Spectral Methods in Causal Dynamical Triangulations
a Numerical Approach to Quantum Gravity

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Quantum Gravity: open problem in theoretical physics:

Manifest difficulties:

- Standard perturbation theory fails to renormalize GR: dimensionful parameters in the Einstein action $\frac{1}{G}$ and $\frac{\Lambda}{G}$ give rise to divergences from the High-Energy (short-scales) sector.
- Gravitational quantum effects unreachable by experiments:
  $$E_{Pl} = \sqrt{\frac{\hbar c}{G}} c^2 \sim 10^{19}\text{GeV}$$ (big bang or black holes)

Two lines of direction in QG approaches

- non-conservative: introduce new short-scale physics “by hand”
- conservative: do not give up on the Einstein theory

Causal Dynamical Triangulations (CDT): conservative approach of non-perturbative renormalization of the Einstein gravity, based on Monte-Carlo simulations.
Lattice regularization

A *regularization* makes the renormalization procedure well posed.

- discretize spacetime introducing a minimal **lattice spacing** ‘a’
- localize dynamical variables on lattice sites
- study how quantities diverge for $a \to 0$
- Cartesian grids approximate Minkowski space
- **Regge triangulations** approximate generic manifolds
Configuration space of CDT

A Lorentzian (causal) structure on $\mathcal{T}$ can be enforced by using a *foliation* of spatial **slices** of constant proper time

- Vertices “live” in slices.
- $d$-simplexes fill spacetime between slices.
- Links can be spacelike with $\Delta s^2 = a^2$, or timelike with $\Delta s^2 = -\alpha a^2$.
- Only a finite number of simplex types.
- The $\alpha$ parameter is used later to perform a Wick-rotation from Lorentzian to Euclidean
Regge formalism: action discretization

Also the EH action must be discretized accordingly ($g_{\mu\nu} \rightarrow T$):

\[
S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G} \left[ \int d^d x \sqrt{|g|} R - 2\Lambda \int d^d x \sqrt{|g|} \right] \]

Total curvature

\[
\downarrow \quad \text{discretization} \quad \downarrow
\]

\[
S_{\text{Regge}}[T] = \frac{1}{16\pi G} \left[ \sum_{\sigma^{(d-2)} \in T} 2\varepsilon_{\sigma^{(d-2)}} V_{\sigma^{(d-2)}} - 2\Lambda \sum_{\sigma^{(d)} \in T} V_{\sigma^{(d)}} \right],
\]

where $V_{\sigma^{(k)}}$ is the $k$-volume of the simplex $\sigma^{(k)}$.

Wick-rotation $iS_{\text{Lor}}(\alpha) \rightarrow -S_{\text{Euc}}(-\alpha)$

\[\Rightarrow \quad \text{Monte-Carlo sampling} \; \mathcal{P}[T] \equiv \frac{1}{Z} \exp (-S_{\text{Euc}}[T])\]
Wick-rotated action in 4D

At the end of the day [Ambjörn et al., arXiv:1203.3591]:

\[ S_{\text{CDT}} = -k_0 N_0 + k_4 N_4 + \Delta(N_4 + N_4^{(4,1)} - 6N_0) \]

- New parameters: \((k_0, k_4, \Delta)\), related respectively to \(G\), \(\Lambda\) and \(\alpha\).
- New variables: \(N_0\), \(N_4\) and \(N_4^{(4,1)}\), counting the total numbers of vertices, pentachorons and type-(4, 1)/(1, 4) pentachorons respectively (\(T\) dependence omitted).

It is convenient to “fix” the total spacetime volume \(N_4 = V\) by fine-tuning \(k_4 \implies \) actually free parameters \((k_0, \Delta, V)\).

Simulations at different volumes \(V\) allow finite-size scaling analysis.
For simulations at fixed volumes $V$ the phase diagram of CDT is 2-dimensional, parametrized by $(k_0, \Delta)$.

**Ultimate goal of CDT**

Find in the phase diagram of CDT a second order critical point with diverging correlation length

$\Rightarrow$ **continuum limit**
Main CDT result

The average of profiles in $C_{dS}$ phase fits well with a de Sitter cosmological model!

($S^4$ in Euclidean space)
Problem: lack of geometric observables

Observables currently employed in CDT

• Spatial volume per slice: $V_s(t)$
  (number of spatial tetrahedra at the slice labeled by $t$)

• Order parameters for transitions:
  • $\text{conj}(k_0) = N_0/N_4$ for the $A|C_{dS}$ transition
  • $\text{conj}(\Delta) = (N_4^{(4,1)} - 6N_0)/N_4$ for the $B|C_b$ transition
  • $\text{OP}_2$ for the $C_b|C_{dS}$ transition
    [Ambjorn et al. arXiv:1704.04373]

• Fractal dimensions:
  • spectral dimension
  • Hausdorff dimension

No observable characterizing geometries at all lattice scales!!
Proposed solution: spectral analysis

Analysis of eigenvalues and eigenvector of the **Laplace-Beltrami operator**: \(-\nabla^2\)

- Spectral analysis on smooth manifolds \((\mathcal{M}, g_{\mu\nu})\):

\[-\nabla^2 f \equiv -\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu f) = \lambda f, \text{ with boundary conditions}\]

Can one hear the shape of a drum?

Example: disk drum
Spectral graph analysis on CDT spatial slices

Observation
One can define graphs associated to spatial slices of triangulations.

• spatial tetrahedra become vertices of associated graph
• adjacency relations between tetrahedra become edges
• The Laplace matrix can be defined on the graph associated to spatial slices as described previously
• Eigenvalue problem $L\vec{f} = \lambda \vec{f}$ solved by numerical routines

2D slice and its dual graph
Laplacian embedding: embedding of graph in $k$-dimensional (Euclidean) space, solution to the optimization problem:

$$\min_{\vec{f}^1,\ldots,\vec{f}^k} \left\{ \sum_{(v,w)\in E} \sum_{s=1}^k [f^s(v) - f^s(w)]^2 \mid \vec{f}^s \cdot \vec{f}^p = \delta_{s,p}, \vec{f}^s \cdot \vec{1} = 0 \forall s, p = 1,\ldots,k \right\},$$

where for each vertex $v \in V$ the value $f^s(v)$ is its $s$-th coordinate in the embedding.

The solution $\{f^s(v)\}_{s=1}^k$ is exactly the (orthonormal) set of the first $k$ eigenvectors of the Laplace matrix $\{e_s(v)\}_{s=1}^k$!
Laplacian embedding: example torus $T^2 = S^1 \times S^1$

For each graph-vertex $v \in V$ plot the tuple of coordinates:

2D: $(e_1(v), e_2(v)) \in \mathbb{R}^2$

3D: $(e_1(v), e_2(v), e_3(v)) \in \mathbb{R}^3$

(a) 2D embedding

(b) 3D projected embedding
The first three eigenstates are not enough to probe the geometry of substructures.
Result: spectral clustering of $C_{dS}$ spatial slices

**Spectral clustering**: recursive application of min-cut procedure

![Spectral clustering diagram](image)

**Observation**: fractality

Self-similar filamentous structures in $C_{dS}$ phase ($S^3$ topology)
Other evidences of fractality: spectral dimension $D_S$

Computed from the return probability for random-walks on manifold or graph: $P_r(\tau) \propto \tau^{-\frac{D_S}{2}} \implies D_S(\tau) \equiv -2\frac{d\log P_r(\tau)}{d\log \tau}$.

- Usual integer value on regular spaces: e.g. $D_S(\tau) = d$ on $\mathbb{R}^d$
- $\tau$-independent fractional value on true fractals
- $\tau$-dependent fractional value on multi-fractals (not self-similar)

Equivalent definition of return probability: $P_r = \frac{1}{|V|} \sum_k e^{-\lambda_n t}$

$\implies$ Nice interpretation of return probability in terms of diffusion processes (random-walks): smaller eigenvalues $\leftrightarrow$ slower modes. The smallest non-zero eigenvalue $\lambda_1$ represents the **algebraic connectivity** of the graph.
The spectral dimension on $C_{dS}$ slices

Compare $P_r$ obtained by explicit diffusion processes or by the LB eigenspectrum

fractional value $D_S(\tau) \simeq 1.6 \implies$ fractal distribution of space.

A spectral analysis of the full spacetime is required.
Comparing spectral gap $\lambda_1$ of $C_{dS}$ and $B$ phases

Histogram of eigenspectra for $C$ phase slices

**Observation**

Unlike $C_{dS}$ phase, $B$ phase has high spectral gap $\rightarrow$ high connectivity (spectral dimension shows multi-fractal behaviour).

$\rightarrow \lambda_1$ could be used as an alternative order parameter of the $B|C$ transition.
Many other results have been obtained by spectral analyzing CDT slices (a paper will soon pop up, so stay tuned!)

Future work

- Implement the spectral analysis of the full spacetime triangulations (not merely spatial slices) ➞ more involved coding based on Finite Element Methods.
- Apply spectral methods to perform Fourier analysis of any local function, like scalar curvature or matter fields living on triangulations simplexes.
- Analyze phase transitions in CDT using spectral observables instead of the ones currently employed.

Expectations
Provide CDT of more meaningful observables to characterize geometries of full spacetimes, especially giving a definition of correlation length ➞ powerful tool for continuum limit analysis!
Thank you for the attention!
Additional slides
Regge formalism: curvature for equilateral triangles (2D)
Monte-Carlo method: sum over causal geometries

Configuration space in CDT: triangulations with causal structure

Lorentzian (causal) structure on $\mathcal{T}$ enforced by means of a foliation of spatial slices of constant proper time.

Path-integral over causal geometries/triangulations $\mathcal{T}$ using Monte-Carlo sampling by performing local updates. E.g., in 2D:

- flipping timelike link
- creating/removing vertex
Continuum limit

The system must forget the lattice discreteness: second-order critical point with divergent correlation length \( \hat{\xi} \equiv \xi / a \rightarrow \infty \)

Asymptotic freedom (e.g. QCD):

\[
\vec{g}_c \equiv \lim_{a \rightarrow 0} \vec{g}(a) = \vec{0}
\]

Asymptotic safety (maybe QG):

\[
\vec{g}_c \equiv \lim_{a \rightarrow 0} \vec{g}(a) \neq \vec{0}
\]
$C_{dS}$: de Sitter phase

- Time-extended distribution of the triangulation/Universe (blob)
- Average of blob profiles over configurations has the same distribution of the **de Sitter cosmological model**: the best description of the physical Universe dominated by dark energy!
- Fluctuations of spatial volume interpreted as quantum effects

Lorentzian: \[- x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\]

\[\Downarrow\]

**analytic continuation** \[\Downarrow\]

Euclidean: \[+ x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\]

De Sitter spatial volume distribution

\[
V_s^{(dS)}(t) = \frac{V_{tot}}{2} \frac{3}{4} \frac{1}{\tilde{s}(V_{tot})^{\frac{1}{4}}} \cos^3 \left( \frac{t}{\tilde{s}(V_{tot})^{\frac{1}{4}}} \right)
\]
Dimensional reduction in CDT

Spectral dimension as diffusion process on the full spacetime:

Dimensional reduction from 4-dimensions at large scales to 2-dimensions at shorter ones, observed in many QG approaches. ['t Hooft, arXiv:gr-qc/9310026; Carlip, arXiv:1605.05694]
Standard definitions of order parameters in CDT

Recall 4D action: \( S = -k_0 N_0 + k_4 N_4 + \Delta (N_4 + N_4^{(4,1)} - 6N_0) \)

- **AC\(_dS\) transition**: \( \text{conj}(k_0) \equiv \frac{N_0}{N_4} \)
- **BC\(_b\) transition**: \( \text{conj}(\Delta) \equiv \frac{N_4^{(4,1)} - 6N_0}{N_4} \)
- **C\(_b\)C\(_dS\) transition**:

\[
\text{OP}_2 = \frac{1}{2} \left[ \left| O_{\text{max}}(t_0) - O_{\text{max}}(t_0 + 1) \right| + \left| O_{\text{max}}(t_0) - O_{\text{max}}(t_0 - 1) \right| \right],
\]

where \( O_{\text{max}}(t) \) is the highest coordination number for vertices in the slice \( t \), and \( t_0 \) is the slice label maximizing \( O_{\text{max}} \) amongst slices, that is \( O_{\text{max}}(t_0) = \max_t O_{\text{max}}(t) \).
Spectral graph analysis

**Graph**: tuple $G = (V, E)$ where

- $V$ set of **vertices** $\nu$
- $E$ set of **edges**, unordered pairs of adjacent vertices $e = (\nu_1, \nu_2)$

Laplace matrix acting on functions of vertices $\vec{f} = (f(\nu)) \in \mathbb{R}^{|V|}$:

$$L = D - A$$

- $D_{\nu, \nu} = \text{“order of the vertex } \nu \text{ (number of departing edges)”}$
- $A_{\nu_1, \nu_2} = 1$ if $(\nu_1, \nu_2) \in E$, zero otherwise
Interpretations of the first eigenvalue and eigenvector

Fiedler value and vector
First (non-null) eigenvalue $\lambda_1$ and associated eigenvector $e_1$. The Fiedler value, or **spectral gap**, $\lambda_1$ measures the connectivity of the graph: the larger, the more connections between vertices.

Applications of the Fiedler vector $e_1$:

- **Min-cut**: minimal set of edges disconnecting the graph if cut
- **Fiedler ordering on regular graphs (like CDT slices)**: core of the Google Search engine, and paramount reason for the Google’s rise to success.
- many others...
3D Laplacian embedding of $T^3$ torus

\[ T^3 \cong T^2 \times S^1 \cong S^1 \times S^1 \times S^1 \]