Pseudoscalar meson octet in term of $SU(N)$ Gauge Invariant Lagrangian

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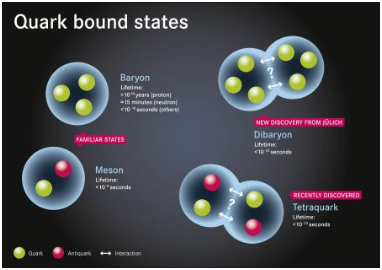
QFC'17, 25-27 Oct. Pisa, Italy

October 25, 2017

Contents

1. Introduction
2. $SU(N)$ gauge invariant Lagrangian
3. Dirac equation and its solution
4. Fermion and total currents
5. Gluodynamic Lagrangian
6. Lagrangian of scalar particles
7. Octet of pseudoscalar mesons
8. Discussion
9. Conclusion

Introduction



Hadronization models :

1) Local parton-hadron duality approach

(i) String model(1+1) tube, Lund model

(ii) Cluster model (fragmentation of the colorless clasters)

2) Dynamic approach

(i) Running gluon mass in the pure gluodynamic picture

ii) Renorm. group approach

Introduction

Bibliography

1) Local parton-hadron duality approach

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(i) String model(1+1) tube, Lund model

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(ii) Cluster model (fragmentation of the colorless clasters)

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Bibliography

2) Dynamic approach

(i) Running gluon mass in the pure gluodynamic picture

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ii) Renorm. group approach J.Braun, L. Fister, J. Pawlowski, and F. Renneck. Phys. Rev.**D 94**, 034016 (2016),

$SU(N)$ gauge invariant Lagrangian

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$$\mathcal{L}_{q-g} = \frac{1}{2}\sum_f \left[\bar{\Psi}_f(\gamma^\mu D_\mu \; - m_f)\Psi_f - \bar{\Psi}_f(\gamma^\mu \overleftarrow{D}_\mu + m_f)\Psi_f\right] \tag{2}$$

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$$\mathcal{L}_g = -\frac{1}{4}F^a_{\mu\nu}\; F^{\mu\nu}_a \tag{3}$$

$$\mathcal{L}_{gh} = -\bar{c}_a(x)\partial^2 c^a(x) - g\bar{c}_a(x)f^{\,a}_{\;bc}\partial^\mu\left(A^b_\mu(x)c^c(x)\right) \tag{4}$$

Dirac equation and its solution

$$\left\{ i\gamma^{\mu} \left(\partial_{\mu} - ig \cdot A^{a}_{\mu}(x)\ T_{a} \right) - m_f \right\} \Psi_f(x) = 0, \qquad (5)$$

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$$\Psi_f(x) = \left\{ T_{l(x_0;x)} \exp \right\} \left\{ igT_a \int dx^{\mu} A_{\mu}^{a} \right\}\ \psi_f(x), \tag{6}$$

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The symbol $\{ T_{l(x_0;x)} \exp \}$ means that the integration is to be carried out along the line from the point x_0 to the point x such that the factors in exponent expansion are chronologically ordered from x_0 to x.

Dirac equation and its solution

$$\Downarrow$$

$$\{i\gamma^\mu \partial_\mu - m_f\} \psi_f(x) = 0. \tag{7}$$

Dirac equation and its solution

$$\Downarrow$$

$$\{i\gamma^{\mu}\partial_{\mu} - m_f\}\,\psi_f(x) = 0. \tag{7}$$

$$\psi_{\sigma cf}(x) = u_{\sigma}(P)\,\frac{\exp(-iP_{\mu}x^{\mu})}{\sqrt{2\varepsilon(\vec{p})}}v_{c,f}(x_0), \quad \varepsilon^2(\vec{p}) = \vec{p}^{\;2} + m_f^2$$

$$P \equiv P^{\mu} = (\varepsilon(\vec{p}), \quad \vec{p}); \bar{u}u = 2m_f, \quad (v^{\dagger})_{c'f'}(x_0)v^{cf}(x_0) = \delta^c_{c'}\delta^f_{f'}. \tag{8}$$

Dirac equation and its solution

$$\Psi_{c,f}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{\sigma\lambda} [u_\sigma(P)\ a_f(P,\sigma,\lambda,c)\ \theta(P^0)\ +$$
$$u_\sigma(-P)\ a_f(-P,\sigma,\lambda,c)\ \theta(-P^0)] \frac{\exp(-iP_\mu x^\mu)}{\sqrt{2\varepsilon(\vec{p})}} \times$$
$$\{T_{l(x_0;x)} \exp\} \left\{ igT_a \int dx^\mu A^a_\mu \right\}\ v_{c,f}(x_0), \qquad (9)$$

Summation with respect to λ means summing over all the possible trajectories of a fermion which connect the points x_0 and x in the Minkowskii space-time. As for $a_f(P,\sigma,\lambda,c)$, $\Rightarrow$

Dirac equation and its solution

They are related to particles or anti-particles, and satisfy the standard anti-commutative relations

$$[a_f(P,\sigma,\lambda,c), a_f^\dagger(P',\sigma',\lambda',c')]_+ = (2\pi)^3\delta(\vec{p}-\vec{p}\,') \; \delta_{\sigma\sigma'}\delta_{cc'}\delta_{ff'}\delta_{\lambda\lambda'}. \quad (10)$$

The δ-symbol with respect to the "variable" λ eliminates interception of the particle trajectories what is the direct consequence of the superposition principle.

Fermion and total currents

Following Schwinger

$$J_a^\mu(x) = g \sum_f \left\{ \bar{\Psi}_f(x)\gamma^\mu T_a \Psi_f(x') \right\}, \qquad x' \to x. \quad (11)$$

$$(T\exp)\left\{ igT_a \int_x^{x'} A_\mu^a dx^\mu \right\} = 1 + igT_a(x'-x)^\mu A_\mu^a(\xi) +$$
$$\frac{i}{2} gT_a(x'-x)^\mu (x'-x)^\nu \partial_\nu A_\mu^a(\xi) + \ldots, \quad \xi \in [x, x']. \qquad (12)$$

$$\Downarrow$$

Fermion and total currents

$$\Downarrow$$

$$J_a^\mu(x) = M^2(N_c, N_f)\left(A_a^\mu(x) - \frac{\partial_\lambda \partial^\mu}{\partial^2} A_a^\lambda(x)\right), \tag{13}$$

Fermion and total currents

$$\Downarrow$$

$$J^{\mu}_{a}(x) = M^2(N_c, N_f)\left(A^{\mu}_{a}(x) - \frac{\partial_\lambda \partial^\mu}{\partial^2} A^{\lambda}_{a}(x)\right), \tag{13}$$

$$M^2(N_c, N_f) = \frac{g^2}{8} \int \frac{d^4P}{(2\pi)^3} \sum_{f,c,\lambda} \sum_{\sigma} \times$$

$$\frac{\partial}{\partial P_\nu} \left\{ n_f(P, \sigma, \lambda, c) \frac{P^\nu \left[\delta(P^0 + \varepsilon(\vec{p})) + \delta(P^0 - \varepsilon(\vec{p}))\right]}{\varepsilon(\vec{p})} \right\} \tag{14}$$

Fermion and total currents

A total current $I_a^\mu(x)$

$$I_a^\mu(x) = M^2\left(A_a^\mu(x) - \frac{\partial_\lambda\partial^\mu}{\partial^2}A_a^\lambda(x)\right) - f_{ab}{}^c A_\nu^b F_c^{\mu\nu}. \qquad (15)$$

The total $I_a^\mu(x)$) is gauge invariant, and satisfies the continuity equation:

$$\partial_\mu I_a^\mu(x) = 0. \qquad (16)$$

Gluodynamic Lagrangian

We substitute the fermion field given by Eq.(9) into Eq. (1).

Such a modification of the Lagrangian results in taking into account only such trajectories of fermions which are governed by the Dirac equation, rather than all the possible ones, which are contained in the Lagrangian (1).

Gluodynamics Lagrangian

We substitute the fermion field given by Eq.(9) into Eq. (1).

$$\mathcal{L} = M^2 A_a^{\mu}(x) \left(A_{\mu}^a(x) - \frac{\partial^{\lambda}\partial_{\mu}}{\partial^2} A_{\lambda}^a(x) \right) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \mathcal{L}_{gh} +$$

$$\partial^{\mu}(x) \lim_{x' \to x} f_{\mu}(x, x')$$

$$\Downarrow$$

$$\mathcal{L} = M^2 A_a^{\mu}(x) \left(A_{\mu}^a(x) - \frac{\partial^{\lambda}\partial_{\mu}}{\partial^2} A_{\lambda}^a(x) \right) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \mathcal{L}_{gh}. \tag{17}$$

Lagrangian of scalar particles

To derive the Lagrangian governing observable particles we need to eliminate arbitrariness in freedom degrees , which always is in the gauge invariant Lagrangians. To do it we fix a gauge, and take the Lorenz one because of its relativistic invariance:

$$\partial_\mu A_a^\mu = 0. \tag{18}$$

In this gauge we have no necessity in the ghost fields to support the unitarity in the perturbative expansion!

$$\Downarrow$$

$$\mathcal{L} = M^2 A_a^\mu(x)\ A_\mu^a(x) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \tag{19}$$

Lagrangian of scalar particles

To derive the scalar particle Lagrangian we assume that fields $A_a^\mu(x)$ take a form:

$$A_a^\mu(x) = \mathfrak{a}_a^\mu + e^\mu \varphi_a(x), \tag{20}$$

where $\varphi_a(x)$ are scalar functions, whereas the constant vectors $\mathfrak{a}_a^\mu$ are assumed to be orthogonal to both the unit vector e_μ and the scalar field gradients $\partial_\mu \varphi_a(x)$:

$$\mathfrak{a}_a^\mu \ e_\mu = 0, \quad \mathfrak{a}_\mu^a \ \partial^\mu \varphi_b(x) = 0, \ \ e_\mu \ e^\mu = -1. \tag{21}$$

The Lorentz gauge (18) results in the additional orthogonality condition:

$$e^\mu \partial_\mu \varphi_a(x) = 0. \tag{22}$$

Lagrangian of scalar particles

Substituting Eq.(20) into the Lagrangian (19), and taking into account Eqs.(18),(21)-(22), we derive after direct calculations:

$$\mathcal{L} = \frac{1}{2}\,(\partial^\mu \varphi^a(x))(\partial_\mu \varphi_a(x)) - \frac{1}{2}(\mathcal{M}^2)^a_b \varphi_a(x)\varphi^b(x), \qquad (23)$$

where $(\mathcal{M}^2)^a_b$ is the matrix of the squared masses which is given by a formula

$$(\mathcal{M}^2)^a_b = 2M^2(N_c, N_f)\delta^a_b - g^2 \mathfrak{a}^\mu_c\ \mathfrak{a}^{c'}_\mu f_b{}^{cs} f^a{}_{c's} \qquad (24)$$

Lagrangian of scalar particles

$$\mathcal{M}^2 = \begin{pmatrix} 2M_1^2+m_1^2 & -\mu_1^2 & -\mu_1^2 & 0 & 0 & 0 & 0 & 0 \\ -\mu_1^2 & 2M_1^2+m_1^2 & -\mu_1^2 & 0 & 0 & 0 & 0 & 0 \\ -\mu_1^2 & -\mu_1^2 & 2M_1^2+m_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2M_2^2+m_2^2 & 0 & -\mu_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M_2^2+m_2^2 & 0 & -\mu_2^2 & 0 \\ 0 & 0 & 0 & -\mu_2^2 & 0 & 2M_2^2+m_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2^2 & 0 & 2M_2^2+m_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2M_3^2+m_3^2 \end{pmatrix}$$

Octet of pseudoscalar mesons

(i) The Lagrangian (23) is very symmetrical, and contains 8 scalar fields

(ii) Because o the mass matrix structure, and following Gell-Mann, we take the conservation of the isospin T and strangeness S.

(iii) Let us arrange these 8 fields in multiplets:
1) the strangenessless pion triple,
2) two kaon doublets at $S = \pm 1$, where $S = -1$ corresponds to antiparticles,
3) η meson with the zeroth isospin and strangeness.

Octet of pseudoscalar mesons

We establish relation of these pseudoscalar mesons to the fields φ_a by means of the complex subscribe $a \Rightarrow (T, S)$:

$$\varphi_a(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ \varphi_4(x) \\ \varphi_5(x) \\ \varphi_6(x) \\ \varphi_7(x) \\ \varphi_8(x) \end{pmatrix} \equiv \begin{pmatrix} \varphi_{\pi^+}(x) \\ \varphi_{\pi^-}(x) \\ \varphi_{\pi^0}(x) \\ \varphi_{K^+}(x) \\ \varphi_{K^-}(x) \\ \varphi_{K^0}(x) \\ \varphi_{\tilde{K}^0}(x) \\ \varphi_{\eta}(x) \end{pmatrix} \equiv \begin{pmatrix} \varphi_{(1,0)}(x) \\ \varphi_{(1,0)}(x) \\ \varphi_{(1,0)}(x) \\ \varphi_{(1/2,1)}(x) \\ \varphi_{(1/2,-1)}(x) \\ \varphi_{(1/2,1)}(x) \\ \varphi_{(1/2,-1)}(x) \\ \varphi_{(0,0)}(x) \end{pmatrix}, \quad (25)$$

T and S are the isospin and strangeness.

Octet of pseudoscalar mesons

After diagonalization

$$\mathcal{L} = \frac{1}{2}\,(\partial^{\mu}\Phi^{a}(x))(\partial_{\mu}\Phi_{a}(x)) - \frac{1}{2}(m_{oct}^{2})_{b}^{a}\Phi_{a}(x)\Phi^{b}(x), \qquad (26)$$

Octet of pseudoscalar mesons

$$(\mathcal{M}^2) \equiv (m_{oct}^2)^a_b = \begin{pmatrix} m_{\pi^+} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{\pi^-} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{\pi^0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{K^+} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{K^-} & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & m_{K^0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{\tilde{K}^0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_\eta \end{pmatrix}$$

Octet of pseudoscalar mesons

$$\mathcal{M}^2 = 2 \cdot \begin{pmatrix} M_1^2 + \frac{m_1^2+\mu_1^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1^2 + \frac{m_1^2+\mu_1^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_1^2 + \frac{m_1^2-2\mu_1^2}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2^2 + \frac{m_2^2-\mu_2^2}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2^2 + \frac{m_2^2-\mu_2^2}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_2^2 + \frac{m_2^2+\mu_2^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_2^2 + \frac{m_2^2+\mu_2^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^2 + \frac{m_3^2}{2} \end{pmatrix}.$$

Octet of pseudoscalar mesons

In the tensor representation of the $SU(3)$ group the meson octet has the well-known form:

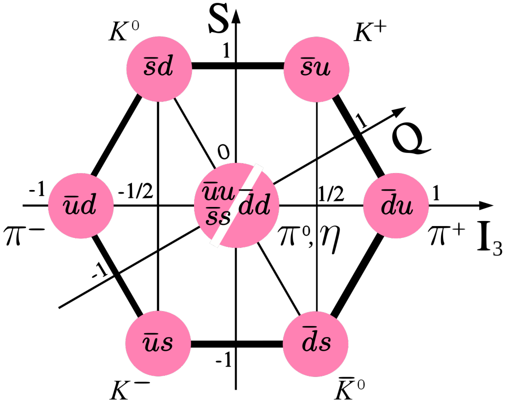
$$\hat{P} = \sum_{0}^{8} T_a \Phi^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi^3 + \frac{\Phi^8}{\sqrt{3}} & \Phi^1 - i\Phi^2 & \Phi^4 - i\Phi^5 \\ \Phi^1 + i\Phi^2 & -\Phi^3 + \frac{\Phi^8}{\sqrt{3}} & \Phi^6 - i\Phi^7 \\ \Phi^4 + i\Phi^5 & \Phi^6 - i\Phi^7 & -\frac{2\Phi^8}{\sqrt{3}} \end{pmatrix}. \quad (28)$$

Octet of pseudoscalar mesons

Or

$$\hat{P} = \sum_{0}^{8} T_a \Phi^a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \tilde{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}. \quad (29)$$

Octet of pseudoscalar mesons



Discussion

a) The lightest mesons (pions)

$$\sqrt{2M_1^2 + m_1^2} = \sqrt{\frac{2m_{\pi^\pm}^2 + m_{\pi^0}^2}{3}} \simeq 1.37 \cdot 10^2 \ MeV;$$
$$\mu_1 = \sqrt{\frac{m_{\pi^\pm}^2 - m_{\pi^0}^2}{3}} \simeq 21.4 \ MeV. \tag{30}$$

Non-diagonal elements , which are in the upper left block in the mass matrix, are sufficiently less than the diagonal ones.

Discussion

b) K -mesons

$$\mu_2 = \sqrt{\frac{m^2_{\tilde{K}^0} - m^2_{K^\pm}}{2}} \simeq 44.5MeV \ll m_{K^\pm} \ , \qquad (31)$$

The similar situation: non-diagonal element , which are in the middler left block in the mass matrix, are sufficiently less than the diagonal ones.

c) η -mesons

Since

$$m_{\pi^\pm} < m_{K^\pm} < m_\eta$$

Then, hereas $M_1 = M_3$, the mass m_3 should be more than m_1 in three times, at least.

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Discussion

c) η -mesons

Since

$$m_{\pi^\pm} < m_{K^\pm} < m_\eta,$$

whereas

$$M_1 = M_3.$$

The mass m_3 should be more than m_1 in three times, at least.

Conclusion

1. The dynamics of fermion and boson fields in the gauge $SU(N)$ model is self-consistently considered beyond the perturbative approximation.
2. On the basis of the developed consideration the pure gluodynamic Lagrangian governing the “massive” gluon field is derived.
3. By breaking the initial $SU(3)$ symmetry in the gluodynamics Lagrangian, the Lagrangian governing the octet of the pseudoscalar mesons is obtained.
4. The meson spectrum is derived .