

Hadronic light-by-light contribution to $(g - 2)_\mu$: dispersive calculation of the 2π -contribution

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Outline

Introduction: $(g - 2)_\mu$ and hadronic light-by-light (HLbL)

Approaches to the calculation of HLbL

Setting up the stage:

Gauge invariance and crossing symmetry for the HLbL tensor

Master Formula

A dispersion relation for HLbL

Numerics

- Pion box contribution

- Pion rescattering contribution

Total

Outlook and Conclusions

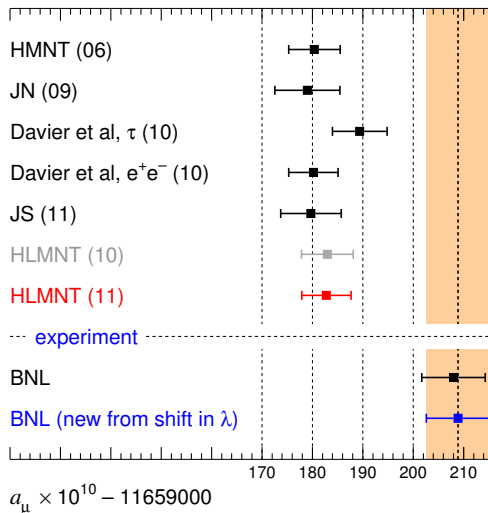
JHEP09(14)091, JHEP09(15)074, 1701.06554, 1702.07347

in collab. with M. Hoferichter, M. Procura and P. Stoffer and

PLB738(2014)6 +B. Kubis

Status of $(g - 2)_\mu$, experiment vs SM

Hagiwara et al. 2012



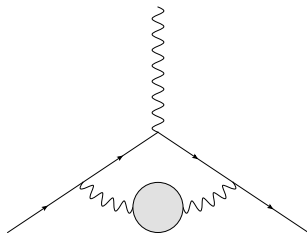
Status of $(g - 2)_\mu$, experiment vs SM

	$a_\mu [10^{-11}]$	$\Delta a_\mu [10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.95	0.04
electroweak, total	153.6	1.0
HVP (LO) [Hagiwara et al. 11]	6 949.	43.
HVP (NLO) [Hagiwara et al. 11]	-98.	1.
HLbL [Jegerlehner-Nyffeler 09]	116.	40.
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.	2.
theory	116 591 855.	59.

Hadronic light-by-light: irreducible uncertainty?

- ▶ Hadronic contributions responsible for most of the theory uncertainty
- ▶ Hadronic vacuum polarization (HVP) can be systematically improved

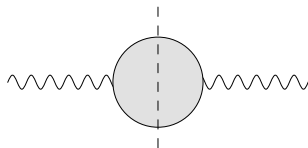
→ talk by Dominguez



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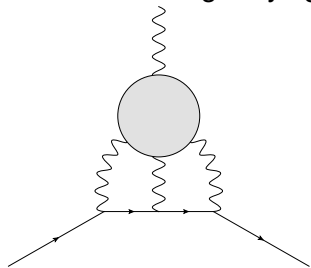
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- ▶ basic principles: unitarity and analyticity
- ▶ direct relation to experiment: total hadronic cross section
 $\sigma_{\text{tot}}(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$
- ▶ dedicated e^+e^- program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)

Hadronic light-by-light: irreducible uncertainty?

- ▶ Hadronic contributions responsible for most of the theory uncertainty
- ▶ Hadronic vacuum polarization (HVP) can be systematically improved → talk by Dominguez
- ▶ Hadronic light-by-light (HLbL) is more problematic:



- ▶ 4-point fct. of em currents in QCD
- ▶ *“it cannot be expressed in terms of measurable quantities”*
- ▶ up to now, only model calculations
- ▶ lattice QCD is making fast progress

Different evaluations of HLbL

Jegerlehner Nyffeler 2009

Table 13

Summary of the most recent results for the various contributions to $a_{\mu}^{\text{lbt;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	-	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	-	-	-	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	-	-	-	0 ± 10	-	-	-
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	-	22 ± 5	-	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	-	-	-	-	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	-	-	-	$2.3 \pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

- ▶ large uncertainties (and differences among calculations) in individual contributions
- ▶ pseudoscalar pole contributions most important
- ▶ second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*)
- ▶ heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)

Approaches to Hadronic light-by-light

► Model calculations

- ENJL Bijnens, Pallante, Prades (95-96)
- NJL and hidden gauge Hayakawa, Kinoshita, Sanda (95-96)
- nonlocal χ QM Dorokhov, Broniowski (08)
- AdS/CFT Cappiello, Cata, D'Ambrosio (10)
- Dyson-Schwinger Goecke, Fischer, Williams (11)
- constituent χ QM Greynat, de Rafael (12)
- resonances in the narrow-width limit Pauk, Vanderhaeghen (14)

► Impact of rigorously derived constraints

- high-energy constraints taken into account in several models above
addressed specifically by Knecht, Nyffeler (01)
- high-energy constraints related to the axial anomaly Melnikov, Vainshtein (04) and Nyffeler (09)
- sum rules for $\gamma^* \gamma \rightarrow X$ Pascalutsa, Pauk, Vanderhaeghen (12)
see also: workshop MesonNet (13)
- low-energy constraints—pion polarizabilities Engel, Ramsey-Musolf (13)

► Lattice

Blum et al. (05,12)

Our approach to hadronic light-by-light

We address the calculation of the hadronic light-by-light tensor

- ▶ model independent \Rightarrow rely on dispersion relations
- ▶ as data-driven as possible
- ▶ takes into account high-energy constraints
[OPE, perturbative QCD]
(work in progress, not discussed here)

Alternative dispersive approach for the μ -FF

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Hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2)$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- ▶ Lorentz invariance: 2 structures
- ▶ gauge invariance: reduction to 1 structure
- ▶ Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- ▶ $\bar{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\bar{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} dt \frac{\text{Im}\bar{\Pi}(t)}{t(t - q^2)}$$

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Easy!

The HLbL tensor (much less easy...)

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz e^{-i(x \cdot q_1 + y \cdot q_2 + z \cdot q_3)} \langle 0 | T \{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \quad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \quad t = (q_1 + q_3)^2 \quad u = (q_2 + q_3)^2$$

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General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi_{ijkl}^4 + \dots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \dots\}$, but in $d = 4$ only
136 are linearly independent

Eichmann et al. (14)

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$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz e^{-i(x \cdot q_1 + y \cdot q_2 + z \cdot q_3)} \langle 0 | T \{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

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Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

⇒ Apply the Bardeen-Tung (68) method + Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

- ▶ 43 basis tensors (BT) two less in $d = 4$
- ▶ 11 additional ones (T) to guarantee basis completeness everywhere
- ▶ of these 54 only 7 are completely independent

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

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$$T_1^{\mu\nu\lambda\sigma} = \epsilon^{\mu\nu\alpha\beta} \epsilon^{\lambda\sigma\gamma\delta} q_{1\alpha} q_{2\beta} q_{3\gamma} q_{4\delta},$$

$$T_4^{\mu\nu\lambda\sigma} = (q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}) (q_4^\lambda q_3^\sigma - q_3 \cdot q_4 g^{\lambda\sigma}),$$

$$T_7^{\mu\nu\lambda\sigma} = (q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}) (q_1 \cdot q_4 (q_1^\lambda q_3^\sigma - q_1 \cdot q_3 g^{\lambda\sigma}) + q_4^\lambda q_1^\sigma q_1 \cdot q_3 - q_1^\lambda q_1^\sigma q_3 \cdot q_4),$$

$$T_{19}^{\mu\nu\lambda\sigma} = (q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}) (q_2 \cdot q_4 (q_1^\lambda q_3^\sigma - q_1 \cdot q_3 g^{\lambda\sigma}) + q_4^\lambda q_2^\sigma q_1 \cdot q_3 - q_1^\lambda q_2^\sigma q_3 \cdot q_4),$$

$$T_{31}^{\mu\nu\lambda\sigma} = (q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}) (q_2^\lambda q_1 \cdot q_3 - q_1^\lambda q_2 \cdot q_3) (q_2^\sigma q_1 \cdot q_4 - q_1^\sigma q_2 \cdot q_4),$$

$$T_{37}^{\mu\nu\lambda\sigma} = (q_3^\mu q_1 \cdot q_4 - q_4^\mu q_1 \cdot q_3) (q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma} (q_4^\nu q_2 \cdot q_3 - q_3^\nu q_2 \cdot q_4) \\ + g^{\nu\sigma} (q_2^\lambda q_3 \cdot q_4 - q_4^\lambda q_2 \cdot q_3) + g^{\lambda\nu} (q_3^\sigma q_2 \cdot q_4 - q_2^\sigma q_3 \cdot q_4)),$$

$$T_{49}^{\mu\nu\lambda\sigma} = q_3^\sigma (q_1 \cdot q_3 q_2 \cdot q_4 q_4^\mu g^{\lambda\nu} - q_2 \cdot q_3 q_1 \cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu (q_1^\lambda q_2 \cdot q_3 - q_2^\lambda q_1 \cdot q_3) \\ + q_1 \cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2 \cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1 \cdot q_4 q_2 \cdot q_4 (q_3^\nu g^{\lambda\mu} - q_3^\mu g^{\lambda\nu})) \\ - q_4^\lambda (q_1 \cdot q_4 q_2 \cdot q_3 q_3^\mu g^{\nu\sigma} - q_2 \cdot q_4 q_1 \cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu (q_1^\sigma q_2 \cdot q_4 - q_2^\sigma q_1 \cdot q_4) \\ + q_1 \cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2 \cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1 \cdot q_3 q_2 \cdot q_3 (q_4^\nu g^{\mu\sigma} - q_4^\mu g^{\nu\sigma})) \\ + q_3 \cdot q_4 ((q_1^\lambda q_4^\mu - q_1 \cdot q_4 g^{\lambda\mu}) (q_3^\nu q_2^\sigma - q_2 \cdot q_3 g^{\nu\sigma}) - (q_2^\lambda q_4^\nu - q_2 \cdot q_4 g^{\lambda\nu}) (q_3^\mu q_1^\sigma - q_1 \cdot q_3 g^{\mu\sigma})).$$

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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions Π_i are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

HLbL contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2).$$

Contribution to a_μ :

$$m := m_\mu$$

$$a_\mu = \frac{-1}{48m} \text{Tr} \left\{ (\not{p} + m) [\gamma^\rho, \gamma^\sigma] (\not{p} + m) \Gamma_{\rho\sigma}^{\text{HLbL}}(p) \right\}$$

$$\Gamma_{\rho\sigma} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m) \gamma^\lambda (\not{p} - \not{q}_2 + m) \gamma^\nu}{((p + q_1)^2 - m^2) ((p - q_2)^2 - m^2)} \times$$

$$\times \left. \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) \right|_{k=0}$$

Thanks to BTT we can take the limit $k_\mu \rightarrow 0$ explicitly here
(no kinematic singularities!)

Master Formula

$$a_{\mu}^{\text{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_{\mu}^2][(p - q_2)^2 - m_{\mu}^2]}$$

- ▶ \hat{T}_i : known kernel functions
- ▶ $\hat{\Pi}_i$: linear combinations of the Π_i
- ▶ 5 integrals can be performed with Gegenbauer polynomial techniques

Master Formula

After performing the 5 integrations:

$$a_{\mu}^{\text{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^{\infty} dQ_1^4 \int_0^{\infty} dQ_2^4 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^{μ} are the **Wick-rotated** four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1| |Q_2| \tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

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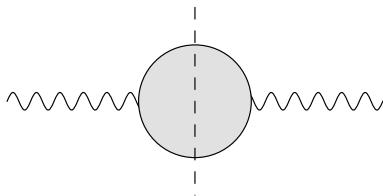
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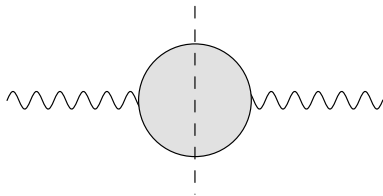
For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\text{Im}\Pi(q^2) \propto \sigma(e^+ e^- \rightarrow \text{hadrons})$$

Setting up the dispersive calculation

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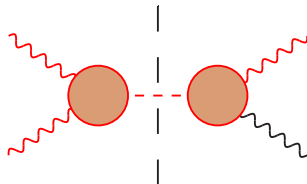
$$\text{Im}\Pi(q^2) \propto \sigma(e^+ e^- \rightarrow \text{hadrons})$$

For HLbL things are more complicated

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$



Pion pole: known

Projection on the BTT basis: done

Our master formula = explicit expressions in the literature

Input: pion transition form factor Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

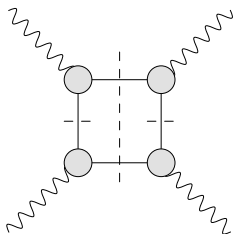
First results of direct lattice calculations Gerardin, Mayer, Nyffeler (2016)

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π -box with the BTT set:

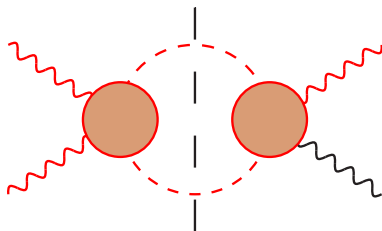


- we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole
- we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

Setting up the dispersive calculation

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The “rest” with 2π intermediate states has cuts only in one channel and will be
calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will not be discussed here

Partial wave expansion for 2π contributions

To complete the program of writing down a dispersion relation for two-pion contributions is not easy:

- ▶ unitarity relations are diagonal in a helicity amplitude basis;
- ▶ the helicity basis relevant for $(g - 2)_\mu$ is the one with one on-shell photon, which has 27 elements;
- ▶ in the limit $q_4^2, q_4^\sigma \rightarrow 0$ of the HLbL tensor the number of independent elements of the BTT set drops from 41 to 27;
- ▶ there is freedom in the choice of this subset (**singly-on-shell basis**);
- ▶ the transformation from helicity amplitudes to the singly-on-shell basis is easy to derive;
- ▶ inverting this relation (a 27×27 matrix) is a lot more complicated - **but we did it, analytically!**;
- ▶ the arbitrariness in the choice of the 27 elements of the singly-on-shell basis does not influence the final result **because of sum rules**

S-wave 2π contributions

$$\hat{\Pi}_4^S = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{-2}{\lambda_{12}(s')(s' - q_3^2)^2} \left(4s' \text{Im}h_{++,+}^0(s') - (s' + q_1^2 - q_2^2)(s' - q_1^2 + q_2^2) \text{Im}h_{00,+}^0(s') \right)$$

$$\hat{\Pi}_5^S = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{-2}{\lambda_{13}(t')(t' - q_2^2)^2} \left(4t' \text{Im}h_{++,+}^0(t') - (t' + q_1^2 - q_3^2)(t' - q_1^2 + q_3^2) \text{Im}h_{00,+}^0(t') \right)$$

$$\hat{\Pi}_6^S = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} du' \frac{-2}{\lambda_{23}(u')(u' - q_1^2)^2} \left(4u' \text{Im}h_{++,+}^0(u') - (u' + q_2^2 - q_3^2)(u' - q_2^2 + q_3^2) \text{Im}h_{00,+}^0(u') \right)$$

$$\hat{\Pi}_{11}^S = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} du' \frac{4}{\lambda_{23}(u')(u' - q_1^2)^2} \left(2 \text{Im}h_{++,+}^0(u') - (u' - q_2^2 - q_3^2) \text{Im}h_{00,+}^0(u') \right)$$

$$\hat{\Pi}_{16}^S = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{4}{\lambda_{13}(t')(t' - q_2^2)^2} \left(2 \text{Im}h_{++,+}^0(t') - (t' - q_1^2 - q_3^2) \text{Im}h_{00,+}^0(t') \right)$$

$$\hat{\Pi}_{17}^S = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{4}{\lambda_{12}(s')(s' - q_3^2)^2} \left(2 \text{Im}h_{++,+}^0(s') - (s' - q_1^2 - q_2^2) \text{Im}h_{00,+}^0(s') \right)$$

Analogous expressions for the D , G and all higher waves have been derived but are too long to be shown

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Numerics

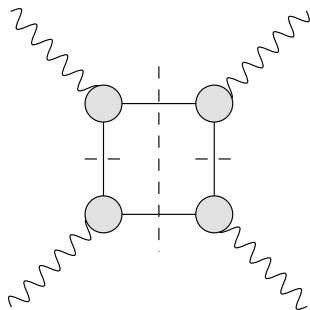
- Pion box contribution
- Pion rescattering contribution

Total

Outlook and Conclusions

Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$



Pion box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy l_i(x, y),$$

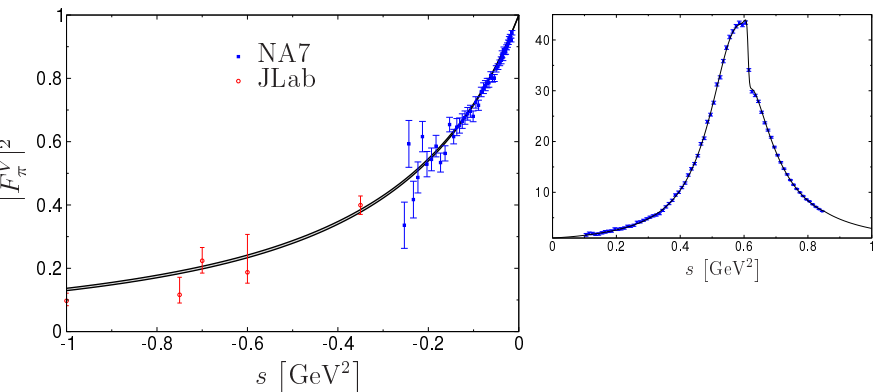
where

$$l_1(x, y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $l_{4,7,17,39,54}$ and

$$\begin{aligned}\Delta_{123} &= M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2, \\ \Delta_{23} &= M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2\end{aligned}$$

Pion box contribution



Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion box contribution

Table 13

Summary of the most recent results for the various contributions to $a_\mu^{\text{lbl;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	-	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	-	-	-	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	-	-	-	0 ± 10	-	-	-
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	-	22 ± 5	-	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	-	-	-	-	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	-	-	-	$2.3 \pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Uncertainties are negligibly small:

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Check of the partial-wave formalism

Comparison of the partial-wave expansion of the pion-box contribution to the full result

J_{\max}	$\delta_{J_{\max}}$	$\Delta_{J_{\max}}$
0	29.2%	55.4%
2	10.4%	20.9%
4	4.3%	11.0%
6	2.4%	6.2%
8	1.5%	3.7%
10	1.0%	2.4%
12	0.7%	1.6%
14	0.6%	1.1%
16	0.4%	0.7%

where

$$\delta_{J_{\max}} := 1 - \frac{a_{\mu, J_{\max}}^{\pi\text{-box, PW}}}{a_{\mu}^{\pi\text{-box}}} \quad \Delta_{J_{\max}} := \frac{\left| a_{\mu, J_{\max}}^{\pi\text{-box, PW}} - a_{\mu}^{\pi\text{-box}} \right|}{\left| a_{\mu}^{\pi\text{-box}} \right|}$$

First evaluation of S - wave 2π -rescattering

Based on:

- ▶ taking the pion pole as the only left-hand singularity
- ▶ \Rightarrow pion vector FF to describe the off-shell behaviour
- ▶ $\pi\pi$ phases obtained with the inverse amplitude method
[realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]
- ▶ numerical solution of the $\gamma^*\gamma^* \rightarrow \pi\pi$ dispersion relation

S -wave contributions:

	a_{μ}^{HLbL} in 10^{-11} units			
cutoff	1 GeV	1.5 GeV	2 GeV	∞
$l = 0$	-9.2	-9.5	-9.3	-8.8
$l = 2$	2.0	1.3	1.1	0.9
sum	-7.3	-8.3	-8.3	-7.9

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S -wave contributions:

$$a_{\mu, J=0}^{\pi\pi, \pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

a_{μ}^{HLbL} in 10^{-11} units

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Recall π -Box:

$$a_{\mu}^{\pi\text{-box}} = -15.9(2) \cdot 10^{-11}$$

First evaluation of S-wave 2π -rescattering

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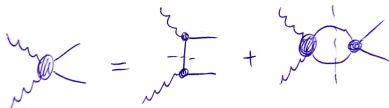
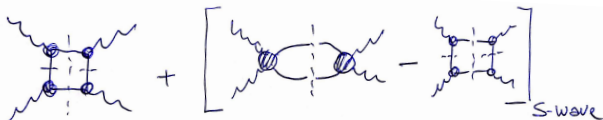
Our first numerical result

Handwritten Feynman diagrams illustrating the decomposition of a four-point function. The first diagram shows a four-point function with wavy external lines and a central box containing four vertices. This is followed by a plus sign and a bracketed sum of two diagrams: the first is a box diagram with two shaded vertices, and the second is a four-point function with a different internal structure. The bracketed sum is labeled "S-wave".

where

Handwritten Feynman diagram showing the decomposition of a shaded vertex. The shaded vertex is equal to the sum of two diagrams: a four-point function with a central box and two shaded vertices, and a four-point function with a shaded vertex and a shaded circle.

Our first numerical result



$$a_{\mu}^{\pi\text{-box}} + a_{\mu, J=0}^{\pi\pi, \pi\text{-pole LHC}} = -24(1) \cdot 10^{-11}$$

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Numerics

- Pion box contribution

- Pion rescattering contribution

Total

Outlook and Conclusions

Conclusions

- ▶ The HLbL contribution to $(g - 2)_\mu$ **can be** expressed in terms of measurable quantities in a **dispersive approach**
- ▶ **master formula**: HLbL contribution to a_μ as triple-integral over **scalar functions** which satisfy dispersion relations
- ▶ the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
 - ▶ single-pion contribution: **pion transition form factor**
 - ▶ pion-box contribution: **pion vector form factor**
 - ▶ 2-pion rescattering: $\gamma^* \gamma^{(*)} \rightarrow \pi\pi$ **helicity amplitudes**
- ▶ I have presented results for the pion-box and the S-wave pion-rescattering contributions:

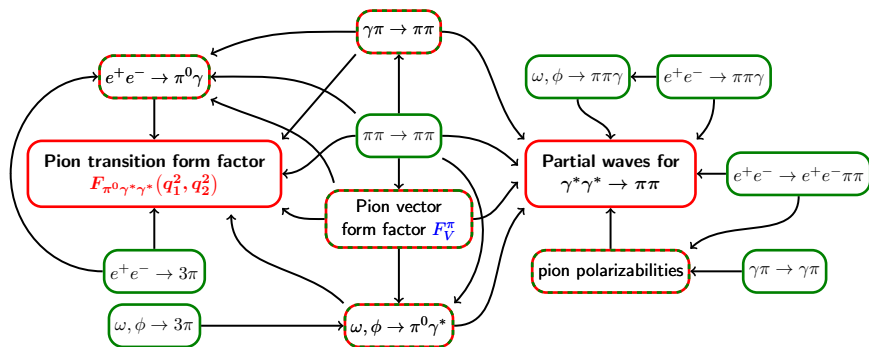
model independence = much reduced uncertainties

Outlook

- ▶ More work is needed to complete the evaluation of contributions of 2π intermediate states
 - ▶ take into account experimental constraints on $\gamma^{(*)}\gamma \rightarrow \pi\pi$
 - ▶ estimate the dependence on the q^2 of the second photon (theoretically, there are no data yet on $\gamma^*\gamma^* \rightarrow \pi\pi$)
 - ▶ \Rightarrow solve the dispersion relation for the **helicity amplitudes of $\gamma^*\gamma^* \rightarrow \pi\pi$** , including a full treatment of the LHC
- ▶ same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- ▶ short-distance constraints need to be derived and imposed

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer [arXiv:1408.2517](https://arxiv.org/abs/1408.2517) (PLB '14)



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among (lattice) theorists and experimentalists

Backup Slides

Detour: the subprocess $\gamma^* \gamma^* \rightarrow \pi \pi$

Consider $\gamma^*(q_1, \lambda_1) \gamma^*(q_2, \lambda_2) \rightarrow \pi^a(p_1) \pi^b(p_2)$:

$$W_{ab}^{\mu\nu}(p_1, p_2, q_1) = i \int d^4x e^{-iq_1 \cdot x} \langle \pi^a(p_1) \pi^b(p_2) | T \{ j_{\text{em}}^\mu(x) j_{\text{em}}^\nu(0) \} | 0 \rangle$$

General tensor decomposition ($q_i, i = 1, \dots, 3, q_3 = p_2 - p_1$):

$$W^{\mu\nu} = g^{\mu\nu} W_1 + \sum_{i,j} q_i^\mu q_j^\nu W_2^{ij}$$

gives **ten independent** scalar functions.

Gauge invariance requires:

$$q_1^\mu W_{\mu\nu} = q_2^\nu W_{\mu\nu} = 0$$

Gauge invariance: Bardeen-Tung-Tarrach approach

Consider the projector

Bardeen, Tung (68)

$$I^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}$$

which satisfies

$$I_\mu^\lambda W_{\lambda\nu} = W_{\mu\lambda} I^\lambda{}_\nu = W_{\mu\nu}, \quad q_1^\mu I_{\mu\nu} = q_2^\nu I_{\mu\nu} = 0$$

and contract it twice with $W_{\mu\nu}$, leaving it invariant:

$$W_{\mu\nu} = I_{\mu\mu'} I_{\nu'\nu} W^{\mu'\nu'} = \sum_{i=1}^5 \bar{T}_{\mu\nu}^i \bar{A}_i = \sum_{i=1}^5 T_{\mu\nu}^i A_i$$

The \bar{A}_i are free of kinematic singularities, but have zeros. To remove the zeros from the $\bar{A}_i \Rightarrow$ **remove the poles** from the $\bar{T}_i^{\mu\nu}$

Gauge invariance: Bardeen-Tung-Tarrach approach

$$T_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu,$$

$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu,$$

$$T_3^{\mu\nu} = q_1^2 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_1^\mu q_1^\nu,$$

$$T_4^{\mu\nu} = q_2^2 q_1 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu - q_1 \cdot q_3 q_2^\mu q_2^\nu,$$

$$T_5^{\mu\nu} = q_1 \cdot q_3 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_3^\nu - q_1 \cdot q_3 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_3^\mu q_1^\nu,$$

This is a basis of gauge-invariant tensors, but for $q_1 \cdot q_2 = 0$ it becomes degenerate: need one more structure:

Tarrach (75)

$$T_6^{\mu\nu} = (q_1^2 q_3^\mu - q_1 \cdot q_3 q_1^\mu) (q_2^2 q_3^\nu - q_2 \cdot q_3 q_2^\nu)$$

Inverse-amplitude method's input

