Efimov Spectrum for N bosons: from few- to many-boson systems

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Universal behavior in few-body systems

Scales
- short-range interactions $\rightarrow r_0$
- natural energy scale $E_n \approx \hbar^2 / mr_0^2$
- shallow states $\rightarrow$ large scattering length $a >> r_0$
- unnatural energy scale $E_s \approx \hbar^2 / ma^2$

Limits
This two scales define two limits:
- scaling limit: $r_0 \rightarrow 0$ (Three-body Thomas collapse)
- unitary limit: $a \rightarrow \infty$ (Three-body Efimov states)
- in both cases the ratio $r_0 / a \rightarrow 0$
Two-body scattering: Low energy limit

In the low energy limit: $k \to 0$, the s-wave phase-shift is determined by the effective range expansion:

$$k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} k^2 + \ldots$$

with $r_{\text{eff}}$ the effective range. For shallow states we can extend this low energy expansion to the complex plane $k \to i\kappa$ remembering that $\kappa \cot \delta_0 = i\kappa$ is a pole of the S-matrix

$$\kappa = \frac{1}{a} + \frac{1}{2} r_{\text{eff}} \kappa^2$$

from which

$$\hbar^2 \kappa^2 / m = E_s = \hbar^2 / ma^2 \left(1 + \frac{r_{\text{eff}}}{a} + \ldots\right)$$
Universal behavior in few-body systems

**Examples**

- **The helium dimer:**
  - $E_d = 1.309$ mk
  - $a = 188.78$ a.u.
  - $r_{\text{eff}} = 13.845$ a.u.
  - $E(a, r_{\text{eff}}) = 1.311$ mk

- **The deuteron:**
  - $E_d = 2.225$ MeV
  - $a^1 = 5.419 \pm 0.007$ fm
  - $r_{\text{eff}}^1 = 1.753 \pm 0.008$ fm
  - $E(a, r_{\text{eff}}) = 2.223$ fm

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Gaussian potential model: A Leading Order Description

$$V(r) = V_0 e^{-(r/r_0)^2}$$

with $V_0$ and $r_0$ fixed to describe $a$ and $E_d$
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with \( V_0 \) and \( r_0 \) fixed to describe \( a \) and \( E_d \)
Universal behavior in few-body systems

When a shallow state exists, a Gaussian potential gives a reasonable description of the low energy regime, bound and scattering states: Continuous Scale invariance
Efimov Spectrum: Discrete Scale Invariance

\[ E^n_3 / (\hbar^2 / ma^2) = \tan^2 \xi \]

\[ E^n_3 + \frac{\hbar^2}{ma^2} = e^{-2(n-n_*)\pi/s_0} e^{\Delta(\xi)/s_0} \frac{\hbar^2 \kappa^2}{m} \]

\[ a(0) \quad a(1) \quad \lambda = 1 \]

\[ \text{sgn}(a)(l/a)^{1/4} \]

\[ \text{sgn}(E)(E/E_l)^{1/4} \]

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Efimov Spectrum for N bosons

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The three-boson system

Zero-Range Equations: Efimov spectrum

\[ \frac{E_3^n}{(\hbar^2/ma^2)} = \tan^2 \xi \]

\[ \kappa_a = e^{\pi(n-n_*)/s_0} e^{-\Delta(\xi)/2s_0} / \cos \xi \]

- The ratio \( E_3^n/E_2 \) defines the angle \( \xi \)
- The three-body parameter \( \kappa_\ast \) defines the energy of the system at the unitary limit \( E_u = \hbar^2 \kappa^2_\ast/m \)
- The product \( \kappa_\ast a \) is a function of \( \xi \) governed by the universal function \( \Delta(\xi) = s_0 \log \left( \frac{E_3^n+E_2}{E_u} \right) \)
- The universal function \( \Delta(\xi) \) is obtained by solving the STM equations (Faddeev equation in the zero-range limit) and is the same for all levels \( n \)
The three-boson system

Finite-Range Equations: Gaussian spectrum

\[ \frac{E_n^3}{E_2} = \tan^2 \xi_n \]
\[ \frac{\kappa_n^*}{\kappa_d} = e^{\pi(n-n_*)/s_0} e^{-\tilde{\Delta}_n(\xi_n)/2s_0} / \cos \xi \]

- The ratio \( \frac{E_n^3}{E_2} \) defines the angle \( \xi_n \)
- The three-body parameter \( \kappa_n^* \) defines the energy of the system at the unitary limit \( E_u = \hbar^2 (\kappa_n^*)^2 / m \)
- The product \( \kappa_n^* a \) is a function of \( \xi_n \) governed by the level function \( \tilde{\Delta}(\xi_n) = s_0 \log \left( \frac{E_n^3 + E_2}{E_u} \right) \)
- The level function \( \tilde{\Delta}(\xi) \) is obtained by solving the Schrödinger equation in the desired region.
\[ \tilde{\Lambda}_3(\xi) \]

- NLG \( \tilde{\Lambda}_3^0(\xi) \)
- NLG \( \tilde{\Lambda}_3^1(\xi) \)
- LG \( \tilde{\Lambda}_3^0(\xi) \)
- LG \( \tilde{\Lambda}_3^1(\xi) \)
- LM2M2 \( \tilde{\Lambda}_3^0(\xi) \)
- LM2M2 \( \tilde{\Lambda}_3^1(\xi) \)
The Helium-Helium system

- The He-He system has been extensively studied in the ’80s.
- Many realistic potentials have been constructed, for example, the Aziz LM2M2 potential.
Ground state properties of helium drops

- The Aziz potential has been used to calculate the ground state energies of drops with $3 \leq N \leq \infty$
- For example in V.R. Pandharipande et al., PRL 50, 1676 (1983) using the GFMC method (The Aziz HFDHE2 potential)
- The motivations for that study were twofold:
  i) To compare theoretical results using potential models with experimental data
  ii) To analyze extrapolation formulas from calculations with fix number of atoms to the infinite system
- The $E/N$ experimental value of liquid Helium ($-7.14K$) was well described. The calculations predicted $-7.11K$ or $-7.02K$ from an extrapolation using results in the range $20 \leq N \leq 112$
Ground state properties of helium drops

\[ E_N / N = (7.02 - 18.8/N^{1/3} + 11.2/N^{2/3}) \text{K} \]

\[ \langle r^2 \rangle^{1/2} [a_0] \]

\( N \) (\( N^{1/3} \) scale)

\( N \)
Two different views

- The helium-helium system is described with great detail using realistic interaction models
- This description is extended up to describe saturation properties

- The helium-helium system is well inside the Efimov window
- The details of the interaction are not important
- Many properties can be described using two parameters:
  i) the two body scattering length $a$
  ii) the three-body parameter $\kappa_*$

- in the context of EFT this corresponds to a LO description
- Can the LO description be extended to describe saturation properties?
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Soft Two-Body Gaussian Potential

Effective low-energy soft potential

- $V(r) = V_0 \, e^{-r^2/r_0^2}$
- Fix $V_0, r_0$ to reproduce $a$ and $E_2$

$V_0 = -1.2344 \, \text{K}, \quad r_0 = 10.0 \, \text{a.u.}$

<table>
<thead>
<tr>
<th>Gaussian</th>
<th>LM2M2</th>
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<tbody>
<tr>
<td>$a_0$ (a.u.)</td>
<td>189.41</td>
</tr>
<tr>
<td>$r_{\text{eff}}$ (a.u.)</td>
<td>13.81</td>
</tr>
<tr>
<td>$E_2$ (mK)</td>
<td>-1.303</td>
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Graph showing $V(r)$ versus $r$ [a.u.] with a minimum at $r_0$ and $a_0$. The graph compares two models: Gaussian and LM2M2.
The helium trimer

Problems in the three-body sector

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Introducing a three-body potential: A Leading Order description

$$W(\rho_{ijk}) = W_0 \ e^{-2\rho_{ijk}^2/\rho_0^2} \ (\rho_{ijk}^2 \propto r_{ij}^2 + r_{jk}^2 + r_{ki}^2)$$

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($W_0$ [K], $\rho_0$ [a.u.])

- (1.474, 10) : -126.4, -2.292
- (0.721, 12) : -126.4, -2.295
- (0.422, 14) : -126.4, -2.299
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### Saturation properties of helium drops

- We define a soft potential model to describe $E_3$
- It consists in a two- plus a three-body term $V = V(i, j) + W(i, j, k)$
- $V(i, j) = V_0 e^{-r_{ij}^2 / r_0^2}$
- $W(i, j, k) = W_0 e^{-\rho_{ijk}^2 / \rho_0^2}$
- $W_0$ is determined from $E_3$
- $\rho_0$ is taken as a parameter
- $E/N$ is calculated for increasing values of $N$ as a function the $\rho_0$
- The saturation properties are determined from a liquid drop formula:
  \[ E_N/N = E_v + E_s x + E_c x^2 \] with \[ x = N^{-1/3} \]
- In general drops with $N$ around 100 is sufficient to determine $E_v$ and $E_s$
drops with $N \leq 10$
drops with $N \leq 10$
drops with \( N \leq 112 \)
drops with $N = 4$
Propagation of universal behavior with $N$

Saturation properties of helium drops

- Using the appropriate value of $\rho_0$
- we obtain (in K):
  $$E_N/N = 6.79 - 18.0x + 9.98x^2$$
- To be compared to the results of the HFDHE2 potential:
  $$E_N/N = 7.02 - 18.8x + 11.2x^2$$
- The experimental result is 7.14 K
- for the surface tension $t = E_s/4\pi r_0^2(\infty)$
  - the experimental value is 0.29 KA$^2$
  - with the gaussian soft potential the result is 0.27 KA$^2$
- Since the potential model is determined only from the two, three and four body sector, we can conjecture that the saturation energy can be extracted from $E_2$, $E_3$ and $E_4$
- Accordingly we can think in a different expansion of $E_N/N$
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- Accordingly we can think in a different expansion of $E_N/N$
For example we propose the following formula:

\[
\frac{E_N}{N} = E_v^{(0)} \frac{1 - (3/N)^{1/4}}{1 + \frac{3E_4}{4E_3}(3/N)} ,
\]

\[E_v^{(0)} = \rho_0 = 8.5 a_0\]
From the equation:

\[
\frac{E_N}{N} = E_V^{(0)} \frac{1 - (3/N)^{1/4}}{1 + \frac{3E_4}{4E_3}(3/N)}
\]

The saturation energy \(E_V^{(0)}\) can be determined using a single value of \(E_N/N\)

using the \(N = 4\) value we obtain

\[
\frac{E_V^{(0)}}{E_4} = 3.602 \left(1 + \frac{9E_4}{16E_3}\right)
\]

Using the GFMC ratio \(E_4/E_3 = 4.55\), it results \(E_V^{(0)}/E_4 = 3.602\)

This analysis suggests the possibility of describing \(E_V^{(0)} = \xi_4E_4\) at the unitary limit with \(\xi_4\) a universal number?
Conclusions

- Helium drops have been studied using soft gaussian two- and three-body potentials.
- The range and depth of the two-body gaussian have been fixed by $a$ and $E_2$.
- The depth of the three-body gaussian was fixed by the trimer energy. Its range was taken as a parameter.
- The soft potential has to follow the original potential in the description of the drops.
- However the original potential is a two-body interaction: the potential energy increases as the number of pairs.
- The potential energy of the soft potential has an attractive two-body term which increases as the number of pairs and a repulsive three-body term increasing as the number of triplets.
Conclusions

- The range of the three-body interaction has been taken as a parameter to set the total energy as close as possible to the original one.
- Using this argument we have extended the leading order description to $N \to \infty$.
- At the end the soft potential has been determined by four parameters: the two body scattering length and $E_2$ (the two-body force) $E_3$ and $E_4$ (the three-body force).
- It seems that the saturation properties are already hidden in these four quantities!