A new axiomatic formulation for quantum theory: the general boundary formulation

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Motivations from Quantum Gravity

The formulation of a quantum theory of gravity leads to technical as well as conceptual difficulties

Quantization problems

- Incompatibility between the foundational principles of GR and QT
- Standard quantization prescriptions require a fixed, non-dynamical background metric
- GR: spacetime is a physical and dynamical system + diff invariance

$\Rightarrow$ Background independent quantum field theory

Interpretational problems

- The problem of time
- Lack of a manifest local description of dynamics

The standard formulation of QT has limitations that obstruct its application in a general relativistic context.

Question

Can we sufficiently extend the standard formulation of QT in order to render it compatible with the symmetries of GR?

- no explicit reference to a background (space)time
- description of physics in a manifestly local way
- ability to reproduce known physics

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YES, using:

- The mathematical framework of topological quantum field theory.
- A generalization of the Born rule.

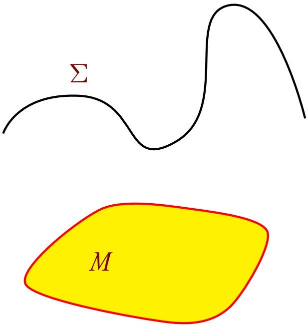
Basic structures and axioms - I

In the GBF algebraic structures are associated to geometric ones.

Geometric structures (representing pieces of spacetime in dimension d):

- hypersurfaces: oriented manifolds of dimension $d-1$
- regions: oriented manifolds of dimension d with boundary

These manifolds may carry additional structure: differentiable, metric, etc.



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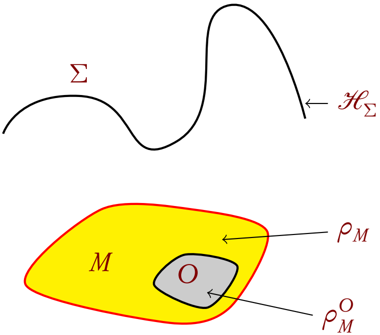
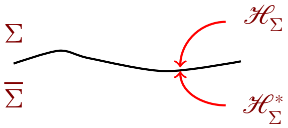


Algebraic structures:

- To Σ a Hilbert space $\mathcal{H}_\Sigma$
- To M a linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$
- As in AQFT, observables are associated to spacetime regions: In a region M, an observable O is a linear map $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$.

Axioms - II

- If $\overline{\Sigma}$ denote Σ with opposite orientation, then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_\Sigma^*$.

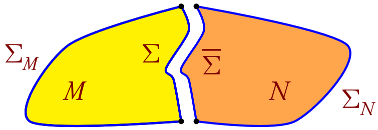


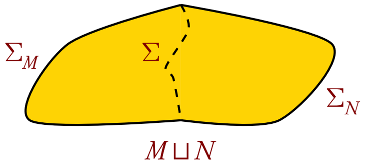
- Decomposition rule: If $\Sigma = \Sigma_1 \cup \Sigma_2$, then $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

- Gluing rule: $M \sqcup N$

$$\begin{aligned}&\rho_{M\sqcup N}(\psi_M \otimes \psi_N)\\ &= \rho_M \circ \rho_N(\psi_M \otimes \psi_N)\\ &= \sum_i \rho_M(\psi_M \otimes \xi_i)\rho_N(\xi_i^* \otimes \psi_N)\end{aligned}$$

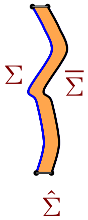
where $\psi_M \in \mathcal{H}_M$, $\psi_N \in \mathcal{H}_N$ and $\{\xi_i\}$ is an ON-basis of $\mathcal{H}_\Sigma$.





Axioms - III

A special region is the *empty region* $\hat{\Sigma}$



$$\rho_{\hat{\Sigma}}(\psi \otimes \eta) = \langle \psi | \eta \rangle$$

Probability interpretation

In quantum theory, probabilities are generally conditional probabilities depending on two type of data: *preparation* and *observation*.

In the GBF, both type of data encoded through closed subspaces of $\mathcal{H}_{\partial M}$:

$$\textit{preparation} : \boxed{\mathcal{S} \subset \mathcal{H}_{\partial M}} \qquad \textit{observation} : \boxed{\mathcal{A} \subset \mathcal{H}_{\partial M}}$$

The probability that the system is described by $\mathcal{A}$ given that it is described by $\mathcal{S}$ is:

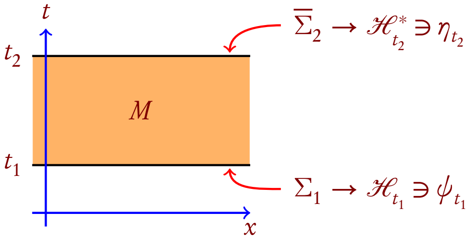
$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_M \circ P_{\mathcal{S}} \circ P_{\mathcal{A}}|^2}{|\rho_M \circ P_{\mathcal{S}}|^2} = \frac{\sum_{i \in J} |\rho_M(\xi_i)|^2}{\sum_{i \in I} |\rho_M(\xi_i)|^2}$$

- $P_{\mathcal{S}}$ and $P_{\mathcal{A}}$ are the orthogonal projectors onto the subspaces.
- $\{\xi_i\}_{i \in I}$ is an ON-basis of $\mathcal{S}$, $\{\xi_i\}_{i \in J}$ is an ON-basis of $\mathcal{A}$.

Recovering of standard transition probability

The expression of the generalized probability reduces to a standard transition probability for a standard transition amplitude.

Spacetime region: $M = [t_1, t_2] \times \mathbb{R}^3$
Boundary: $\partial M = \Sigma_{t_1} \cup \overline{\Sigma}_{t_2}$
State space: $\mathcal{H}_{\partial M} = \mathcal{H}_{t_1} \otimes \mathcal{H}^*_{t_2}$



$$\rho_{[t_1,t_2]}(\psi_{t_1} \otimes \eta_{t_2}) = \langle \eta | U(t_1, t_2) | \psi \rangle$$

preparation: $\mathcal{S} = \psi \otimes \mathcal{H}_{t_2} \subset \mathcal{H}_{\partial M}$ observation: $\mathcal{A} = \mathcal{H}_{t_1} \otimes \eta \subset \mathcal{H}_{\partial M}$

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_M \circ P_{\mathcal{S}} \circ P_{\mathcal{A}}|^2}{|\rho_M \circ P_{\mathcal{S}}|^2} = \frac{|\rho_M(\psi \otimes \eta)|^2}{1} = |\langle \eta | U(t_1, t_2) | \psi \rangle|^2$$

Observables

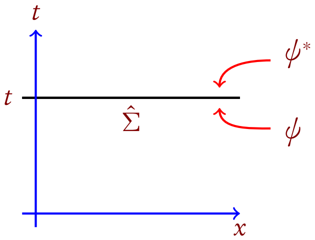
The observable map ρ_M^O allows to define the expectation value of the observable O depending on the preparation of the system encoded in the subspace $\mathcal{S} \subset \mathcal{H}_{\partial M}$ as

$$\langle O \rangle_{\mathcal{S}} = \frac{|\rho_M \circ \rho_M^O \circ P_{\mathcal{S}}|^2}{|\rho_M \circ P_{\mathcal{S}}|^2} = \frac{\sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M^O(\xi_i)}{\sum_{i \in I} |\rho_M(\xi_i)|^2}$$

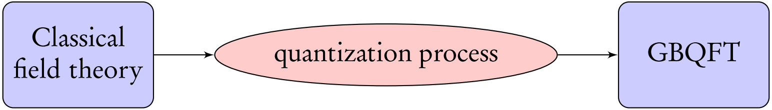
where $\{\xi_i\}_{i \in I}$ is an ON-basis of $\mathcal{S}$.

This formula reduces to the standard expectation value in empty region $\hat{\Sigma}$ at time t, setting $\mathcal{S} = \psi \otimes \mathcal{H}^*$:

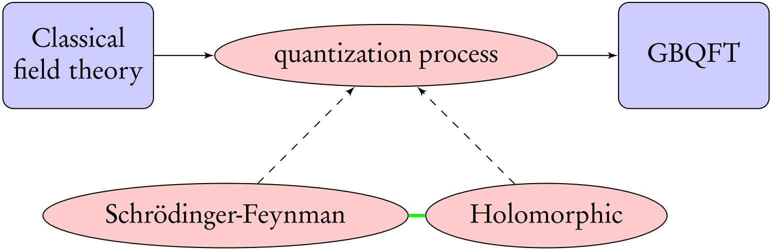
$$\langle O \rangle_{\mathcal{S}} = \frac{\rho_{\hat{\Sigma}}^O(\psi \otimes \psi^*)}{1} = \langle \psi | O | \psi \rangle$$



GBF and QFT



GBF and QFT



Results:

- An isomorphism can be constructed between the Hilbert spaces in the two representations
- The GBF axioms are satisfied by these quantization prescriptions

Schrödinger-Feynman quantization

- Schrödinger representation + Feynman path integral quantization
  The state space $\mathcal{H}_\Sigma$ for a hypersurface Σ is the space of functions on field configurations K_Σ on Σ.
- Inner product,

$$\langle\psi_2|\psi_1\rangle = \int_{K_\Sigma} \mathcal{D}\varphi\, \psi_1(\varphi)\overline{\psi_2(\varphi)}.$$

- Amplitude for a region M, $\psi \in \mathcal{H}_{\partial M}$,

$$\rho_M(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi\, \psi(\varphi) Z_M(\varphi), \quad \text{where} \quad Z_M(\varphi) = \int_{K_M, \phi|_{\partial M}=\varphi} \mathcal{D}\phi\, e^{\mathrm{i}S_M(\phi)}.$$

- A classical observable F in M is modelled as a function on K_M. The quantization of F is the linear map $\rho_M^F : \mathcal{H}_{\partial M} \to \mathbb{C}$ defined as

$$\rho_M^F(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi\, \psi(\varphi) Z_M^F(\varphi), \quad \text{where} \quad Z_M^F(\varphi) = \int_{K_M, \phi|_{\partial M}=\varphi} \mathcal{D}\phi\, F(\phi) e^{\mathrm{i}S_M(\phi)}.$$

Holomorphic quantization

- Linear field theory: L_Σ is the vector space of solutions near the hypersurface Σ.
- L_Σ carries a non-degenerate symplectic structure ω_Σ and a complex structure $J_\Sigma : L_\Sigma \to L_\Sigma$ compatible with the symplectic structure:

$$J_\Sigma^2 = -\mathrm{id}_\Sigma \qquad \text{and} \qquad \omega_\Sigma(J_\Sigma(\cdot), J_\Sigma(\cdot)) = \omega_\Sigma(\cdot,\cdot).$$

- J_Σ and ω_Σ combine to a real inner product $g_\Sigma(\cdot,\cdot) = 2\omega_\Sigma(\cdot, J_\Sigma \cdot)$ and to a complex inner product $\{\cdot,\cdot\}_\Sigma = g_\Sigma(\cdot,\cdot) + 2\mathrm{i}\omega_\Sigma(\cdot,\cdot)$ which makes L_Σ into a complex Hilbert space.
- The Hilbert space $\mathscr{H}_\Sigma$ associated with Σ is the space of holomorphic functions on L_Σ with the inner product

$$\langle \psi, \psi' \rangle_\Sigma = \int_{L_\Sigma} \overline{\psi(\phi)} \psi'(\phi) \exp\left(-\frac{1}{2} g_\Sigma(\phi,\phi) \right) \mathrm{d}\mu(\phi),$$

where μ is a (fictitious) translation-invariant measure on L_Σ.

Holomorphic quantization (II)

- The amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ associated with the spacetime region M for a state $\psi \in \mathcal{H}_{\partial M}$ is given by

$$\rho_M(\psi) = \int_{L_\Sigma} \psi(\phi) \exp\left(-\frac{1}{4} g_{\partial M}(\phi, \phi)\right) \mathrm{d}\mu_{\tilde{M}}(\phi).$$

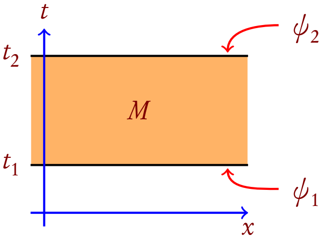
- The observable map associated to a classical observable F in a region M is

$$\rho_M^F(\psi) = \int_{L_\Sigma} \psi(\phi) F(\phi) \exp\left(-\frac{1}{4} g_{\partial M}(\phi, \phi)\right) \mathrm{d}\mu_{\tilde{M}}(\phi).$$

Klein-Gordon theory in Minkowski

The S-matrix technique is used to describe interacting QFT:

Spacetime region:
$M = [t_1, t_2] \times \mathbb{R}^3$
Boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
State space: $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$



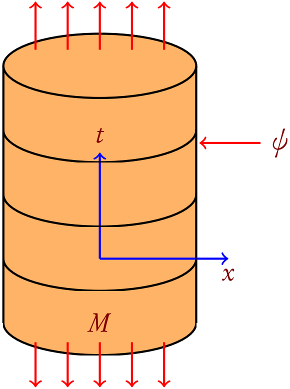
Assume interaction is relevant only between the initial time t_1 and the final time t_2. The S-matrix is the asymptotic limit of the amplitude between free states at early and at late time:

$$\langle \psi_2 | \mathscr{S} | \psi_1 \rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \langle \psi_2 | U_{int}(t_1, t_2) | \psi_1 \rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \rho^U_{[t_1, t_2] \times \mathbb{R}^3}(\psi_1 \otimes \psi_2^*)$$

Spatially asymptotic S-matrix

Similarly, we can describe interacting QFT via a spatially asymptotic amplitude. Assume interaction is relevant only within a radius R from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$\mathscr{S}(\psi) = \lim_{R\to\infty} \rho_R^U(\psi)$$



Result

The S-matrices are equivalent when both are valid.
Cross-symmetry of S-matrix is a prediction of the GBF.

Conclusions

- The GBF is a new versatile formulation of quantum theory:
  - offers new perspectives on QT
  - can treat situations where standard techniques fail
  - may solve conceptual problems of QG
- Many results have been obtained:
  - quantisation prescriptions [Oeckl 2008,2012]
  - general structure of the vacuum state and complex structure for GBQFT in curved spaces [DC 2009,DC, Dohse 2017]
  - unitarity of evolution for QFT in curved space and new representations for Feynman propagators and S-matrices [DC, Oeckl 2007,2009]
  - evanescent states [in progress]
  - GBQFT in Euclidean space [DC, Oeckl 2008], de Sitter (derivation of the Polyakov propagator) [DC 2015], Anti de Sitter (S-matrix from hypercylinder geometry) [DC, Dohse, Oeckl 2012], Rindler [DC, Raetzel 2013], general curved space [DC, Dohse 2017]
  - Unruh effect [DC, Raetzel 2013], Casimir effect [in progress]
- QG: the GBF is already used in some approach (Spin foam models), 3d gravity is topological
- It is a work in progress... '