Non-Analytic Behaviors of $(SU(N_c))$ Fermi Liquid

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Outline

Introduction

What is not analytic?

Why is it not analytic?

Crossover Between $T$ and $H$

$SU(N_c)$

Conclusion
Some history

- Non-analytic specific heat of liquid $^3\text{He}$ in late 1960's
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- Theory well established
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  - Implication on itinerant magnetism
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Some history

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- Non-analytic magnetic response by the same physics
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- Past effort focused on specific heat or spin susceptibility
- Uncertainties in interaction parameters prevent precise comparison
**Why quantum gas?**

- **Theoretically:**
  - We know the interaction well!
  - Dilute regime allows for simple perturbation theory
Why quantum gas?

- **Theoretically:**
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- **Experimentally:**
  - Access to equation of state
    - Not confined to only specific heat or susceptibility
    - New perspective to an old problem
Why quantum gas?

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- **Experimentally:**
  - Access to equation of state
    - Not confined to only specific heat or susceptibility
    - New perspective to an old problem
  - Possible SU($N$) enhancement for large $N$
    - $N = 6$ for $^{173}$Yb and $N = 10$ for $^{87}$Sr
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Free energy at finite $T$

\[ \frac{\Omega}{V} \sim E_F k_F^3 \left[ \omega_0 + \left( \frac{T}{E_F} \right)^2 + \left( \frac{T}{E_F} \right)^4 + \ldots \right] \]

- Ginzburg-Landau free energy is analytic
Free energy at finite $T$

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- Fourth order term is already beyond FL
  - needs properties of systems away from FS
Free energy at finite $T$

\[
\frac{\Omega}{V} \sim E_F k_F^3 \left[ \omega_0 + \left( \frac{T}{E_F} \right)^2 + \left( \frac{T}{E_F} \right)^4 \log \left( \frac{T}{E_F} \right) \right]
\]

- Ginzburg-Landau free energy is analytic
- Fourth order term is already beyond FL
  - needs properties of systems away from FS
- But FL with interaction universally yields a logarithmic term
Free energy at finite $T$ (II)

- $(\text{free energy} - \text{leading analytic terms})/t^4$
- $(\log t + \ldots) \text{ v.s. } \log t$

$(t = T/E_F)$

PRA 95, 033619 (2017)
Free energy at finite $T$ (II)

- $(\text{free energy} - \text{leading analytic terms})/t^4$
- $(\log t + \ldots) \text{ v.s. } \log t$
- Shows eventual failure of FL

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Free energy at finite $T$ (II)

$\frac{(\text{free energy - leading analytic terms})}{t^4}$

$\log t + \ldots$ v.s. $\log t$

Shows eventual failure of FL

When $H \neq 0$: additional crossover!

$t = T/E_F$

PRA 95, 033619 (2017)
When both $T$ and $H$ are non-zero

$$\frac{\Omega}{V} \sim E_F k_F^3 \left[ \cdots - t^4 \log t + h^4 \log |h| + F_4(t, h) + \ldots \right]$$

$(t = T/E_F, \ h = H/E_F)$

- $H \to$ Zeeman energy of spins
  - Shift in spin-dependent chemical potential
  - Tuned by density difference in quantum gas
When both $T$ and $H$ are non-zero

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- $H \rightarrow$ Zeeman energy of spins
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  - Tuned by density difference in quantum gas
- $F_4(t, h)$ is a scaling function

$$F_4(t, h) = \begin{cases} 
  t^4 f_t(t/h) & \text{if } t/h \ll 1 \\
  h^4 f_h(h/t) & \text{if } t/h \gg 1
\end{cases}$$
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- These limits correspond to specific heat/susceptibility
- What if $t/h \sim 1$?
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Particle-hole pair

- Bosonic, can have zero energy
  - Perturbation series has IR problem
  - Relevant IR scale acts as cutoff
Particle-hole pair

- Bosonic, can have zero energy
  - Perturbation series has IR problem
  - Relevant IR scale acts as cutoff
- $T$ smears out sharp Fermi surface: cutoff candidate
- If $a \neq b$: gapped by $H \neq 0$. Another scale!
Ring and ladder diagrams

- Strings of particle-hole pairs. Left: ring; right: ladder
**Ring and ladder diagrams**

- Strings of particle-hole pairs. Left: ring; right: ladder
- Not full-fledged IR divergence; non-analytic at zero energy
Ring and ladder diagrams

- Strings of particle-hole pairs. Left: ring; right: ladder
- Not full-fledged IR divergence; non-analytic at zero energy
- Ladder diagram is sensitive to both $T$ and $H$
  - IR cutoff by $\max(T, H)$
  - Crossover
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Our model

- Non-relativistic Fermi gas
- Contact interaction characterized by scattering length \( a \)
- Dilute limit: 2nd order perturbation theory
Thermodynamic potential

\[ \frac{\Omega}{V} = \frac{k_F^5}{12\pi m} \left\{ \ldots + (k_F a)^2 \left[ -\frac{\pi^2}{20} t^4 \log t + \frac{1}{32\pi^2} h^4 \log |h| + F_4(t, h) \right] + \ldots \right\} \]

ladder type non-analyticity
Thermodynamic potential

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Closed form for the crossover function \( F_4 \):

\[ F_4(t, h) = \frac{3}{8\pi^2} \int_0^{\infty} dx \frac{x^2}{e^{x/t} - 1} (x + h) \log \left| \frac{x}{x + h} \right| + (h \rightarrow -h) \]
The analytic line

- $t^4 \log t$ and $h^4 \log h$ of opposite sign: “competing”
- Along the line $\frac{t}{h} = 4\sqrt{\frac{5}{8}}\pi$, the thermodynamic potential is a completely \textit{analytic} function
Approaching $T = 0$ at constant $H$

- Perhaps most relevant experimentally
- Competition of $T$ and $H$ as IR scale
Approaching $T = 0$ at constant $H$ (II)

![Graph showing the free energy and reference log(t) as functions of $t/h$. The graph illustrates the crossover behavior with a straight line representing $t^4 \log t$.](image)

- straight line: $t^4 \log t$
Approaching $T = 0$ at constant $H$ (II)

- straight line: $t^4 \log t$
- $t$ gets smaller:
- crosses over to $t^4 \log h$
Approaching $H = 0$ at constant $T$

\[ h^4 \log |h| \]

\[ h^4 \log t \]
Approaching $H = 0$ at constant $T$ II

![Graph showing free energy and reference log(h) vs h/t](image-url)
The overall “phase diagram”

- Near axes: \( \log t \) or \( \log h \) dominates all
- middle: need \( F_4 \) in full
- two sets of crossover
- analytic line where the two non-analytic terms cancel
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Consider an even $N_c$

Spin $1, \ldots, N_c/2$ has chemical potential $\mu_a$

Spin $(N_c/2 + 1), \ldots, N_c$ has chemical potential $\mu_b$
Simple generalization of SU(2) and Large-$N_c$ enhancement

- Consider an even $N_c$
- Spin $1,\ldots,N_c/2$ has chemical potential $\mu_a$
- Spin $(N_c/2 + 1),\ldots,N_c$ has chemical potential $\mu_b$
- In practice, load the two sets of spins at two densities
Simple generalization of SU(2) and Large-$\mathcal{N}_c$ enhancement

- Consider an even $\mathcal{N}_c$
- Spin $1, \ldots, \mathcal{N}_c/2$ has chemical potential $\mu_a$
- Spin $(\mathcal{N}_c/2 + 1), \ldots, \mathcal{N}_c$ has chemical potential $\mu_b$
- In practice, load the two sets of spins at two densities

\[
\frac{\Omega}{V} \sim \mathcal{N}_c \, E_F k_F^3 \left\{ \cdots + \frac{\mathcal{N}_c}{2} \left[ t^4 \log t + h^4 \log |h| + F_4(t, h) \right] + \ldots \right\}
\]
Analytic part of SU($N_c$) Fermi gas is no longer even

$$\frac{\Omega}{V} \sim E_F k_F^3 \left\{ \omega_0 + t^2 + \text{tr}(H^2) + \ldots \right\}$$

- SU(2) gas has $H \rightarrow -H$ symmetry
  - Free energy is even.
Analytic part of SU($N_c$) Fermi gas is no longer even

$$\frac{\Omega}{V} \sim E_F k_F^3 \left\{ \omega_0 + t^2 + \text{tr}(H^2) + \ldots \right\}$$

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  - Free energy is even.
- Generalized SU($N$) magnetic field $\rightarrow N \times N$ Hermitian traceless matrix
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\frac{\Omega}{V} \sim E_F k_F^3 \left\{ \omega_0 + t^2 + \text{tr}(H^2) + \text{tr}(H^3) + \ldots \right\}
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- SU(2) gas has $H \rightarrow -H$ symmetry
  - Free energy is even.
- Generalized SU($N$) magnetic field $\rightarrow N \times N$ Hermitian traceless matrix
- Symmetry is lost!
Analytic part of SU($N_c$) Ferri gas is no longer even

\[ \frac{\Omega}{V} \sim E_F k_F^3 \left\{ \omega_0 + t^2 + \text{tr}(H^2) + \text{tr}(H^3) + \ldots \right\} \]

- SU(2) gas has $H \rightarrow -H$ symmetry
  - Free energy is even.
- Generalized SU($N$) magnetic field $\rightarrow N \times N$ Hermitian traceless matrix
- Symmetry is lost!
  - $H \sim \text{diag}(-(N-1), 1, 1, \ldots)$ v.s.
  - $H \sim \text{diag}((N-1), -1, -1, \ldots)$
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Fermi liquid in the normal phase is non-analytic near $T = H = 0$. 

Interesting crossover behaviors reflected in equation of state accessible with cold quantum gas experiment. 

$SU(N_c)$ pseudo-spin may enhance the interaction effect but also adds new difficulty because $H \leftrightarrow -H$ is not a symmetry in general.
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- Interesting crossover behaviors reflected in equation of state
  - accessible with cold quantum gas experiment
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