Euclidean Partons?

Giancarlo Rossi, Massimo Testa

The Classical Light-Cone approach

Inclusive DIS *cross-section*

$$W(q^2, q \cdot p) = \int d^4x \, e^{iq\cdot x} \langle p|j(x)j(0)|p\rangle$$

In the Bjorken limit

$$q^2 \to -\infty$$
$$\omega = -\frac{q^2}{2p \cdot q}$$

the process is dominated by the Light-Cone singularities

$$\langle p|j(x)j(0)|p\rangle \approx_{x^2\approx 0}$$
$$\Delta(x^2) \sum_{n=0}^{+\infty} \alpha_n(\mu^2 x^2) x^{\mu_1} \ldots x^{\mu_n} \langle p|\tilde{O}^{(n)}_{\mu_1\ldots\mu_n}(0)|p\rangle$$

where $\tilde{O}^{(n)}_{\mu_1\ldots\mu_n}(0)$ denote a renormalized version of

$$O^{(n)}_{\mu_1\ldots\mu_n}(0) = \phi(0)\partial_{\mu_1} \ldots \partial_{\mu_n}\phi(0)$$

We stress the point that the

$O^{(n)}_{\mu_1 \ldots \mu_n}(0)$'s are **not multiplicatively renormalizable**

This is what makes it difficult to compute directly the moments of the structure functions

After renormalization we have

$$\langle p|\tilde{O}^{(n)}_{\mu_1 \ldots \mu_n}(0)|p\rangle =$$
$$= A^{(n)}(\mu)\, p_{\mu_1} \ldots p_{\mu_n} + B^{(n)}(\mu) p_{\mu_1} \ldots g_{\mu_i \mu_j} \ldots p_{\mu_n}$$

$$W(q^2, q \cdot p) \to_{Bj} \frac{\omega f(\omega, q^2)}{-q^2}$$

We will consider $-q^2 = \mu^2$ (Evolution)

We have

$$A^{(n)}(\mu) = \int_{-1}^{+1} d\omega f(\omega, \mu^2)\omega^n = \int_{-\infty}^{+\infty} d\omega f(\omega, \mu^2)\omega^n$$

The $A^{(n)}(\mu)$ are measured quantities and can be computed on the lattice as matrix elements of appropriately renormalized local operators.

Partons from Lattice QCD

The deep inelastic scattering process cannot be simulated in the euclidean region starting with the currents, but a very interesting proposal by Xiangdong Ji uses the bilocal operators

The basic formula of the approach is based on the $P_z \to \infty$ limit

$$f(\omega) = \lim_{P_z \to \infty} \tilde{F}(\omega, P_z)$$

where

$$\tilde{F}(\omega, P_z) = \frac{P_z}{2\pi} \int_{-\infty}^{+\infty} dz e^{iP_z z\omega} \langle P_z | \phi(0)\phi(z) | P_z \rangle =$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tilde{z} e^{i\tilde{z}\omega} \langle P_z | \phi(0)\phi(\tilde{z}/P_z) | P_z \rangle \,,$$

where $\tilde{z} \equiv P_z z$

How can we be sure that $f(\omega)$ is the correct structure function?
It must satisfy **necessary** (and **sufficient**) conditions

1. $f(\omega)$ must be u.v. finite;
2. the support of $f(\omega)$ must be contained in $(-1, +1)$
3. Its moments must be related to the matrix elements of the renormalized local operators generated by the bilocal

$$A^{(n)}(\mu) = \int_{-1}^{+1} d\omega f(\omega, \mu^2)\omega^n = \int_{-\infty}^{+\infty} d\omega f(\omega, \mu^2)\omega^n$$

As for the condition 1, $\langle P_z|\phi(0)\phi(z)|P_z\rangle$ can be easily made u.v. finite through an harmless logarithmic wave function renormalization.
After that it becomes a well defined distribution and is only logarithmically divergent as $z \to 0$.
Therefore the Fourier transform of the renormalized bilocal, $\tilde{F}(\omega, P_z)$, is u.v. finite

Condition 2 on the support is difficult to check
We will assume it is satisfied

Condition 3 is more tricky

We start from the definition

$$\tilde{F}(\omega, P_z) = \frac{P_z}{2\pi} \int_{-\infty}^{+\infty} dz e^{izP_z\omega} \langle P_z | \phi(0)\phi(z) | P_z \rangle =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tilde{z} e^{i\tilde{z}\omega} \langle P_z | \phi(0)\phi(\tilde{z}/P_z) | P_z \rangle$$

and invert it

$$\langle P_z|\phi(0)\phi(\tilde{z}/P_z)|P_z\rangle = \int_{-\infty}^{+\infty} d\omega e^{-i\tilde{z}\omega}\tilde{F}(\omega, P_z)$$

We can take the n-th derivative with respect to $\tilde{z}$ at $\tilde{z} = 0$

$$(-i)^n \int_{-\infty}^{+\infty} d\omega\ \omega^n \tilde{F}(\omega, P_z) = \frac{1}{(P_z)^n}\langle P_z|\phi(0)\frac{\partial^n \phi}{\partial z^n}(0)|P_z\rangle\,,$$

which clearly shows the origin of the u.v. divergencies coming from power divergent trace terms
This argument shows that, even if $\langle P_z|\phi(0)\phi(z)|P_z\rangle$ is only logarithmically divergent as $z \to 0$, the moments of $\tilde{F}(\omega, P_z)$ will, in general, exhibit power divergencies: the moments are not quantities of a distribution-theoretical nature.
A simple example of what happens is provided by

$$\langle P_z|\phi(0)\phi(\tilde{z}/P_z)|P_z\rangle \approx \log|z|$$

which shows how the bare local operators are more and more divergent with increasing n

Matching

The approach proposed by Ji does not identify directly the Fourier transform of the bilocal with the physical structure function. In fact there one starts with the Fourier transform in the presence of the regulator $\Lambda \approx 1/a$

$$\tilde{F}(\omega, P_z, \Lambda) = \frac{P_z}{2\pi} \int_{-\infty}^{+\infty} dz e^{iP_z \omega z} \langle P_z | \phi(0) \phi(z) | P_z \rangle |_\Lambda$$

a quantity denoted as a Quasi-PDF. As already discussed the moments of the Quasi-PDF are u.v. divergent

A "matching procedure" is then applied to $\tilde{F}(\omega, P_z, \Lambda)$ through a condition of the form

$$F(\omega, \mu) = \int_{\omega}^{+\infty} \frac{dx}{x} Z(\frac{\omega}{x}, \Lambda, P_z) \tilde{F}(x, P_z, \Lambda)$$

where $Z(\frac{\omega}{x}, \Lambda, P_z)$ is computed in perturbation theory through the requirement that $F(x, \mu)$ be u.v. finite

However the convolution property of the Mellin transform implies

$$\int_0^{+\infty} d\omega F(\omega,\mu)\omega^n = \int_0^{+\infty} dx x^n Z(x,\Lambda) \int_0^{+\infty} dx \tilde{F}(x,P_z,\Lambda)x^n =$$
$$\equiv Z_n\left(\frac{\Lambda}{\mu}\right) \int_0^{+\infty} dx \tilde{F}(x,P_z,\Lambda)x^n$$

which reads

$$\int_0^{+\infty} d\omega F(\omega,\mu)\omega^n = \frac{Z_n(\Lambda/\mu)}{(P_z)^n} \langle P_z|\phi(0)\frac{\partial^n\phi}{\partial z^n}(0)|P_z\rangle|_\Lambda$$

This clearly shows the multiplicative nature of the matching condition. The problem is that the Z_n should be the renormalization constants which make the operators

$$Z_n(\Lambda/\mu)\,\phi(0)\frac{\partial^n\phi}{\partial z^n}(0)$$

finite. However these operators are not multiplicatively renormalizable due to the presence of divergent trace terms, which require actual subtractions and not only multiplications

IN CONCLUSION

THE STRATEGY TO COMPUTE STRUCTURE FUNCTIONS FROM LATTICE QCD STILL REQUIRES SOME CONSIDERATION

THANK YOU FOR YOUR ATTENTION

Truncation

Suppose that, in order to solve this problem, we truncate the structure functions saying that we only consider the restriction of the structure function to the interval $\omega \in (-1, +1)$.

In other words we compute the moments as $\int_{-1}^{+1} d\omega e^{-i\tilde{z}\omega} \tilde{F}(\omega, P_z)$.

We have

$$\int_{-1}^{+1} d\omega e^{-i\tilde{z}\omega} \tilde{F}(\omega, P_z) = \frac{1}{2\pi} \int_{-1}^{+1} d\omega \int_{-\infty}^{+\infty} d\tilde{z}' e^{-i(\tilde{z}-\tilde{z}')\omega} \langle P_z | \phi(0)\phi(\tilde{z}'/P_z) | P_z \rangle =$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} d\tilde{z}' \, \frac{\sin(\tilde{z}-\tilde{z}')}{\tilde{z}-\tilde{z}'} \langle P_z | \phi(0)\phi(\tilde{z}'/P_z) | P_z \rangle$$

so that the computation of a moment restricted to $(-1, +1)$ corresponds to the matrix element

$$(-i)^n \int_{-1}^{+1} d\omega \, \omega^n \tilde{F}(\omega, P_z) =$$

$$= \frac{d^n}{d\tilde{z}^n} \Big[\frac{1}{\pi} \int_{-\infty}^{+\infty} d\tilde{z}' \, \frac{\sin(\tilde{z}-\tilde{z}')}{\tilde{z}-\tilde{z}'} \langle P_z | \phi(0)\phi(\tilde{z}'/P_z) | P_z \rangle \Big]_{\tilde{z}=0}$$

which is not local any more.