Quantized vortices in two-dimensional ultracold Fermi gases

Luca Salasnich

Dipartimento di Fisica e Astronomia "Galileo Galilei" and CNISM, Università di Padova
INO-CNR, Research Unit of Sesto Fiorentino, Consiglio Nazionale delle Ricerche

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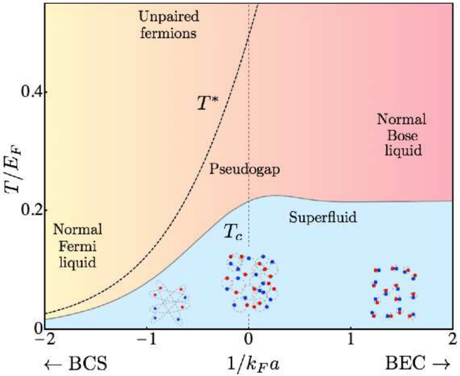
Summary

- BCS-BEC crossover in 3D and 2D
- 2D equation of state
- Zero-temperature 2D results
- Quantized vortices and 2D superfluid density
- Conclusions

BCS-BEC crossover in 3D and 2D (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic ^{40}K or ^{6}Li atoms.**[1]



This crossover is obtained using a Fano-Feshbach resonance to change the 3D s-wave scattering length a_F of the inter-atomic potential.

[1] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

BCS-BEC crossover in 3D and 2D (II)

Recently also the 2D BEC-BEC crossover has been achieved experimentally[2] with a **Fermi gas of two-component ^{6}Li atoms**. In 2D attractive fermions always form biatomic molecules with bound-state energy

$$\epsilon_B \simeq \frac{\hbar^2}{m {a_F}^2} , \tag{1}$$

where a_F is the 2D s-wave scattering length, which is experimentally tuned by a Fano-Feshbach resonance.

The fermionic single-particle spectrum is given by

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} , \tag{2}$$

where Δ_0 is the energy gap and μ is the chemical potential: $\mu > 0$ corresponds to the BCS regime while $\mu < 0$ corresponds to the BEC regime. Moreover, in the deep BEC regime $\mu \to -\epsilon_B/2$.

[2] V. Makhalov et al. PRL **112**, 045301 (2014); M.G. Ries et al., PRL **114**, 230401 (2015); I. Boettcher et al., PRL **116**, 045303 (2016).

2D equation of state (I)

To study the 2D BCS-BEC crossover we adopt the formalism of functional integration[3]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature T, in a 2-dimensional volume L^2, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{ -\frac{S}{\hbar} \right\} , \tag{3}$$

where ($\beta \equiv 1/(k_B T)$ with k_B Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \ \mathcal{L} \tag{4}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \mathbf{g} \, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{5}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[3] N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999)

2D equation of state (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r},\tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_e = \bar{\psi}_s \left[\hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta}\, \psi_\downarrow\, \psi_\uparrow + \Delta \bar{\psi}_\uparrow\, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{\mathbf{g}} \; . \qquad (6)$$

We investigate the effect of fluctuations of the pairing field $\Delta(\mathbf{r},t)$ around its mean-field value Δ_0 which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r},\tau) = \Delta_0 + \eta(\mathbf{r},\tau) \; , \qquad (7)$$

where $\eta(\mathbf{r},\tau)$ is the complex field which describes pairing fluctuations.

2D equation of state (III)

In particular, we are interested in the grand potential Ω, given by

$$\Omega = -\frac{1}{\beta} \ln(\mathcal{Z}) \simeq -\frac{1}{\beta} \ln(\mathcal{Z}_{mf} \mathcal{Z}_g) = \Omega_{mf} + \Omega_g \ , \tag{8}$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp\left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\} \tag{9}$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\eta, \bar{\eta}] \ \exp\left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\} \tag{10}$$

is the partition function of Gaussian pairing fluctuations.

2D equation of state (IV)

After functional integration over quadratic fields, one finds that the mean-field grand potential reads[4]

$$\Omega_{mf} = -\frac{\Delta_0^2}{\mathbf{g}} L^2 + \sum_{\mathbf{k}} \left(\frac{\hbar^2 k^2}{2m} - \mu - E_{sp}(\mathbf{k}) - \frac{2}{\beta} \ln \left(1 + e^{-\beta\, E_{sp}(\mathbf{k})} \right) \right) \quad (11)$$

where

$$E_{sp}(\mathbf{k}) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta_0^2} \quad (12)$$

is the spectrum of fermionic single-particle excitations.

[4] A. Altland and B. Simons, Condensed Matter Field Theory (Cambridge Univ. Press, 2006).

2D equation of state (V)

The Gaussian grand potential is instead given by

$$\Omega_g = \frac{1}{2\beta} \sum_Q \ln \det(\mathbf{M}(Q)) \ , \tag{13}$$

where $\mathbf{M}(Q)$ is the inverse propagator of Gaussian fluctuations of pairs and $Q = (\mathbf{q}, i\Omega_m)$ is the 4D wavevector with $\Omega_m = 2\pi m/\beta$ the Matsubara frequencies and $\mathbf{q}$ the 3D wavevector.[5]

The sum over Matsubara frequencies is quite complicated and it does not give a simple expression. An approximate formula[6] is

$$\Omega_g \simeq \frac{1}{2} \sum_{\mathbf{q}} E_{col}(\mathbf{q}) + \frac{1}{\beta} \sum_{\mathbf{q}} \ln\left(1 - e^{-\beta\, E_{col}(\mathbf{q})}\right) \ , \tag{14}$$

where

$$E_{col}(\mathbf{q}) = \hbar\, \omega(\mathbf{q}) \tag{15}$$

is the spectrum of bosonic collective excitations with $\omega(\mathbf{q})$ derived from

$$\det(\mathbf{M}(\mathbf{q}, \omega)) = 0 \ . \tag{16}$$

[5] R.B. Diener, R. Sensarma, M. Randeria, PRA **77**, 023626 (2008).
[6] E. Taylor, A. Griffin, N. Fukushima, Y. Ohashi, PRA **74**, 063626 (2006).

2D equation of state (VI)

In our approach (Gaussian pair fluctuation theory[7]), given the grand potential

$$\Omega(\mu, L^2, T, \Delta_0) = \Omega_{mf}(\mu, L^2, T, \Delta_0) + \Omega_g(\mu, L^2, T, \Delta_0) \; , \qquad (17)$$

the energy gap Δ_0 is obtained from the (mean-field) gap equation

$$\frac{\partial \Omega_{mf}(\mu, L^2, T, \Delta_0)}{\partial \Delta_0} = 0 \; . \qquad (18)$$

The number density n is instead obtained from the number equation

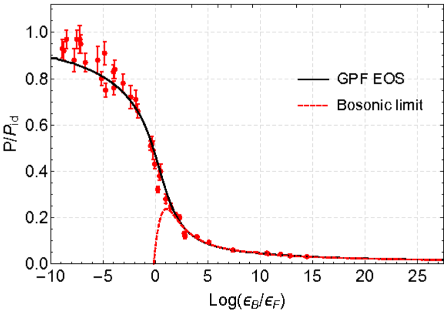
$$n = -\frac{1}{L^2} \frac{\partial \Omega(\mu, L^2, T, \Delta_0(\mu, T))}{\partial \mu} \qquad (19)$$

taking into account the gap equation, i.e. that Δ_0 depends on μ and T: $\Delta_0(\mu, T)$. Notice that the Nozieres and Schmitt-Rink approach[8] is quite similar but in the number equation it forgets that Δ_0 depends on μ.

[7] H. Hu, X-J. Liu, P.D. Drummond, EPL **74**, 574 (2006).

[8] P. Nozieres and S. Schmitt-Rink, JLTP **59**, 195 (1985).

Zero-temperature 2D results (I)



Scaled pressure P/P_{id} vs scaled binding energy ϵ_B/ϵ_F. Notice that $P = -\Omega/L^2$ and P_{id} is the pressure of the ideal 2D Fermi gas. Filled squares with error bars: experimental data of Makhalov *et al.* [9]. Solid line: the regularized Gaussian theory[10]. Dashed line: Popov equation of state of bosons with mass $m_B = 2m$.

[9] V. Makhalov et al. PRL **112**, 045301 (2014)

[10] G. Bighin and LS, PRB **93**, 014519 (2016); G. Bighin and LS, J. Supercond. Novel Magn. **29**, 3103 (2016).

Zero-temperature 2D results (II)

In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular[11], the binding energy $\epsilon_B > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length a_F as

$$\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m {a_F}^2} \, , \tag{20}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength $\mathbf{g}$ of s-wave pairing is related to the binding energy $\epsilon_B > 0$ of a fermion pair in vacuum by the expression[12]

$$-\frac{1}{\mathbf{g}} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_B} \, . \tag{21}$$

[11] C. Mora and Y. Castin, 2003, PRA **67**, 053615.
[12] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

Zero-temperature 2D results (III)

At zero temperature, including Gaussian fluctuations, the pressure is

$$P = -\frac{\Omega}{L^2} = \frac{mL^2}{2\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 + P_g(\mu, L^2, T=0) \ , \qquad (22)$$

with

$$P_g(\mu, L^2, T=0) = -\frac{1}{2}\sum_{\mathbf{q}} E_{col}(\mathbf{q}) \ . \qquad (23)$$

In the full 2D BCS-BEC crossover, from the regularized version of Eq. (13), we obtain numerically the zero-temperature pressure[13]
Notice that the energy of bosonic collective excitations becomes

$$E_{col}(\mathbf{q}) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda\frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)} \qquad (24)$$

in the deep BEC regime, with $\lambda = 1/4$ and $mc_s^2 = \mu + \epsilon_B/2$.

[13] G. Bighin and LS, PRB **93**, 014519 (2016).

Zero-temperature 2D results (IV)

In the deep BEC regime of the **2D BCS-BEC crossover**, where the chemical potential μ becomes strongly negative, the corresponding regularized pressure (dimensional regularization [14]) reads

$$P = \frac{m}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \ \ln\left(\frac{\epsilon_B}{2(\mu + \frac{1}{2}\epsilon_B)}\right) . \qquad (25)$$

This is exactly the Popov equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$, mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length a_B of composite boson as

$$\boxed{a_B = \frac{1}{2^{1/2}e^{1/4}} \ a_F \ .} \qquad (26)$$

The value $a_B/a_F = 1/(2^{1/2}e^{1/4}) \simeq 0.551$ is in full agreement with $a_B/a_F = 0.55(4)$ obtained by Monte Carlo calculations[15].

[14] LS and F. Toigo, PRA **91**, 011604(R) (2015); LS, PRL **118**, 130402 (2017).
[15] G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011).

Quantized vortices and 2D superfluid density (I)

At the beginning we have written the pairing field as

$$\Delta(\mathbf{r},\tau) = \Delta_0 + \eta(\mathbf{r},\tau) \ , \tag{27}$$

where $\eta(\mathbf{r},\tau)$ is the complex field of pairing fluctuations.
A quite different approach[16] is the following

$$\Delta(\mathbf{r},\tau) = (\Delta_0 + \sigma(\mathbf{r},\tau)) \ \ e^{i\theta(\mathbf{r},\tau)} \ , \tag{28}$$

where $\sigma(\mathbf{r},\tau)$ is the real field of amplitude fluctuations and $\theta(\mathbf{r},\tau)$ is the angular field of phase fluctuations.
However, Taylor-expanding the exponential of the phase, one has

$$(\Delta_0 + \sigma(\mathbf{r},\tau)) \ \ e^{i\theta(\mathbf{r},\tau)} = \Delta_0 + \sigma(\mathbf{r},\tau) + i \ \Delta_0 \ \theta(\mathbf{r},\tau) + ... \ . \tag{29}$$

Thus, at the Gaussian level, we can write

$$\eta(\mathbf{r},\tau) = \sigma(\mathbf{r},\tau) + i \ \Delta_0 \ \theta(\mathbf{r},\tau) \ . \tag{30}$$

[16] LS, P.A. Marchetti, and F. Toigo, PRA **88**, 053612 (2013).

Quantized vortices and 2D superfluid density (II)

After functional integration over $\sigma(\mathbf{r},\tau)$, the Gaussian action becomes

$$S_g = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \, \left\{ \frac{J}{2} (\nabla\theta)^2 + \frac{\chi}{2} \left(\frac{\partial\theta}{\partial\tau}\right)^2 \right\} \tag{31}$$

where J is the phase stiffness and χ is the compressibility. The superfluid density is related to the phase stiffness J by the simple formula

$$n_s = \frac{4m}{\hbar^2} J \, . \tag{32}$$

At the Gaussian level J depends only on fermionic single-particle excitations $E_{sp}(k)$.[17] **Beyond the Gaussian level** also bosonic collective excitations $E_{col}(q)$ contribute.[18] Thus, we assume the following Landau-type formula

$$n_s(T) = n - \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)}+1)^2} - \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)}-1)^2} \, . \tag{33}$$

[17] E. Babaev and H.K. Kleinert, PRB **59**, 12083 (1999).
[18] L. Benfatto, A. Toschi, and S. Caprara, PRB **69**, 184510 (2004).

Quantized vortices and 2D superfluid density (III)

It is important to stress that the compactness of the phase angle $\theta(\mathbf{r})$ implies that

$$\oint_{\mathcal{C}} \boldsymbol{\nabla}\theta(\mathbf{r}) \cdot d\mathbf{r} = 2\pi \sum_i q_i \ , \tag{34}$$

where q_i is the integer number associated to **quantized vortices** ($q_i > 0$) and **antivortices** ($q_i < 0$) encircled by $\mathcal{C}$. One can write[19]

$$\boldsymbol{\nabla}\theta(\mathbf{r}) = \boldsymbol{\nabla}\theta_0(\mathbf{r}) - \boldsymbol{\nabla} \wedge (\mathbf{u}_z \ \psi_v(\mathbf{r})) \tag{35}$$

where $\boldsymbol{\nabla}\theta_0(\mathbf{r})$ has zero circulation (no vortices) while $\psi_v(\mathbf{r})$ encodes the contribution of **quantized vortices and anti-vortices**, namely

$$\psi_v(\mathbf{r}) = \sum_i q_i \ln\left(\frac{|\mathbf{r} - \mathbf{r}_i|}{\xi}\right) \ , \tag{36}$$

where $\mathbf{r}_i$ is the position of the i-th vortex and ξ is a cutoff length.

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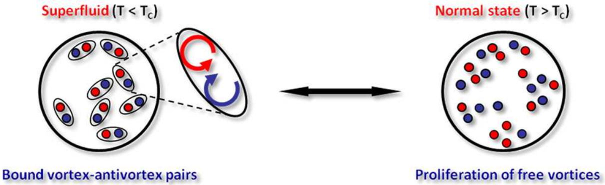
[19] Alternatively, one has $\theta(\mathbf{r}) = \theta_0(\mathbf{r}) + \theta_v(\mathbf{r})$ with $\theta_v(\mathbf{r}) = \sum_i q_i \arctan\left(\frac{y-y_i}{x-x_i}\right)$ because $\boldsymbol{\nabla}\arctan(y/x) = -\boldsymbol{\nabla} \wedge (\mathbf{u}_z \ \ln(\sqrt{x^2+y^2}/\xi))$.

Quantized vortices and 2D superfluid density (IV)

The analysis of **Kosterlitz** and **Thouless**[20] on the 2D gas of quantized vortices shows that:

- As the temperature T increases vortices start to appear in vortex-antivortex pairs (mainly with $q = \pm 1$).
- The pairs are bound at low temperature until at the critical temperature $T_c = T_{BKT}$ an unbinding transition occurs above which a proliferation of free vortices and antivortices is predicted.
- The phase stiffness J and the vortex energy μ_v are renormalized.
- The renormalized superfluid density $n_{s,R} = J_R(4m/\hbar^2)$ decreases by increasing the temperature T and jumps to zero at $T_c = T_{BKT}$.



[20] J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).

Quantized vortices and 2D superfluid density (V)

The renormalized phase stiffness J_R is obtained from the bare one J by solving the Kosterlitz renormalization group equations[21].

$$\frac{\mathrm{d}}{\mathrm{d}\ell}K(\ell) = -4\pi^3 K(\ell)^2 y(\ell)^2 \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}\ell}y(\ell) = (2-\pi K(\ell))\, y(\ell) \tag{38}$$

for the running variables $K(\ell)$ and $y(\ell)$, as a function of the adimensional scale ℓ subjected to the initial conditions $K(\ell=0)=J/\beta$ and $y(\ell=0)=\exp(-\beta\mu_v)$, with $\mu_v=\pi^2 J/4$ the **vortex energy**.[22] The renormalized phase stiffness is then

$$J_R = \beta\ K(\ell=+\infty)\ , \tag{39}$$

and the corresponding renormalized superfluid density reads

$$n_{s,R} = \frac{4m}{\hbar^2} J_R\ . \tag{40}$$

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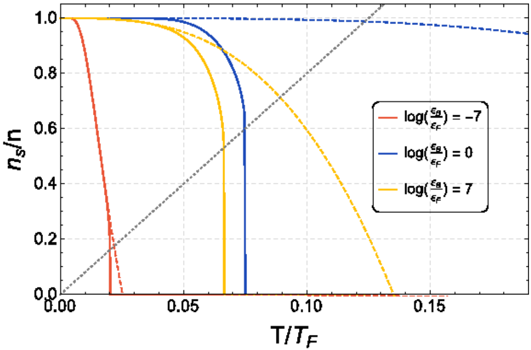
[21] D.R. Nelson and J.M. Kosterlitz, PRL **39**, 1201 (1977)

[22] W. Zhang, G.D. Lin, and L.M. Duan, PRA **78**, 043617 (2008).

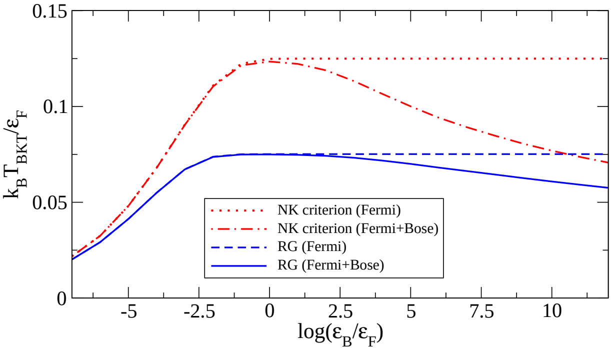
Quantized vortices and 2D superfluid density (VI)



Superfluid fraction n_s/n vs scaled temperature T/T_F in the 2D BEC-BEC crossover.[23] Solid lines: renormalized superfluid density. Dashed lines: bare superfluid density. $T_F = \epsilon_F/k_B$ is the Fermi temperature. Gray dotted line: Kosterlitz-Nelson condition $k_B T = (\pi/2)J(T) = (\hbar^2\pi/(8m))n_s(T)$.

[23] G. Bighin and LS, Sci. Rep. **7**, 45702 (2017).

Quantized vortices and 2D superfluid density (VII)



Theoretical predictions for the Berezinskii-Kosterlitz-Thouless (BTK) critical temperature T_{BKT}. Red lines obtained by using[24] the Nelson-Kosterlitz (NK) criterion on the bare superfluid density: $k_B T_{BKT} = (\hbar^2 \pi/(8m)) n_s(T_{BKT})$. Blue lines obtained by solving[25] the renormalization group (RG) equations of Kosterlitz.

[24] G. Bighin and LS, PRB **93**, 014519 (2016).

[25] G. Bighin and LS, Sci. Rep. **7**, 45702 (2017).

Conclusions

- After regularization[26] beyond-mean-field Gaussian fluctuations give remarkable effects for superfluid fermions in the 2D BCS-BEC crossover at zero temperature:
  – logarithmic behavior of the equation of state in the deep BEC regime
  – good agreement with (quasi) zero-temperature experimental data
- Also at finite temperature beyond-mean-field effects, with the inclusion of **quantized vortices and antivortices**, become relevant in the strong-coupling regime of 2D BCS-BEC crossover:
  – bare n_s and renormalized $n_{s,R}$ superfluid density
  – Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT}
- Finite-range effects of the inter-atomic potential could be included within an effective-field-theory (EFT) approach.[27]

[26] For a recent comprehensive review see LS and F. Toigo, Phys. Rep. **640**, 1 (2016).
[27] EFT for 2D dilute bosons: LS, PRL **118**, 130402 (2017).

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