Spectral Methods in Causal Dynamical Triangulations

a Numerical Approach to Quantum Gravity

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Quantum Gravity: open problem in theoretical physics:

Manifest difficulties:

- Standard perturbation theory fails to renormalize GR: dimensionful parameters in the Einstein action $\frac{1}{G}$ and $\frac{\Lambda}{G}$ give rise to divergences from the High-Energy (short-scales) sector.
- Gravitational quantum effects unreachable by experiments: $E_{Pl} = \sqrt{\frac{\hbar c}{G}} c^2 \simeq 10^{19} GeV$ (big bang or black holes)

Two lines of direction in QG approaches

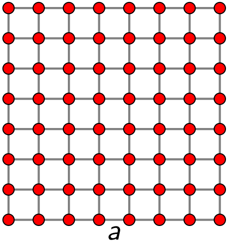
- non-conservative: introduce new short-scale physics "by hand"
- conservative: do not give up on the Einstein theory

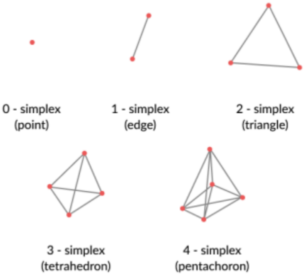
Causal Dynamical Triangulations (CDT): conservative approach of non-perturbative renormalization of the Einstein gravity, based on Monte-Carlo simulations.

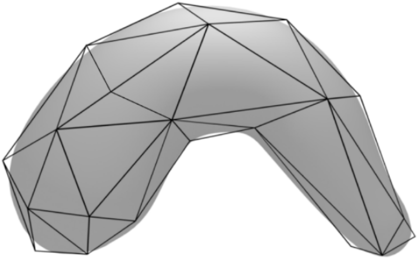
Lattice regularization

A *regularization* makes the renormalization procedure well posed.

- discretize spacetime introducing a minimal **lattice spacing** ‘a’
- localize dynamical variables on lattice sites
- study how quantities diverge for $a \to 0$
- Cartesian grids approximate Minkowski space
- **Regge triangulations** approximate generic manifolds



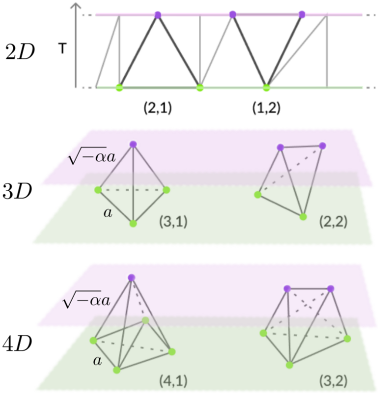




Configuration space of CDT

A Lorentzian (causal) structure on $\mathcal{T}$ can be enforced by using a *foliation* of spatial **slices** of constant proper time

- Vertices "live" in slices.
- d-simplexes fill spacetime between slices.
- Links can be spacelike with $\Delta s^2 = a^2$, or timelike with $\Delta s^2 = -\alpha a^2$.
- Only a finite number of simplex types.
- The α parameter is used later to perform a Wick-rotation from Lorentzian to Euclidean



Regge formalism: action discretization

Also the EH action must be discretized accordingly ($g_{\mu\nu} \to \mathcal{T}$):

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\left[\underbrace{\int d^d x\, \sqrt{|g|}\, R}_{\text{Total curvature}} - 2\Lambda \underbrace{\int d^d x\, \sqrt{|g|}}_{\text{Total volume}}\right]$$

$\Downarrow$ discretization $\Downarrow$

$$S_{Regge}[\mathcal{T}] = \frac{1}{16\pi G}\left[\sum_{\sigma^{(d-2)}\in\mathcal{T}} 2\varepsilon_{\sigma^{(d-2)}} V_{\sigma^{(d-2)}} - 2\Lambda \sum_{\sigma^{(d)}\in\mathcal{T}} V_{\sigma^{(d)}}\right],$$

where $V_{\sigma^{(k)}}$ is the k-volume of the simplex $\sigma^{(k)}$.

Wick-rotation $iS_{Lor}(\alpha) \to -S_{Euc}(-\alpha)$

$\Longrightarrow$ Monte-Carlo sampling $\mathcal{P}[\mathcal{T}] \equiv \frac{1}{Z}\exp\left(-S_{Euc}[\mathcal{T}]\right)$

Wick-rotated action in 4D

At the end of the day [Ambjörn et al., arXiv:1203.3591]:

$$S_{CDT} = -k_0 N_0 + k_4 N_4 + \Delta(N_4 + N_4^{(4,1)} - 6N_0)$$

- New parameters: (k_0, k_4, Δ), related respectively to G, Λ and α.
- New variables: N_0, N_4 and $N_4^{(4,1)}$, counting the total numbers of vertices, pentachorons and type-$(4,1)/(1,4)$ pentachorons respectively ($\mathcal{T}$ dependence omitted).

It is convenient to "fix" the total spacetime volume $N_4 = V$ by fine-tuning $k_4 \implies$ actually free parameters (k_0, Δ, V).

Simulations at different volumes V allow finite-size scaling analysis.

For simulations at fixed volumes V the phase diagram of CDT is 2-dimensional, parametrized by (k_0, Δ).

Ultimate goal of CDT

Find in the phase diagram of CDT a second order critical point with diverging correlation length

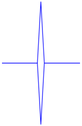
$\Longrightarrow$ **continuum limit**

Phase diagram of CDT in 4D

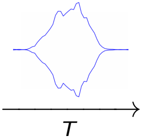
phase	spatial volume per slice

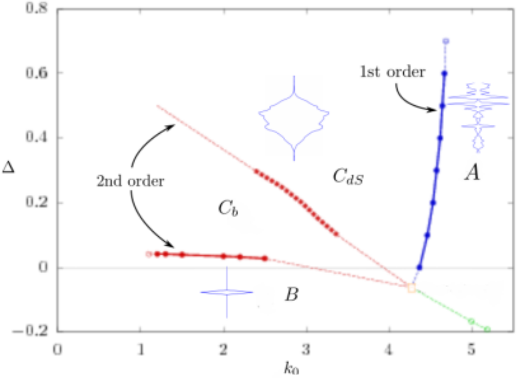
A:

B:



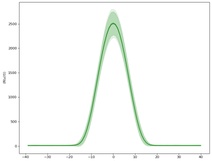
C_{dS}/C_b:





Main CDT result

The average of profiles in C_{dS} phase fits well with a de Sitter cosmological model!
(S^4 in Euclidean space)



Problem: lack of geometric observables

Observables currently employed in CDT

- Spatial volume per slice: $V_s(t)$
  (number of spatial tetrahedra at the slice labeled by t)
- Order parameters for transitions:
  - $\text{conj}(k_0) = N_0/N_4$ for the $A|C_{dS}$ transition
  - $\text{conj}(\Delta) = (N_4^{(4,1)} - 6N_0)/N_4$ for the $B|C_b$ transition
  - OP_2 for the $C_b|C_{dS}$ transition
    [Ambjorn et al. arXiv:1704.04373]
- Fractal dimensions:
  - spectral dimension
  - Hausdorff dimension

No observable characterizing geometries at all lattice scales!!

Proposed solution: spectral analysis

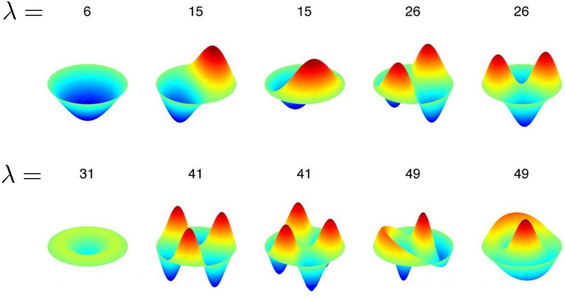
Analysis of eigenvalues and eigenvector of the **Laplace-Beltrami operator**: $-\nabla^2$

- Spectral analysis on smooth manifolds $(\mathcal{M}, g_{\mu\nu})$:

$$-\nabla^2 f \equiv -\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu f) = \lambda f, \text{ with boundary conditions}$$

Can one hear the shape of a drum?

Example:
disk drum

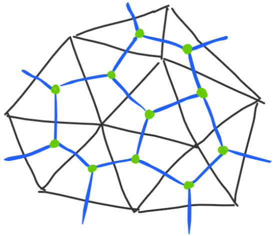


Spectral graph analysis on CDT spatial slices

Observation

One can define graphs associated to spatial slices of triangulations.

- spatial tetrahedra become vertices of associated graph
- adjacency relations between tetrahedra become edges
- The Laplace matrix can be defined on the graph associated to spatial slices as described previously
- Eigenvalue problem $L\vec{f} = \lambda\vec{f}$ solved by numerical routines



2D slice and its dual graph

Laplacian embedding

Laplacian embedding: embedding of graph in k-dimensional (Euclidean) space, solution to the optimization problem:

$$\min_{\vec{f}^1,\ldots,\vec{f}^k} \Big\{ \sum_{(v,w)\in E} \sum_{s=1}^{k} [f^s(v) - f^s(w)]^2 \mid \vec{f}^s \cdot \vec{f}^p = \delta_{s,p},\ \vec{f}^s \cdot \vec{1} = 0\ \forall s, p = 1, \ldots, k \Big\},$$

where for each vertex $v \in V$ the value $f^s(v)$ is its s-th coordinate in the embedding.

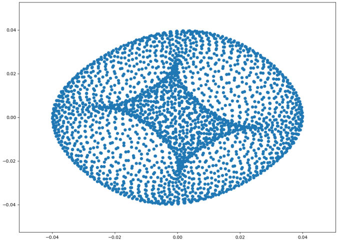
The solution $\{f^s(v)\}_{s=1}^k$ is exactly the (orthonormal) set of the first k eigenvectors of the Laplace matrix $\{e_s(v)\}_{s=1}^k$!

Laplacian embedding: example torus $T^2 = S^1 \times S^1$

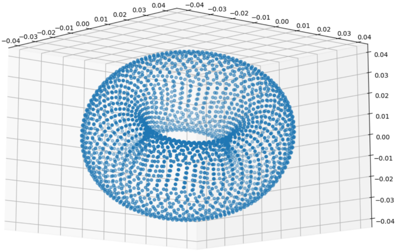
For each graph-vertex $v \in V$ plot the tuple of coordinates:

2D: $(e_1(v), e_2(v)) \in \mathbb{R}^2$

3D: $(e_1(v), e_2(v), e_3(v)) \in \mathbb{R}^3$

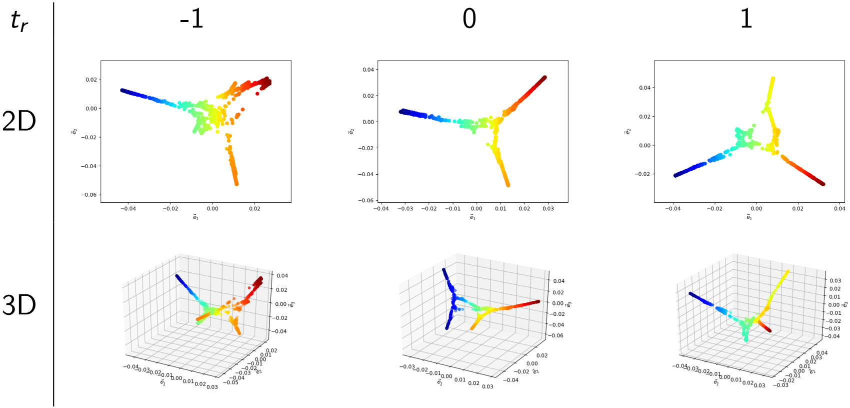


(a) 2D embedding



(b) 3D projected embedding

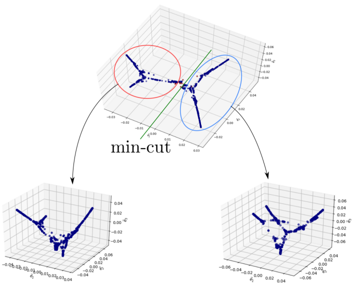
Laplacian embedding of spatial slices in C_{dS} phase



The first three eigenstates are not enough to probe the geometry of substructures

Result: spectral clustering of C_{dS} spatial slices

Spectral clustering: recursive application of min-cut procedure



Qualitative picture (2D)

Observation: fractality

Self-similar filamentous structures in C_{dS} phase (S^3 topology)

Other evidences of fractality: spectral dimension D_S

Computed from the return probability for random-walks on manifold or graph: $P_r(\tau) \propto \tau^{-\frac{D_S}{2}} \implies D_S(\tau) \equiv -2\frac{d \log P_r(\tau)}{d \log \tau}$.

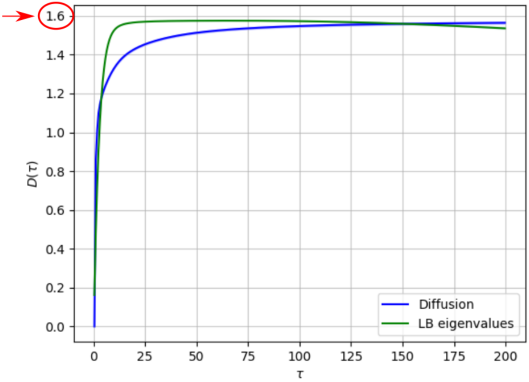
- Usual integer value on regular spaces: e.g. $D_S(\tau) = d$ on $\mathbb{R}^d$
- τ-independent fractional value on true fractals
- τ-dependent fractional value on multi-fractals (not self-similar)

Equivalent definition of return probability: $P_r = \frac{1}{|V|}\sum_k e^{-\lambda_n t}$

$\implies$ Nice interpretation of return probability in terms of diffusion processes (random-walks): smaller eigenvalues $\leftrightarrow$ slower modes. The smallest non-zero eigenvalue λ_1 represents the **algebraic connectivity** of the graph.

The spectral dimension on C_{dS} slices

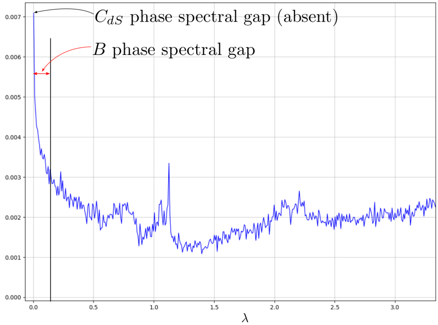
Compare P_r obtained by explicit diffusion processes or by the LB eigenspectrum



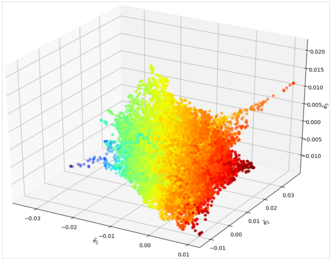
fractional value $D_S(\tau) \simeq 1.6 \implies$ fractal distribution of space.

A spectral analysis of the full spacetime is required.

Comparing spectral gap λ_1 of C_{dS} and B phases



Histogram of eigenspectra for C phase slices



3D embedding of slice in B phase ($V = 40k$, $\lambda_1 \simeq 0.11$)

Observation

Unlike C_{dS} phase, B phase has high spectral gap $\implies$ high connectivity (spectral dimension shows multi-fractal behaviour).

$\implies$ λ_1 could be used as an alternative order parameter of the $B|C$ transition.

Many other results have been obtained by spectral analyzing CDT slices (a paper will soon pop up, so stay tuned!)

Future work

- Implement the spectral analysis of the full spacetime triangulations (not merely spatial slices)
  $\Longrightarrow$ more involved coding based on Finite Element Methods.
- Apply spectral methods to perform Fourier analysis of any local function, like scalar curvature or matter fields living on triangulations simplexes.
- Analyze phase transitions in CDT using spectral observables instead of the ones currently employed.

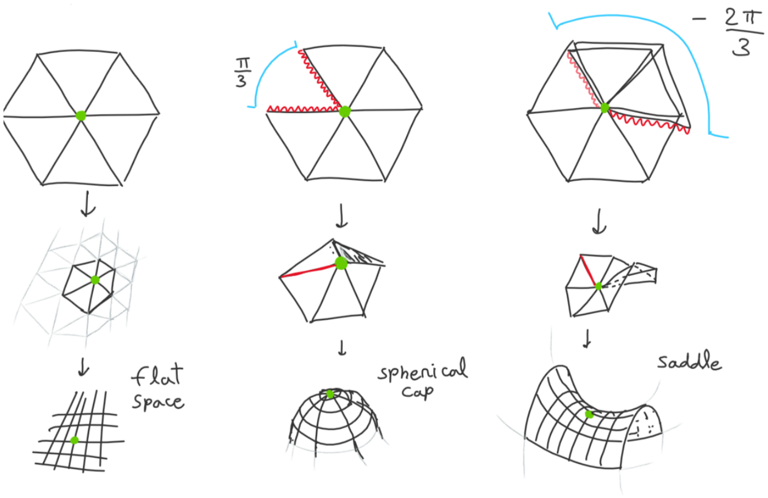
Expectations

Provide CDT of more meaningful observables to characterize geometries of full spacetimes, especially giving a definition of correlation length $\Longrightarrow$ powerful tool for continuum limit analysis!

Thank you for the attention!

Additional slides

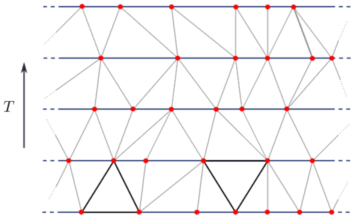
Regge formalism: curvature for equilateral triangles (2D)



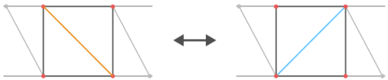
Monte-Carlo method: sum over causal geometries

Configuration space in CDT: triangulations with **causal structure**

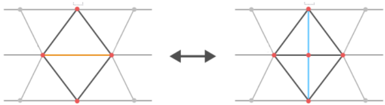
Lorentzian (causal) structure on $\mathcal{T}$ enforced by means of a *foliation* of spatial **slices** of constant proper time.



Path-integral over causal geometries/triangulations $\mathcal{T}$ using Monte-Carlo sampling by performing local updates. E.g., in 2D:



flipping timelike link

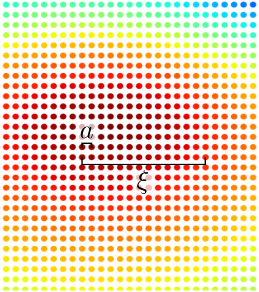


creating/removing vertex

Continuum limit

Continuum limit

The system must forget the lattice discreteness: second-order critical point with divergent correlation length $\hat{\xi} \equiv \xi/a \to \infty$

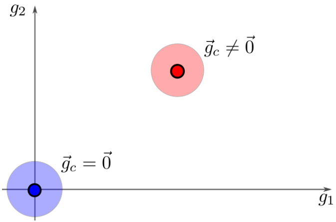


Asymptotic freedom (e.g. QCD):

$$\vec{g}_c \equiv \lim_{a \to 0} \vec{g}(a) = \vec{0}$$

Asymptotic safety (maybe QG):

$$\vec{g}_c \equiv \lim_{a \to 0} \vec{g}(a) \neq \vec{0}$$



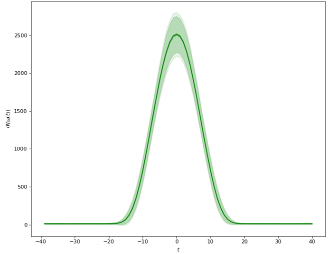
C_{dS}: de Sitter phase

- Time-extended distribution of the triangulation/Universe (blob)
- Average of blob profiles over configurations has the same distribution of the **de Sitter cosmological model**: the best description of the physical Universe dominated by dark energy!
- Fluctuations of spatial volume interpreted as quantum effects

Lorentzian: $-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$

$\Downarrow$ analytic continuation $\Downarrow$

Euclidean: $+x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$

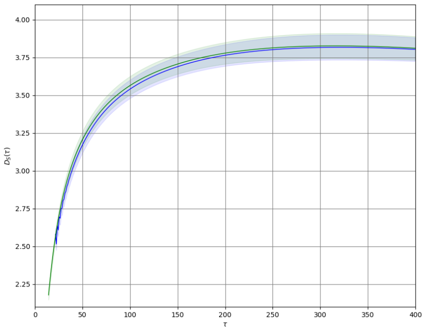


De Sitter spatial volume distribution

$$V_s^{(dS)}(t) = \frac{V_{tot}}{2}\frac{3}{4}\frac{1}{\widetilde{s}(V_{tot})^{\frac{1}{4}}}\cos^3\left(\frac{t}{\widetilde{s}(V_{tot})^{\frac{1}{4}}}\right)$$

Dimensional reduction in CDT

Spectral dimension as diffusion process on the full spacetime:



Dimensional reduction from 4-dimensions at large scales to 2-dimensions at shorter ones, observed in many QG approaches. ['t Hooft, arXiv:gr-qc/9310026; Carlip, arXiv:1605.05694]

Standard definitions of order parameters in CDT

Recall 4D action: $S = -k_0 N_0 + k_4 N_4 + \Delta(N_4 + N_4^{(4,1)} - 6N_0)$

- AC_{dS} transition: $\text{conj}(k_0) \equiv \frac{N_0}{N_4}$
- BC_b transition: $\text{conj}(\Delta) \equiv \frac{N_4^{(4,1)} - 6N_0}{N_4}$
- $C_b C_{dS}$ transition:

$$\text{OP}_2 = \frac{1}{2}\left[\left|O_{max}(t_0) - O_{max}(t_0+1)\right| + \left|O_{max}(t_0) - O_{max}(t_0-1)\right|\right],$$

where $O_{max}(t)$ is the highest coordination number for vertices in the slice t, and t_0 is the slice label maximizing O_{max} amongst slices, that is $O_{max}(t_0) = \max_t O_{max}(t)$.

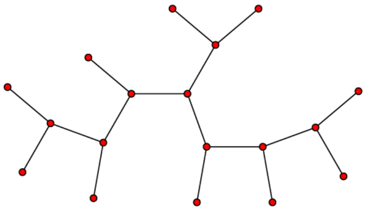
Spectral graph analysis

Graph: tuple $G = (V, E)$ where

- V set of **vertices** v
- E set of **edges**, unordered pairs of adjacent vertices $e = (v_1, v_2)$

Laplace matrix acting on functions of vertices $\vec{f} = (f(v)) \in \mathbb{R}^{|V|}$:

$$L = D - A$$



- $D_{v,v} =$ "order of the vertex v (number of departing edges)"
- $A_{v_1,v_2} = 1$ if $(v_1, v_2) \in E$, zero otherwise

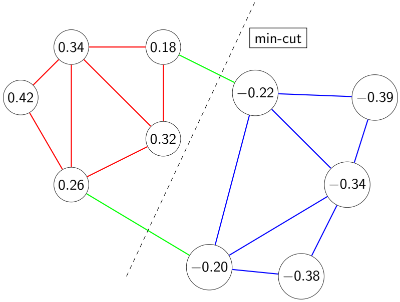
Interpretations of the first eigenvalue and eigenvector

Fiedler value and vector

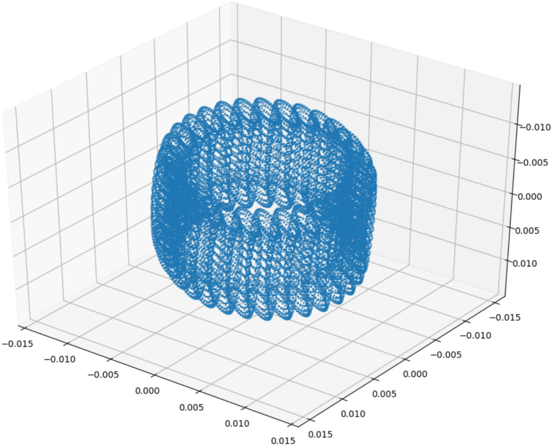
First (non-null) eigenvalue λ_1 and associated eigenvector e_1.
The Fiedler value, or **spectral gap**, λ_1 measures the connectivity of the graph: the larger, the more connections between vertices.

Applications of the Fiedler vector e_1:

- Min-cut: minimal set of edges disconnecting the graph if cut
- Fiedler ordering on regular graphs (like CDT slices): core of the Google Search engine, and paramount reason for the Google's rise to success.
- many others...



3D Laplacian embedding of T^3 torus



$$T^3 \cong T^2 \times S^1 \cong S^1 \times S^1 \times S^1$$