geqslant \epsilon\} \overset{n \to \infty}{\to} 0, \forall \epsilon > 0$

Backup slide 2:
Central Limit Theorem (3)

$CLP1: X_{1,}^{(n)} X_{2,\ldots}^{(n)}, X_n^{(n)} independent:$

Infinitely divisible laws

$$X \sim \sum_{k=1}^{n} X_k^{(n)} \qquad \varphi(u) = (\varphi_n(u))^n$$

$CLP2: X_{1,}^{(n)} X_{2,\ldots}^{(n)}, X_n^{(n)} independent+$

$\exists a_n, b_n \text{ and } X_k \; s.t.\; X_k^{(n)} = \frac{1}{a_n}(X_k - \frac{b_n}{n}):$

Self-Decomposable laws

$$X \sim aX' + Y_a \qquad \varphi(u) = \varphi(au)\varphi_a(u)$$

$$a \in (0,1) \qquad X' \sim X$$

$CLP3: X_{1,}^{(n)} X_{2,\ldots}^{(n)}, X_n^{(n)} i.i.d.:$

Stable laws

$$c_1 X_1 + c_2 X_2 \sim aX + b$$

$$e^{ibu}\varphi(au) = \varphi(c_1 u)\varphi_a(c_2 u)$$

Symmetric stable laws: $\varphi(u) = e^{-b|u|^{\alpha}}$

Backup slide 3:
Central Limit Theorem (4)

Example of construction of a triangular array

$$P(\lambda)=\frac{\lambda^k e^{-\lambda}}{k!} \quad \textit{Poisson law} \longrightarrow \varphi_P(u)=e^{\lambda(e^{iu}-1)}$$

$$B(k;N,p)=\binom{N}{k} p^k (1-p^k)^{N-k} \quad \textit{Binomial law} \longrightarrow \varphi_B(u)=(1-p+pe^{it})^N$$

$$\textit{Triangular array}: \; X_k^{(n)} \sim B(N=1, p=\frac{\lambda}{n})$$

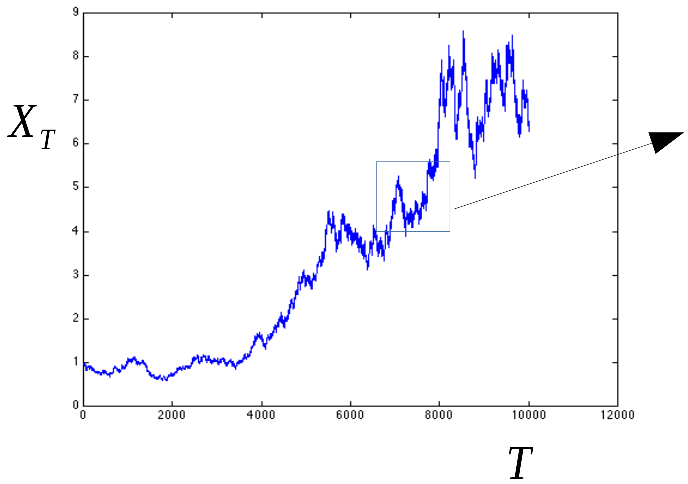
$$S_n = X_1^{(n)} + X_2^{(n)} + \dots\dots\dots + X_n^{(n)} \sim B(n, \frac{\lambda}{n}) \xrightarrow{n \to \infty} P(\lambda)$$

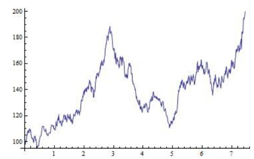
$$X_1^{(1)} \sim B(1,\lambda) \longrightarrow \varphi(u)=(1-\lambda+\lambda e^{it}) \xrightarrow[\textit{rescaling}]{\textit{centering}} \varphi(u)=e^{ibt}(1-\lambda+\lambda e^{iat})$$

No a and *b make the job*

$$X_1^{(2)} \sim B\left(1,\frac{\lambda}{2}\right) \longrightarrow \varphi(u)=\left(1-\frac{\lambda}{2}+\frac{\lambda}{2} e^{it}\right)$$

Backup slide 4: Stationary and Self Similar Process (4)





Process defined through the law of its increments

Levy process:

- $X_{t=0}=0$
- *Independent and Stationary Increments*
- *Continuity in probability*

$$X_t=\sum_{k=1}^{n} X_{\frac{kt}{n}}-X_{(k-1)\frac{t}{n}}$$

ϕ : *law of the increment*

φ : *law of* X_T

Stationary increment: $\phi_{s,t}=\phi_{\tau=t-s}$

Stationary process from an infinitely divisible law: $\phi_t(u)=(\varphi(u))^{t/T}$ *iff* φ *is infinitely divisible*

Self-similar process: $\phi_{as,at}(u)=\phi_{s,t}(bu)$

Self-similar process from a self-decomposable law:

$$\phi_{s,t}(u)=\frac{\varphi\left(\left(\frac{t}{T}\right)^H u\right)}{\varphi\left(\left(\frac{s}{T}\right)^H u\right)}$$

iff φ *is self-decomposable*

$b=a^H$

N. C. Petroni, Phys. A ***387*** *(8-9), 1875-1894 (2008)*