Canonical simulations of heavy-dense QCD without a sign problem

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Motivation for canonical formulation of QCD

- Consider the grand-canonical partition function of QCD:

$$Z_{\mathrm{GC}}^{\mathrm{QCD}}(\mu) = \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t(\mu)$$

- The sign problem of QCD is a manifestation of huge cancellations between different states:
  - all states are present for any μ and T
  - some states need to cancel out at different μ and T

- In the canonical formulation:

$$Z_{\mathrm{C}}^{\mathrm{QCD}}(N_Q) = \mathrm{Tr}_{N_Q}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t^{(N_Q)}$$

  - dimension of Fock space tremendously reduced
  - less cancellations necessary
  - e.g. $Z_{\mathrm{C}}^{\mathrm{QCD}}(N_Q) = 0$ for $N_Q \neq 0 \bmod N_c$

Motivation for canonical formulation of QCD

Canonical transfer matrices can be obtained explicitly!

- based on the dimensional reduction of the QCD fermion determinant [Alexandru, Wenger '10; Nagata, Nakamura '10]
- identification of transfer matrices [Steinhauer, Wenger '14]

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Outline:

- Definition of the transfer matrices for canonical QCD
- Explicit calculation in the heavy-dense limit
- Solution of the sign problem in the strong coupling limit
- Solution for the Potts model away from strong coupling

Dimensional reduction of QCD

- Consider the Wilson fermion matrix for a single quark with chemical potential μ:

$$M_{\pm}(\mu) = \begin{pmatrix} B_0 & P_+ A_0^+ & & & \pm P_- A_{L_t-1}^- \\ P_- A_0^- & B_1 & P_+ A_1^+ & & \\ & P_- A_1^- & B_2 & \ddots & \\ & & \ddots & \ddots & \\ & & & & P_+ A_{L_t-2}^+ \\ \pm P_+ A_{L_t-1}^+ & & & P_- & B_{L_t-1} \end{pmatrix}$$

  - B_t are (spatial) Wilson Dirac operators on time-slice t,
  - Dirac projectors $P_{\pm} = \frac{1}{2}(\mathbb{I} \mp \Gamma_4)$,
  - temporal hoppings are

$$A_t^+ = e^{+\mu} \cdot \mathbb{I}_{4\times 4} \otimes \mathcal{U}_t = \left(A_t^-\right)^{-1}$$

  - all blocks are $(4 \cdot N_c \cdot L_s^3 \times 4 \cdot N_c \cdot L_s^3)$-matrices

Dimensional reduction of QCD

- Reduced Wilson fermion determinant is given by

$$\det M_{p,a}(\mu) = \prod_t \det Q_t^+ \cdot \det\left[\mathbb{I} \pm \mathcal{T}\right]$$

where $\mathcal{T}$ is a product of transfer matrices given by

$$\mathcal{T} = e^{+\mu L_t} \prod_t \mathcal{U}_{t-1}^+ \cdot \left(Q_t^-\right)^{-1} \cdot Q_t^+ \cdot \mathcal{U}_t^-$$

with

$$Q_t^\pm = B_t P_\pm + P_\mp, \qquad \mathcal{U}_t^\pm = \mathcal{U}_t P_\pm + P_\mp$$

- Fugacity expansion yields with $N_Q^{\max} = 2 \cdot N_c \cdot L_s^3$

$$\det M_a(\mu) = \sum_{N_Q=-N_Q^{\max}}^{N_Q^{\max}} e^{\mu N_Q/T} \cdot \det M_{N_Q}$$

Canonical formulation of QCD

Canonical transfer matrices of QCD

$$\det M_{N_Q} = \prod_t \det Q_t^+ \cdot \sum_A \det \mathcal{T}^{\not{A}\not{A}} = \mathrm{Tr} \prod_t \mathcal{T}_t^{(N_Q)}$$

- sum is over all index sets $A \in \{1, 2, \ldots, 2N_Q^{\max}\}$ of size N_Q,
- i.e. the trace over the minor matrix of rank N_Q of $\mathcal{T}$

- Provides a complete temporal factorization of the fermion determinant.

Relation between quark and baryon number

- Consider $\mathbb{Z}(N_c)$-transformation by $z_k = e^{2\pi i \cdot k/N_c} \in \mathbb{Z}(N_c)$:

$$U_4(x) \to U_4(x)' = (1 + \delta_{x_4,t} \cdot (z_k - 1)) \cdot U_4(x)$$

- Hence, $\mathcal{U}_{x_4}$ transforms as $\mathcal{U}_{x_4} \to \mathcal{U}'_{x_4} = z_k \cdot \mathcal{U}_{x_4}$, while for all others $\mathcal{U}'_{t \neq x_4} = \mathcal{U}_{t \neq x_4}$.
- As a consequence we have

$$\begin{aligned} \det \mathcal{M}_{N_Q} \to \det \mathcal{M}'_{N_Q} &= \prod_t \det Q_t^+ \cdot \sum_A \det(z_k \cdot \mathcal{T})^{\bcancel{A}\bcancel{A}} \\ &= z_k^{-N_Q} \cdot \det \mathcal{M}_{N_Q} \end{aligned}$$

  and summing over z_k therefore yields

$$\det \mathcal{M}_{N_Q} = 0 \qquad \text{for } N_Q \neq 0 \bmod N_c$$

  - reduces cancellations by factor of N_c

Heavy-dense limit of grand-canonical QCD

- The heavy-dense approximation in general consists of taking the limit $\kappa \equiv (2m+8)^{-1} \to 0$, $\mu \to \infty$ while keeping $\kappa e^{+\mu}$ fixed.
- Better: just drop the spatial hopping terms, but keep forward and backward hopping in time:
  - system of static quarks and antiquarks
- Multiplying fermion matrix by 2κ we have

$$B_t \to \mathbb{I}, \qquad A_t^{\pm} \to 2\kappa \cdot A_t^{\pm} = 2\kappa e^{\pm\mu} \cdot \mathbb{I}_{4\times 4} \otimes \mathcal{U}_t^{\cdot/\dagger}$$

and the reduced Wilson fermion matrix in the HD limit

$$\det M_{p,a}^{HD} = \prod_{\bar{x}} \det\left[\mathbb{I} \pm (2\kappa e^{+\mu})^{L_t} P_{\bar{x}}\right]^2 \det\left[\mathbb{I} \pm (2\kappa e^{-\mu})^{L_t} P_{\bar{x}}^{\dagger}\right]^2$$

Heavy-dense limit of canonical QCD

- The canonical determinants are given by the trace over the minor matrix $\mathcal{M}$,

$$\det M_k^{HD} = (2\kappa)^{2N_c L_s^3 L_t} \cdot \mathrm{Tr}\,\mathcal{M}_k \left[\left((2\kappa)^{+L_t} \cdot P_+ \mathcal{P} + (2\kappa)^{-L_t} \cdot P_- \mathcal{P} \right) \right]$$

where $\mathcal{P}$ denotes the Polyakov loops $\mathcal{P}_{\bar{x},\bar{y}} = \mathbb{I}_{4\times4} \otimes P_{\bar{x}} \cdot \delta_{\bar{x},\bar{y}}$.

- For SU(3), the expressions of traces of minor matrices $\mathcal{M}$ are

$$\mathrm{Tr}\,\mathcal{M}_{k=0}(P_{\bar{x}}) = \det P_{\bar{x}} = 1\,,$$

$$\mathrm{Tr}\,\mathcal{M}_{k=1}(P_{\bar{x}}) = \sum_{i=1}^{3} \mathcal{M}(P_{\bar{x}})_{ii} = \mathrm{Tr}\,P_{\bar{x}}^{\dagger}\,,$$

$$\mathrm{Tr}\,\mathcal{M}_{k=2}(P_{\bar{x}}) = \sum_{i=1}^{3} \mathcal{M}(P_{\bar{x}})_{ii} = \mathrm{Tr}\,P_{\bar{x}}\,,$$

$$\mathrm{Tr}\,\mathcal{M}_{k=3}(P_{\bar{x}}) = 1\,.$$

Heavy-dense limit of canonical QCD

- Canonical determinant describing no quarks w.r.t. $N_Q^{\max}$:

$$\det M^{HD}_{N_Q^{\max}} = 1 \qquad \Leftrightarrow \qquad \text{quenched case}$$

- Canonical determinant describing a single quark, i.e. $N_Q = 1$:

$$\det M^{HD}_{N_Q^{\max}-1} = \left((2\kappa)^{L_t} + (2\kappa)^{-L_t}\right) \cdot \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}}$$

- For $N_Q = 2$ quarks:

$$\det M^{HD}_{N_Q^{\max}-2}/\Omega \propto 2 \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} + \left(4 \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} - 3 \sum_{\bar{x}} \left(\mathrm{Tr}\, P_{\bar{x}}\right)^2 + 2\,\mathrm{Tr}\, P^{\dagger}_{\bar{x}}\right)$$

- Both determinants vanish under global $\mathbb{Z}(3)$-transformations.

Heavy-dense limit of canonical QCD

- Canonical determinant $N_Q = 3$ quarks:

$$\det M^{HD}_{N_Q^{\max}3}/\Omega = h_3 \cdot \left(4\sum_{\bar{x}} \operatorname{Tr} P^\dagger_{\bar{x}} \sum_{\bar{y}} \operatorname{Tr} P_{\bar{y}} - 3\sum_{\bar{x}} \operatorname{Tr} P_{\bar{x}} \operatorname{Tr} P^\dagger_{\bar{x}} + 2L_s^3 \right)$$
$$+ h_1 \left(4\sum_{\bar{x}} \operatorname{Tr} P^\dagger_{\bar{x}} \sum_{\bar{y}} \operatorname{Tr} P_{\bar{y}} + 2\sum_{\bar{x}} (\operatorname{Tr} P_{\bar{x}})^2 \sum_{\bar{y}} \operatorname{Tr} P_{\bar{y}} \right.$$
$$\left. +4\sum_{\bar{x}} \operatorname{Tr} P_{\bar{x}} \sum_{\bar{y}\neq\bar{x}} \operatorname{Tr} P_{\bar{y}} \sum_{\bar{z}} \operatorname{Tr} P_{\bar{z}} \right)$$

  - describes the propagation of mesons and baryons

- Invariant under global $\mathbb{Z}(3)$-transformations

- Suffers from a severe sign problem, unless
  - all $P_{\bar{x}}$ align $\iff$ deconfined phase
  - global $\mathbb{Z}(3)$ is promoted to a local one $\iff$ strong coupling

Canonical single site determinants in the heavy-dense limit

- For numerical simulations we need the canonical determinants on single sites for arbitrary $k = N_Q$.
- From the reduced determinant we obtain

$$\det M_k^{HDSS} = (2\kappa)^{2N_c L_t} \cdot \mathrm{Tr}\,\mathcal{M}_k \left[\left((2\kappa)^{+L_t} \cdot P_+ \mathcal{P} + (2\kappa)^{-L_t} \cdot P_- \mathcal{P} \right) \right]$$

  - $\mathcal{P}$ is just a $4N_c \times 4N_c$ blockmatrix containing 4 copies of $P_{\bar{x}}$ along the diagonal
  - quark number index now runs over $k = 0, \ldots, 12$

- In the following, suppress $\Omega^{SS} = (2\kappa)^{2N_c L_t}$ and define

$$\det M_k^{HDSS} = \Omega^{SS} z_k$$

Canonical single site determinants in the heavy-dense limit

- Canonical determinants on single site (with $z_k^{HDSS} = (z_{12-k}^{HDSS})^*$):

$z_{k=0}^{HDSS} = 1$

Canonical single site determinants in the heavy-dense limit

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$z_{k=1}^{HDSS} = h_1 \cdot 2\,\mathrm{Tr}\,P_{\bar{x}}^{\dagger}$

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$$z_{k=2}^{HDSS} = h_2 \cdot \left\{2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right)^2\right\} + \left(\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right)^2$$

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$$z_{k=3}^{HDSS} = h_3 \cdot 2\left\{1 + \mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^\dagger\right\} + h_1\left\{2\,\mathrm{Tr}\,P_{\bar{x}}^\dagger\left(2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)^2\right)\right\}$$

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$$z_{k=4}^{HDSS} = h_4\left\{2\,\mathrm{Tr}\,P_{\bar{x}}^\dagger + (\mathrm{Tr}\,P_{\bar{x}})^2\right\} + h_2 \cdot 4\left\{\mathrm{Tr}\,P_{\bar{x}}^\dagger\left(1 + \mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)\right\} + \left(2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)^2\right)^2$$

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$$z_{k=5}^{HDSS} = h_5 \cdot 2\,\mathrm{Tr}\,P_{\bar{x}} + h_3 \left\{\left(2\,\mathrm{Tr}\,P_{\bar{x}}^\dagger + (\mathrm{Tr}\,P_{\bar{x}})^2\right) 2\,\mathrm{Tr}\,P_{\bar{x}}^\dagger\right\} + h_1 \cdot 2\left\{\left(1 + \mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)\left(2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)^2\right)\right\}$$

Canonical single site determinants in the heavy-dense limit

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$$z_{k=6}^{HDSS} = h_6 + h_4 \cdot 4\,\mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^\dagger + h_2\left\{\left(2\,\mathrm{Tr}\,P_{\bar{x}}^\dagger + (\mathrm{Tr}\,P_{\bar{x}})^2\right) \times \left(2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)^2\right)\right\} + 4\left(1 + \mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^\dagger\right)^2$$

Canonical single site determinants in the heavy-dense limit

- Relation $z_k^{HDSS} = \left(z_{12-k}^{HDSS}\right)^*$ implies $z_{k=6}^{HDSS} \in \mathbb{R}$, but in fact

$$z_{k=6}^{HDSS} = h_6 + h_4 \cdot 4\,\mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^{\dagger} + h_2 \left\{ \left(2\,\mathrm{Tr}\,P_{\bar{x}}^{\dagger} + (\mathrm{Tr}\,P_{\bar{x}})^2 \right) \times \left(2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right)^2 \right) \right\} + 4\left(1 + \mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right)^2$$

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$$\geq h_6 + 4 > 0$$

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$$\geq h_6 + 4 > 0$$

- Almost true for $z_{k=3}^{HDSS}$: only is non-positive

$$z_{k=3}^{HDSS} = h_3 \cdot 2\left\{1 + \mathrm{Tr}\,P_{\bar{x}}\,\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right\} + h_1 \left\{2\,\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\left(2\,\mathrm{Tr}\,P_{\bar{x}} + \left(\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right)^2\right)\right\}$$

  - only $\left(\mathrm{Tr}\,P_{\bar{x}}^{\dagger}\right)^3$ can become complex
  - suppressed by a factor $h_1/h_3 \sim (2\kappa)^{\pm 2L_t}$

Canonical single site determinants in the heavy-dense limit

- On a single site, $\mathbb{Z}(N_c)$-transformations projects onto

$$z_k^{HDSS} = 0 \qquad \text{for} \quad k \neq 0 \bmod N_c \,.$$

  - this is what happens in the strong coupling limit $\beta \to 0$

- Nontrivial determinants integrated over all values of $P_{\bar{x}}$:

$$\int dP_{\bar{x}} \det M_k^{HDSS} = \Omega^{SS} \begin{cases} 1\,, & k = 0, 12 \\ 4h_3 + 6h_1\,, & k = 3, 9 \\ h_6 + 4h_4 + 10h_2 + 20\,, & k = 6 \end{cases}$$

- Provides benchmark for numerical simulations:
  - no sign problem in the canonical formulation

Canonical single site determinants in the heavy-dense limit

- More interesting are $N_f = 2$ quark flavours:
  - canonical sectors have definite isospin or baryon charge (or both)
  - for simplicity assume degenerate masses $\kappa_u = \kappa_d = \kappa$
  - relabel $q \in \{-6, -5, \ldots, +5, +6\} \leftarrow k \in \{0, 1, \ldots, 12\}$
- Generically, in the grand-canonical case one has

$$\det \mathcal{M}^{HDSS}(\mu_u) \cdot \det \mathcal{M}^{HDSS}(\mu_d) = \sum_{q_u=-6}^{6} e^{\mu_u q_u L_t} \det M_{q_u}^{HDSS} \cdot \sum_{q_d=-6}^{6} e^{\mu_d q_d L_t} \det M_{q_d}^{HDSS}$$

while for fixed isospin charge only $n_I = q_u - q_d$ contribute

$$\det \mathcal{M}_{n_I}^{HDSS} = \sum_{\substack{q_u, q_d=-6 \\ n_I = q_u - q_d}}^{6} \det M_{q_u}^{HDSS} \cdot \det M_{q_d}^{HDSS}$$

Canonical single site determinants in the heavy-dense limit

- $\mathbb{Z}(N_c)$-symmetry implies the constraint $n_q \equiv q_u + q_d = 0 \bmod N_c$.
- Using $\det \mathcal{M}_{n_I}^{HDSS} \cdot (\Omega^{SS})^2 \cdot z_{n_I}$ and $L = \mathrm{Tr}\, P_{\bar{x}}$ we find

$z_{n_I=-12} = z_{-6} \cdot z_{+6} = 1$ $\qquad$ $n_q = 0$

- Canonical determinants for fixed isospin number

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- Using $\det \mathcal{M}_{n_I}^{HDSS} \cdot (\Omega^{SS})^2 \cdot z_{n_I}$ and $L = \mathrm{Tr}\, P_{\bar{x}}$ we find

$$z_{n_I=-12} = z_{-6} \cdot z_{+6} = 1 \qquad n_q = 0$$

$$z_{n_I=-11} = 0$$

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$$z_{n_I=-11} = 0$$

$$z_{n_I=-10} = z_{-5} \cdot z_{+5} = 4h_1^2 |L|^2 \qquad n_q = 0$$

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$$z_{n_I=-12} = z_{-6} \cdot z_{+6} = 1 \qquad n_q = 0$$
$$z_{n_I=-11} = 0$$
$$z_{n_I=-10} = z_{-5} \cdot z_{+5} = 4h_1^2|L|^2 \qquad n_q = 0$$
$$z_{n_I=-9} = z_{-6} \cdot z_{+3} + z_{-3} \cdot z_{+6} = 2\mathrm{Re}\, z_{-3} \qquad n_q = -3, +3$$

- Canonical determinants for fixed isospin number

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- Using $\det \mathcal{M}_{n_I}^{HDSS} \cdot (\Omega^{SS})^2 \cdot z_{n_I}$ and $L = \mathrm{Tr}\, P_{\bar{x}}$ we find

$$
\begin{aligned}
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z_{n_I=-9} &= z_{-6} \cdot z_{+3} + z_{-3} \cdot z_{+6} = 2\mathrm{Re}\, z_{-3} && n_q = -3, +3 \\
&= 4h_3 \left\{1 + |L|^2\right\} + 4h_1 \left\{2|L|^2 + \mathrm{Re}\left[L^3\right]\right\}
\end{aligned}
$$

- Canonical determinants for fixed isospin number

Canonical single site determinants in the heavy-dense limit

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&= 4h_2^2|L|^2 + 4h_2(h_2+1)\mathrm{Re}\left[L^3\right] + (h_2+1)^2|L|^4
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z_{n_I=-3} &= {\color{red} z_{-6} \cdot z_{-3}} + z_{-3} \cdot z_0 + z_0 \cdot z_{+3} + z_{+3} \cdot z_{+6} && {\color{red} n_q = -9}, -3, +3, +9
\end{aligned}
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z_{n_I=-3} &= z_{-6} \cdot z_{-3} + z_{-3} \cdot z_0 + z_0 \cdot z_{+3} + z_{+3} \cdot z_{+6} && n_q = -9, -3, +3, +9 \\
z_{n_I=-2} &= z_{-4} \cdot z_{-2} + z_{-1} \cdot z_{+1} + z_{+2} \cdot z_{+4} && n_q = -6, 0, 6 \\
z_{n_I=-1} &= z_{-5} \cdot z_{-4} + z_{-2} \cdot z_{-1} + z_{+1} \cdot z_{+2} + z_{+4} \cdot z_{+5} && n_q = -9, -3, +3, +9 \\
z_{n_I=0} &= z_{-6} \cdot z_{-6} + z_{-3} \cdot z_{-3} + z_0 \cdot z_0 + z_{+3} \cdot z_{+3} + z_{+6} \cdot z_{+6} && \\
& && n_q = -12, -6, 0, +6, +12
\end{aligned}
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z_{n_l=-2} &= z_{-4} \cdot z_{-2} + z_{-1} \cdot z_{+1} + z_{+2} \cdot z_{+4} && n_q = -6, 0, 6 \\
z_{n_l=-1} &= z_{-5} \cdot z_{-4} + z_{-2} \cdot z_{-1} + z_{+1} \cdot z_{+2} + z_{+4} \cdot z_{+5} && n_q = -9, -3, +3, +9 \\
z_{n_l=0} &= z_{-6} \cdot z_{-6} + z_{-3} \cdot z_{-3} + z_0 \cdot z_0 + z_{+3} \cdot z_{+3} + z_{+6} \cdot z_{+6} && \\
& && n_q = -12, -6, 0, +6, +12
\end{aligned}
$$

- Fixing in addition $n_q = 0$ yields $z_{n_l} \geq 0$ positive, else almost.

Canonical single site determinants in the heavy-dense limit

- Similarly for fixed baryon number:

$z_{n_B=-12} = z_{-6} \cdot z_{-6}$

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z_{n_B=0} &= && = \textstyle\sum_{k=-6}^{+6} z_k \cdot z_{-k} \\
& && \geq 0
\end{aligned}
$$

The heavy-dense strong coupling limit $\beta \to 0$

- In the strong coupling limit the global $\mathbb{Z}(N_c)$-transformations are promoted to local ones:
  - define triality by the net number of $P_{\bar{x}}$ and $P^\dagger_{\bar{x}}$
  - only contributions with triality-0 survive:

$$
\begin{aligned}
1 &\qquad \text{empty site} \\
\mathrm{Tr}\, P_{\bar{x}} \cdot \mathrm{Tr}\, P^\dagger_{\bar{x}} &\qquad \text{single meson} \\
\left(\mathrm{Tr}\, P_{\bar{x}} \cdot \mathrm{Tr}\, P^\dagger_{\bar{x}}\right)^2 &\qquad \text{two mesons} \\
\left(\mathrm{Tr}\, P_{\bar{x}}\right)^3 &\qquad \text{baryon} \\
\left(\mathrm{Tr}\, P^\dagger_{\bar{x}}\right)^3 &\qquad \text{antibaryon}
\end{aligned}
$$

  - baryonic contributions complex, but very small compared to rest

The heavy-dense strong coupling limit $\beta \to 0$

- Partition function becomes a summation over all baryon configurations $n_B(\bar{x})$ with (essentially) positive contributions:

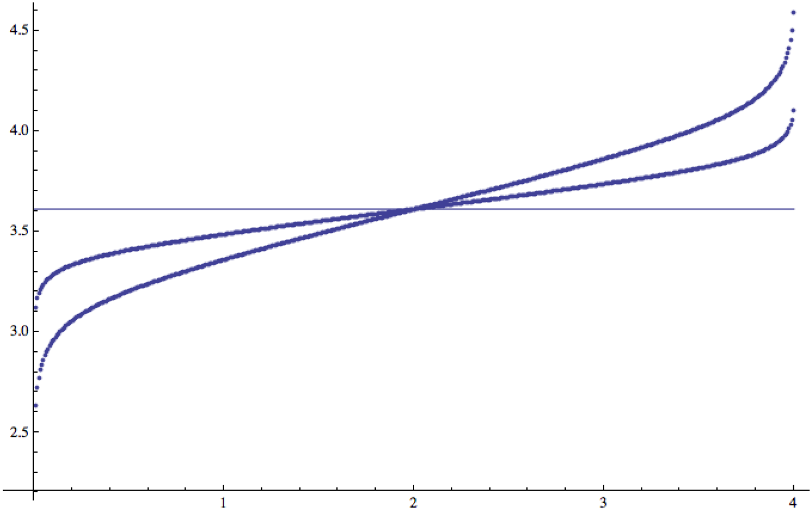
$$Z_C(N_B) = (2\kappa)^{2N_c L_t L_s^3} \cdot \sum_{\{n_B\}, |n_B| = N_B} \int \mathcal{D}U \prod_{\bar{x}} \det \mathcal{M}^{HDSS}_{n_B(\bar{x})}[\mathrm{Tr}\, P_{\bar{x}}]$$

  - $\mathcal{D}U$ can of course be integrated analytically,
  - but also possible to simulate by Monte Carlo

Sign problem is solved in the strong coupling limit!

The heavy-dense strong coupling limit $\beta \to 0$

- Baryon chemical potential as a function of baryon number:



The sign problem strikes back at $\beta > 0$

- Cf. e.g. canonical determinant for $n_f = 3$ quarks:

$$\det \mathcal{D}^{HD}_{n_f=3}/\Omega = h_3 \cdot \left(4 \sum_{\bar{x}} \mathrm{Tr}\, P^\dagger_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} - 3 \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}}\, \mathrm{Tr}\, P^\dagger_{\bar{x}} + 2 L_s^3 \right)$$

$$+ h_1 \left(4 \sum_{\bar{x}} \mathrm{Tr}\, P^\dagger_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} + 2 \sum_{\bar{x}} (\mathrm{Tr}\, P_{\bar{x}})^2 \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} \right.$$

$$\left. + 4 \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}} \sum_{\bar{y} \neq \bar{x}} \mathrm{Tr}\, P_{\bar{y}} \sum_{\bar{z}} \mathrm{Tr}\, P_{\bar{z}} \right)$$

  - describes the propagation of mesons and baryons

- Invariant under global $\mathbb{Z}(3)$-transformations

- Suffers from a severe sign problem, unless
  - all $P_{\bar{x}}$ align $\Longleftrightarrow$ deconfined phase
  - global $\mathbb{Z}(3)$ is promoted to a local one $\Longleftrightarrow$ strong coupling

Possible solution for $\beta > 0$

- Use the 3-state Potts model in 3d as a proxy for the effective Polyakov loop action of heavy-dense QCD.
- Canonical partition function for N_Q quarks:

$$Z_C(N_q) = \sum_{\{n\},|n|=N_Q} \int \mathcal{D}z \exp(-S[z]) \cdot \prod_x f[z_x, n_x]$$

  - Polyakov loops are represented by the Potts spins $z_x \in \mathbb{Z}(3)$
  - standard nearest-neighbour interaction

$$S[z] = -\beta \sum_{\langle xy \rangle} \delta_{z_x, z_y}$$

  - local quark occupation number $n_x \leq n_x^{\max}$ with $|n| = N_Q$
  - use the simple local fermionic weights

$$f[z, n] = z^n$$

The 3-state Potts model in $d = 3$ dimensions

Canonical partition function

$$Z_C(N_Q) = \sum_{\{n\}} \int \mathcal{D}z \exp(\beta \sum_{\langle xy \rangle} \delta_{z_x, z_y}) \prod_x z_x^{n_x}$$

- Action is manifestly complex $\Rightarrow$ fermion sign problem!
- Global $\mathbb{Z}(3)$ symmetry ensures $Z_C(N_Q \neq 0 \bmod 3) = 0$:
  - projection onto integer baryon numbers
- In the limit $\beta \to 0$, the global $\mathbb{Z}(3)$ becomes a local one:
  - projection onto integer baryon numbers on single sites

$$n_x = 0 \bmod 3 \quad (\text{limit}\, \beta \to 0)$$

  - sign problem is absent

The 3-state Potts model in $d = 3$ dimensions

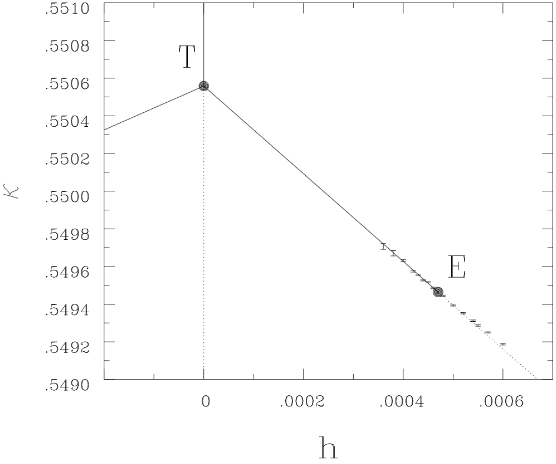
Canonical partition function

$$Z_C(N_Q) = \sum_{\{n\}} \int \mathcal{D}z \exp\Big(\beta \sum_{\langle xy \rangle} \delta_{z_x, z_y}\Big) \prod_x z_x^{n_x}$$

- Action is manifestly complex ⇒ fermion sign problem!
- Global $\mathbb{Z}(3)$ symmetry ensures $Z_C(N_Q \neq 0 \bmod 3) = 0$:
  - projection onto integer baryon numbers
- At $\beta > 0$ sign problem can be solved using cluster algorithm:
  - only clusters with integer baryon number are nonzero
    ⇒ confinement
  - quarks can move freely within the cluster
    ⇒ deconfinement within cluster

Physics of the 3-state Potts model

- Phase diagram in the $(e^{\mu}, \gamma) \equiv (h, \kappa)$-plane:
  [Alford, Chandrasekharan, Cox and Wiese 2001]



- deconfinement phase transition at $T = (0, 0.550565(10))$

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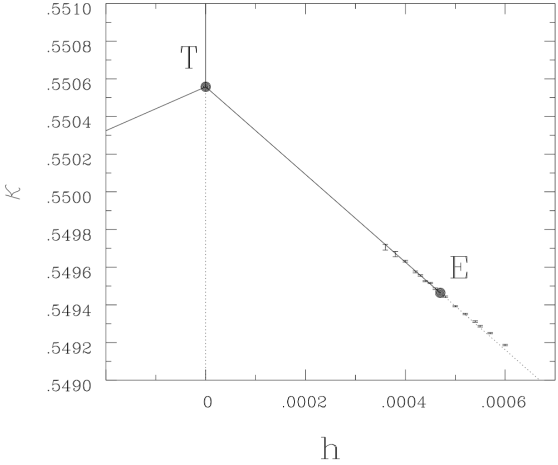

  - line of first order phase transitions from T to E

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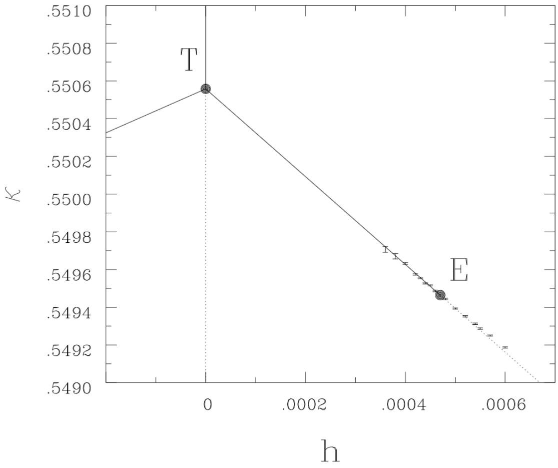
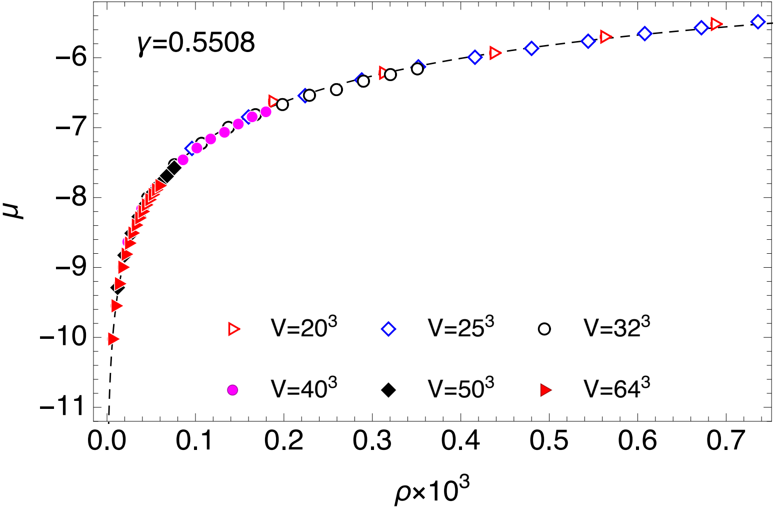


  - critical endpoint $E = (0.000470(2), 0.549463(13))$

Canonical formulation of the 3-state Potts model

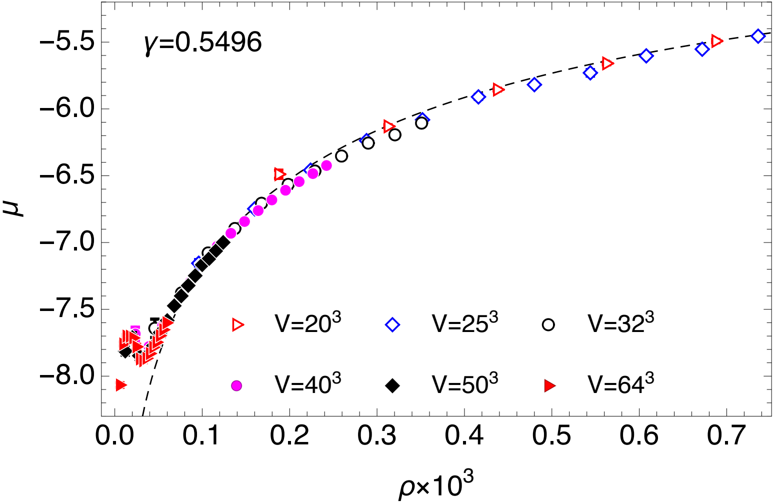
- Canonical simulation results in the deconfined phase:



  - description in terms of a gas of (free) quarks

Canonical formulation of the 3-state Potts model

- Results from below the deconfinement transition:



- transition from the confined into the deconfined phase

Canonical formulation of the 3-state Potts model

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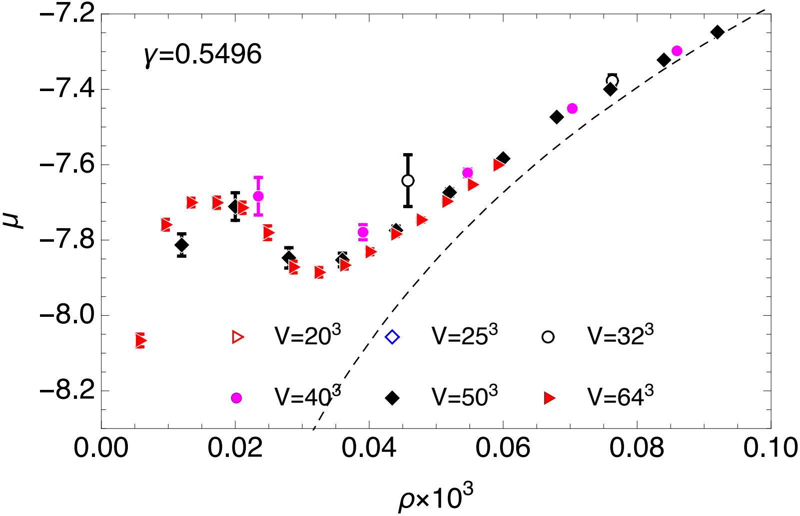

  - typical signature of a 1st order phase transition

Canonical formulation of the 3-state Potts model

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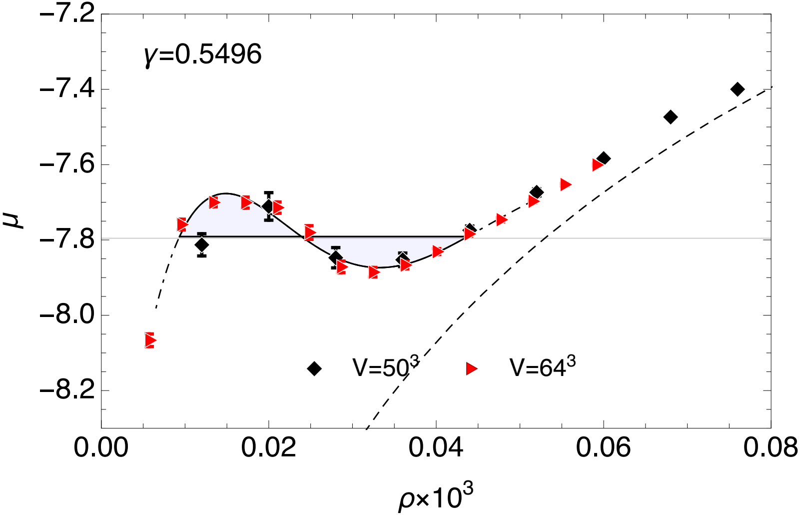

  - Maxwell construction yields critical μ_c

Canonical formulation of the 3-state Potts model

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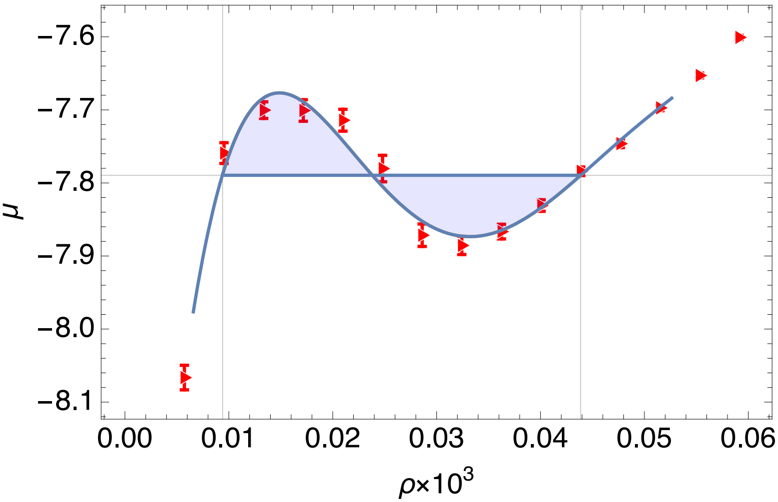

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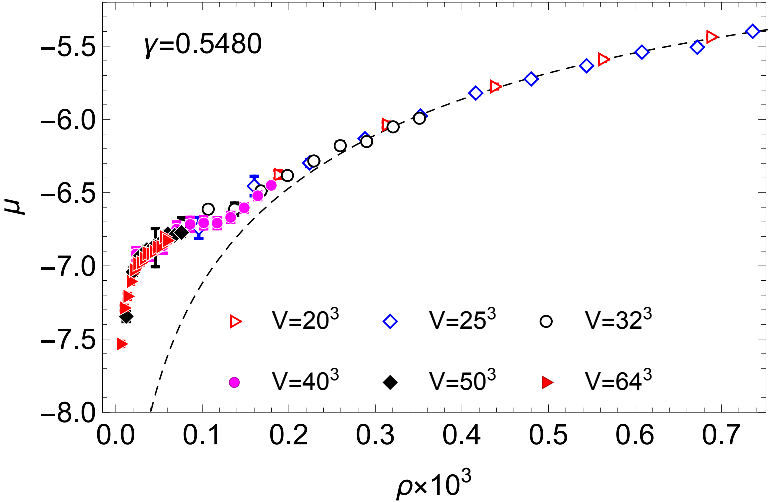

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Canonical formulation of the 3-state Potts model

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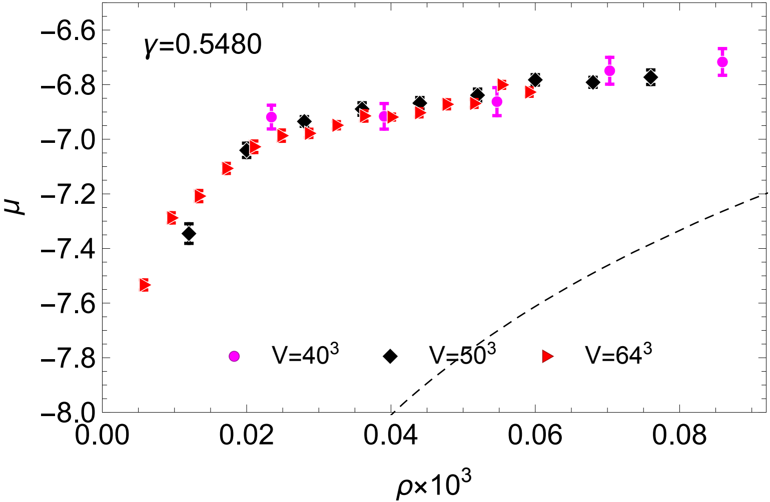

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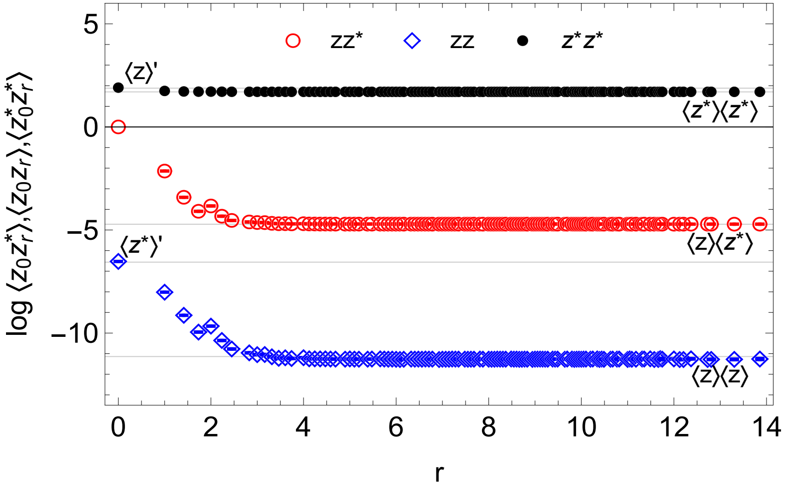
Canonical formulation of the 3-state Potts model

- (Anti)Quark-(anti)quark potentials at low temperature:



- confined phase: $\gamma = 0.3$ for $N_Q = 24$, $V = 16^3$, i.e. $\rho = 5.9 \cdot 10^{-3}$

Canonical formulation of the 3-state Potts model

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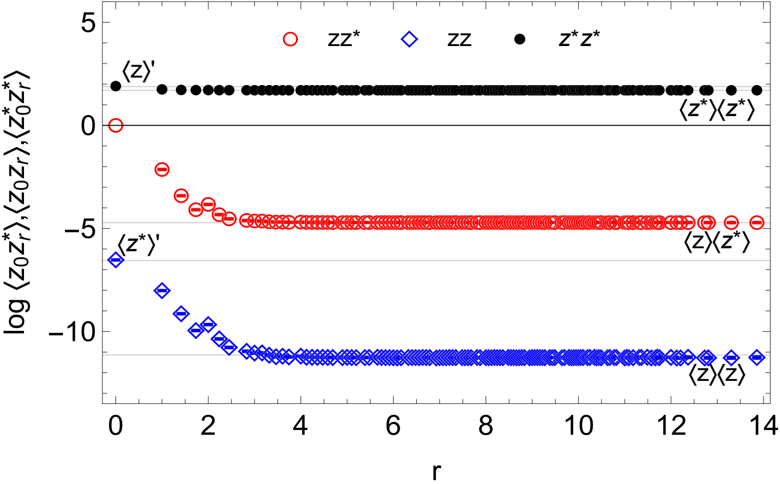

- values at $r = 0$ and $r \to \infty$ match $\langle z \rangle$, $\langle z^* \rangle$, $\langle z^* \rangle \langle z^* \rangle, \ldots$

Summary and outlook

- Canonical QCD can be obtained from transfer matrices defined directly in the canonical sectors of QCD
- In the heavy-dense limit, the fermionic contributions to the canonical partition functions can be derived exactly
- The fermion sign problem is absent at $\beta \to 0$:
  - simulations in the heavy-dense limit are possible
- Sign problem solved by cluster algorithm for $\beta > 0$ in the Potts model:
  - quarks confined in clusters, but move freely within
  - at $\beta \to 0$ clusters are confined to single sites only
  - deconfinement $\Leftrightarrow$ appearance of a percolating cluster

Summary and outlook

- The solution provides an appealing physical picture:

Good algorithms reflect true physics insight!

  - quarks confined in clusters, but move freely within
  - at $\gamma \to 0$ clusters are confined to single sites only
  - deconfinement corresponds to appearance of a percolating cluster

- Extension to Polyakov loop models could be possible:
  - mechanism at work at $\beta = 0$
  - extend it to $\beta > 0$