Quarkonium spectral functions at finite temperature on large quenched lattices and towards the continuum limit

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in collaboration with

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XQCD 2017

Aula Magna Fratelli Pontecorvo, Pisa, Italy

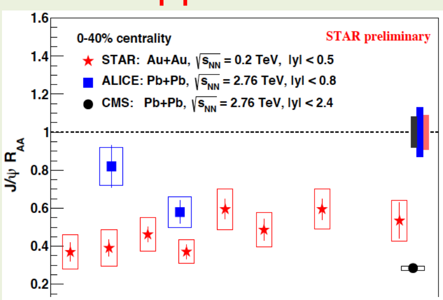
June 28, 2017

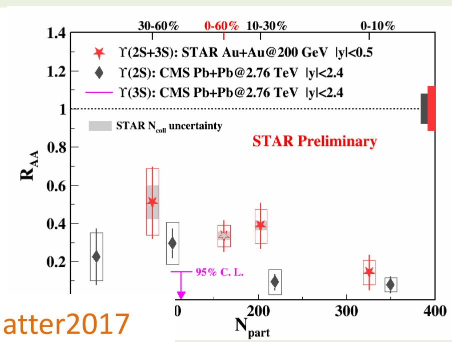
Motivation

- Quarkonium spectral functions (SPFs)
  – have all information about in-medium properties of quarkonia

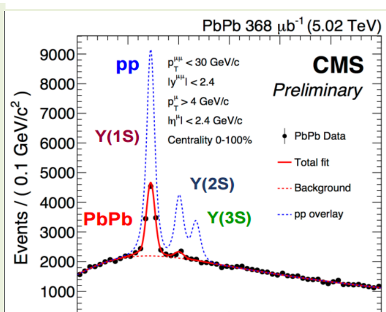
Quarkonium dissociation temperature

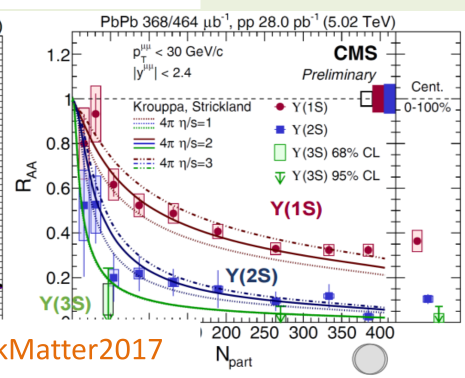
→ Important to understand quarkonium suppression





STAR Collaboration@QuarkMatter2017





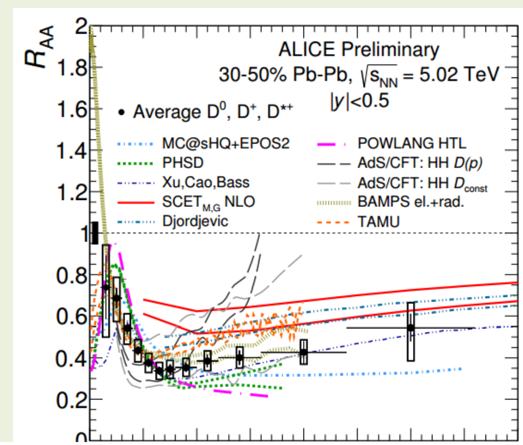
CMS Collaboration@QuarkMatter2017

Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega\to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^{V}(\omega, \mathbf{0})}{\omega}$$

$\rho_{ii}^{V}(\omega)$: vector SPF

→ Important input for heavy quark transport models



ALICE Collaboration@QuarkMatter2017

Quarkonium correlation and spectral functions

Euclidian (imaginary time) meson correlation function

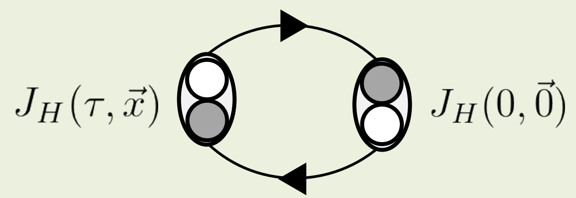
$$G_H(\tau,\vec{p}) \equiv \int d^3x e^{-i\vec{p}\cdot\vec{x}} \langle J_H(\tau,\vec{x}) J_H(0,\vec{0})\rangle$$

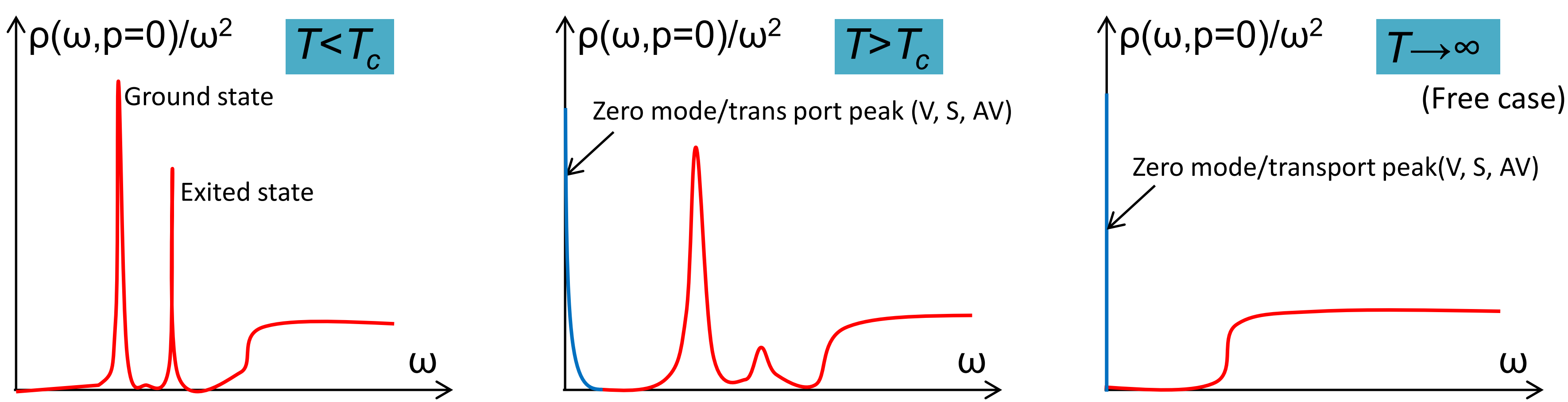
$$= \int_0^\infty \frac{d\omega}{2\pi} \boxed{\rho_H(\omega,\vec{p})} K(\omega,\tau)$$

Spectral function

$$J_H(\tau,\vec{x}) \equiv \bar{\psi}(\tau,\vec{x})\Gamma_H \psi(\tau,\vec{x})$$

$$K(\omega,\tau) \equiv \frac{\cosh[\omega(\tau - 1/2T)]}{\sinh(\omega/2T)}$$

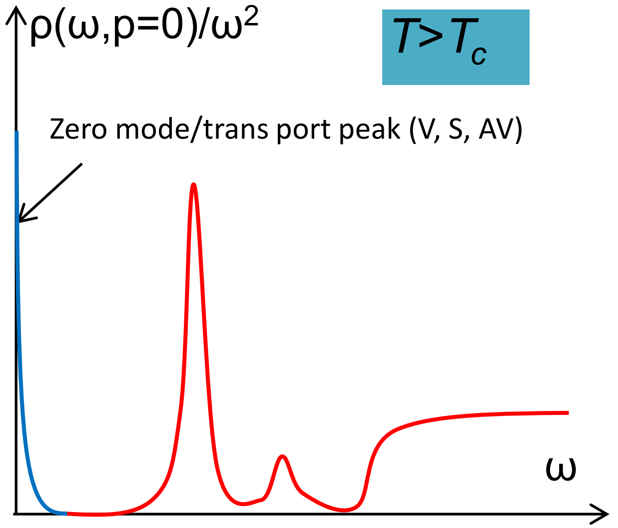




Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega\to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^V(\omega,\mathbf{0})}{\omega}$$

$\rho_{ii}^V(\omega)$: spatial component of vector spectral function



χ_{00} : Quark number susceptibility

$\rho_{00}^V(\omega) = 2\pi\chi_{00}\omega\delta(\omega)$ ➡ $G_{00}^V(\tau) = T\chi_{00}$

D is related to the slope of the vector spectral function around zero frequency.

This study

- Finite temperature lattice QCD simulations
  - large and fine isotropic lattices (N_σ = 192, a ≃ 0.009 fm)
  - N_τ = 96, 48 → T = 0.75T_c, 1.5T_c
  - quenched approximation (no dynamical quark)
  - both charm and bottom valence quarks treated relativistically
  - vector (V) channel

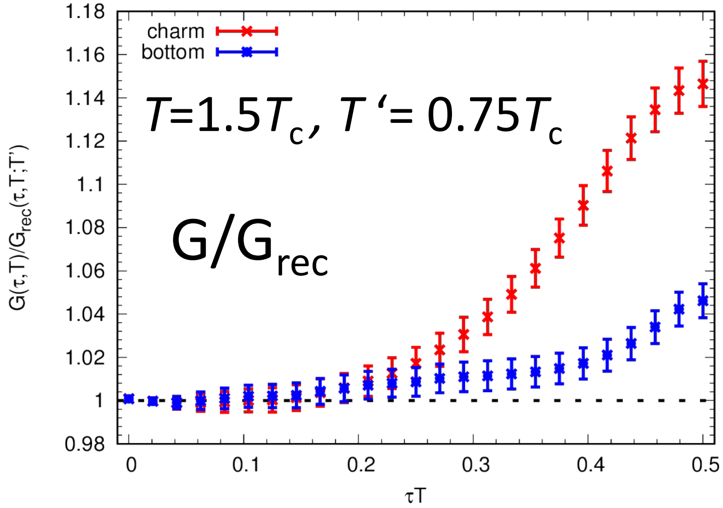
- Investigating quarkonium SPFs (and heavy quark diffusion)
  - indirectly with reconstructed correlators
  - directly by using both MEM and stochastic methods

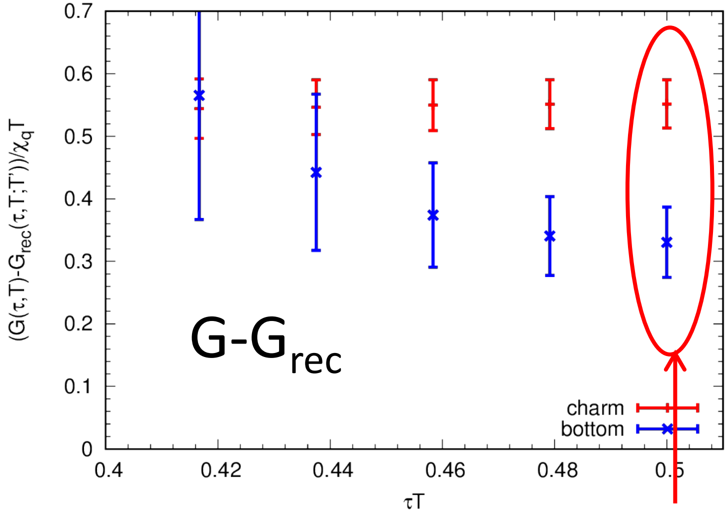
M. Asakawa, T. Hatsuda and Y. Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459-508

H.-T. Shu *et al*, PoS LATTICE 2015, 180 (2016)
HO, PoS LATTICE 2015, 175 (2016)

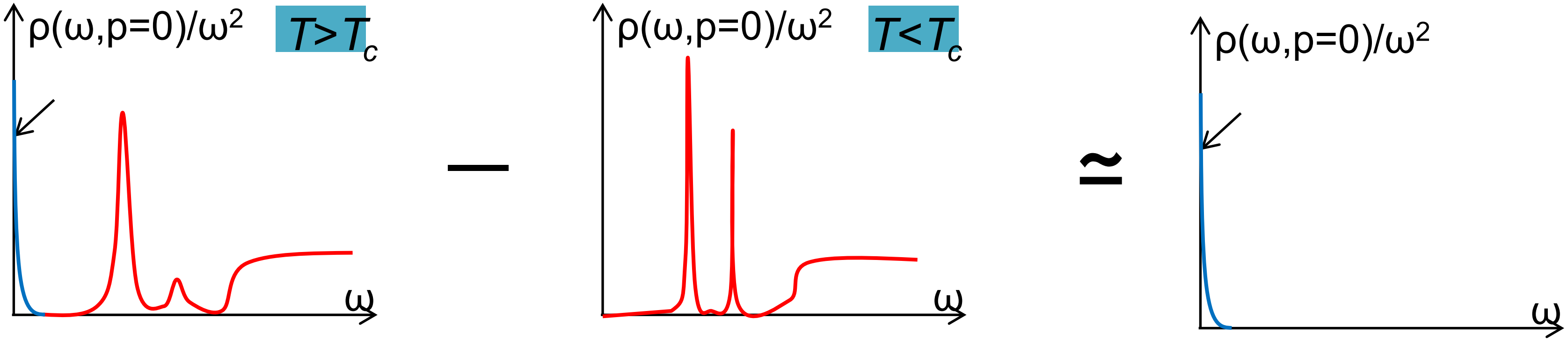
Estimation of the heavy quark diffusion coefficient

Reconstructed correlator: $G_{\rm rec}(\tau,T;T') = \int \frac{d\omega}{2\pi}\rho(\omega,T')K(\omega,\tau,T)$





Contribution from the transport peak is assumed to be dominant for $G - G_{\rm rec}$ at τT = ½.



Estimation of the heavy quark diffusion coefficient

Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega \to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^{V}(\omega, \mathbf{0})}{\omega}$$

H.-T. Ding *et al.*, PRD 86 (2012) 014509

Ansatz: $\rho_{ii}^{V}(\omega << T) = 2\chi_{00} \frac{T}{M} \frac{\omega\eta}{\omega^2 + \eta^2} \qquad \eta \equiv \frac{T}{MD}$

Charm: for M = 1—1.5 GeV

$2\pi TD \simeq 0.5$—0.7 at $1.5T_c$

Bottom: for M = 4—5 GeV

No solution at 1.5*Tc*

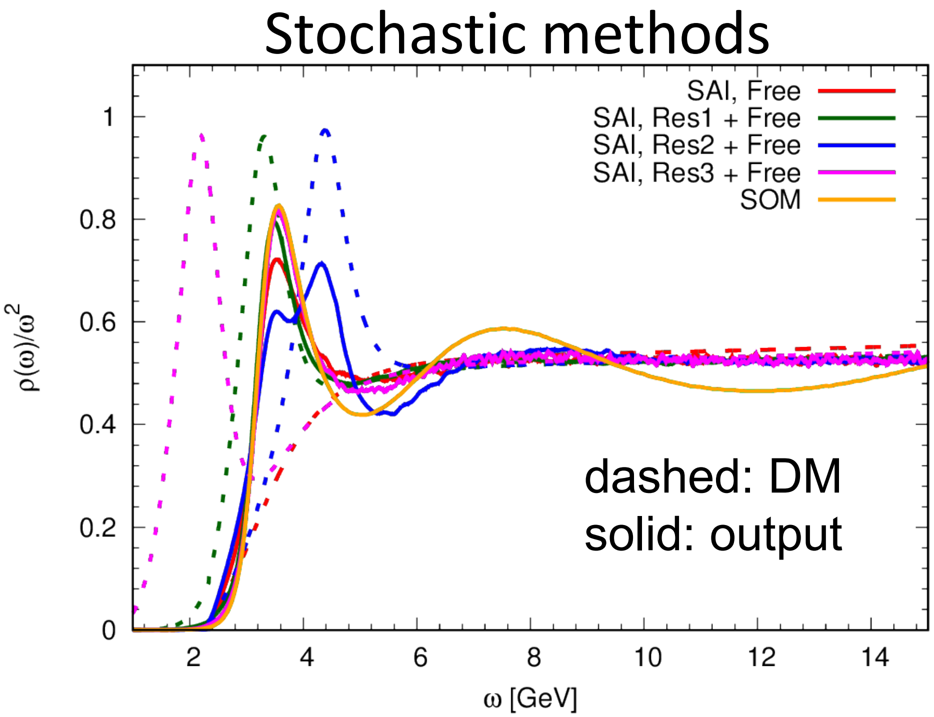
Reconstruction of spectral functions

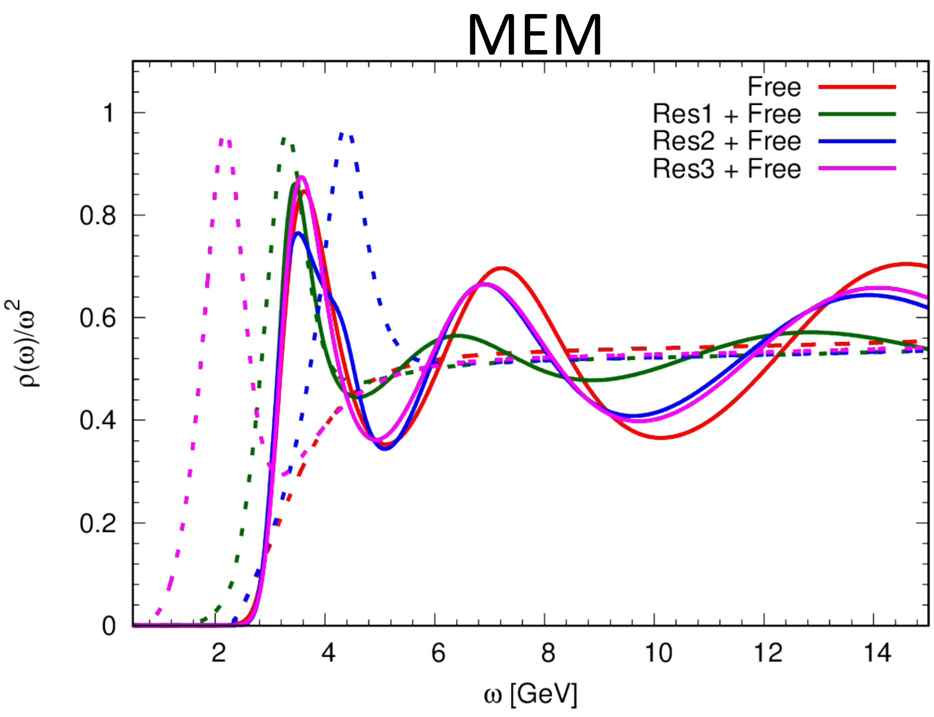
- Known to be an ill-posed problem
  → Simple χ^2-fitting does not work.
- Maximum entropy method (MEM) M. Asakawa, T. Hatsuda and Y. Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459-508
  - Based on Bayes' theorem
  - Prior knowledge (Default models) → Shannon-Jaynes entropy
  - Analytically minimizing χ^2 term + entropy term → a most likely solution
- Stochastic methods
  - Also based on Bayes' theorem
  - Stochastically finding a free-energy minimum = a most likely solution
  - Default models can be introduced
    → Stochastic Analytical Inference (SAI) K.S.D. Beach, arXiv:cond-mat/0403055 S. Fuchs *et al.*, PRE81, 056701 (2010)
  - There is also a default-model-free method
    → Stochastic Optimization Method (SOM) A. S. Mishchenko et al., Phys. Rev. B62, 6317 (2000)

There is also another type of Bayesian methods proposed recently.

Y. Burnier and A. Rothkopf, PRL 111 (2013) 18, 182003

Default model (DM) dependence of the charmonium SPF at 0.75T_c





DM = Free, Res(1-3) + Free

Peak location: Res1 ~ J/Ψ mass, Res2 > J/Ψ mass, Res3 < J/Ψ mass

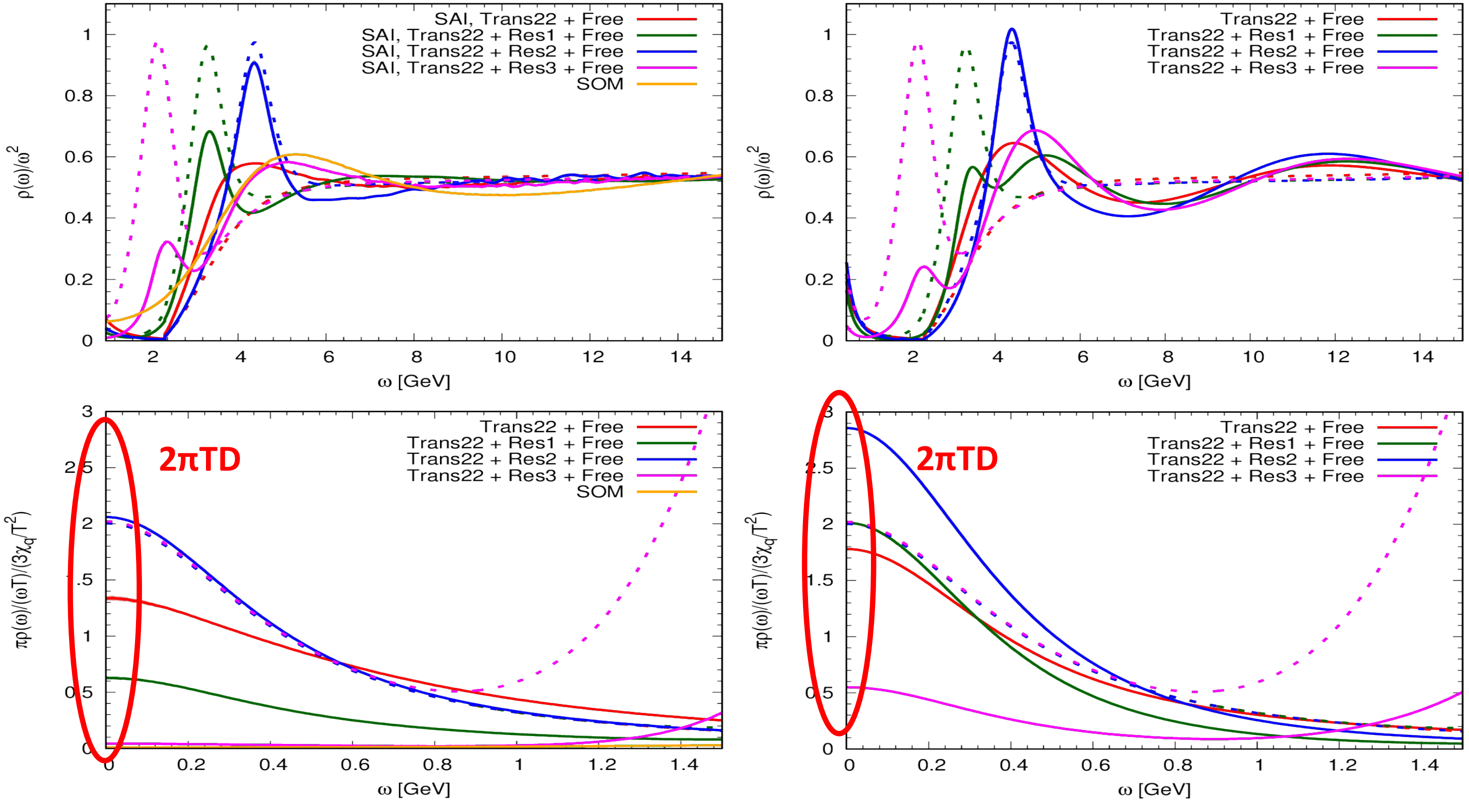
Continuum part behaves differently between the stochastic methods and MEM.
Location of the first peak ~ J/Ψ mass, small DM dependence
→ There is a stable bound state corresponding to J/Ψ.

DM dependence of the charmonium SPF at $1.5T_c$ (1)

Stochastic methods | MEM

High ω

Low ω



DM = Trans + Free, Trans + Res(1-3) + Free (Trans is fixed)

The resonance peak is unstable and highly sensitive to DMs.

→ J/Ψ seems to melt already $T < 1.5T_c$.

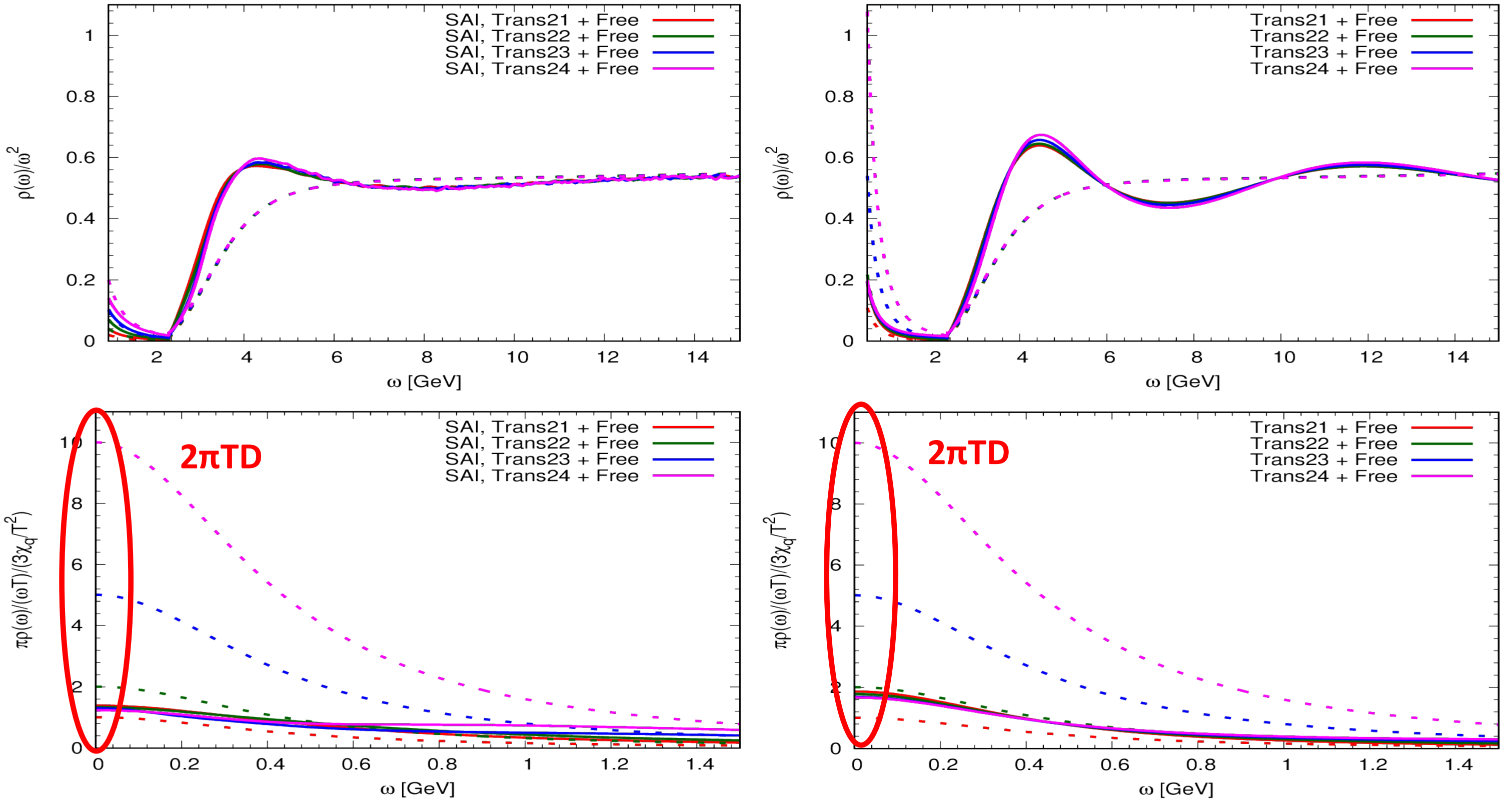
The transport peak is also sensitive to DMs.

DM dependence of the charmonium SPF at 1.5T_c (2)

Stochastic methods | MEM

High ω

Low ω



DM = Trans(21-24) + Free (width of the transport peak is fixed)

Trans21: 2πTD ~ 1, Trans22: 2πTD ~ 2, Trans23: 2πTD ~ 5, Trans24: 2πTD ~ 10

Both high and low frequency parts have small DM dependence.

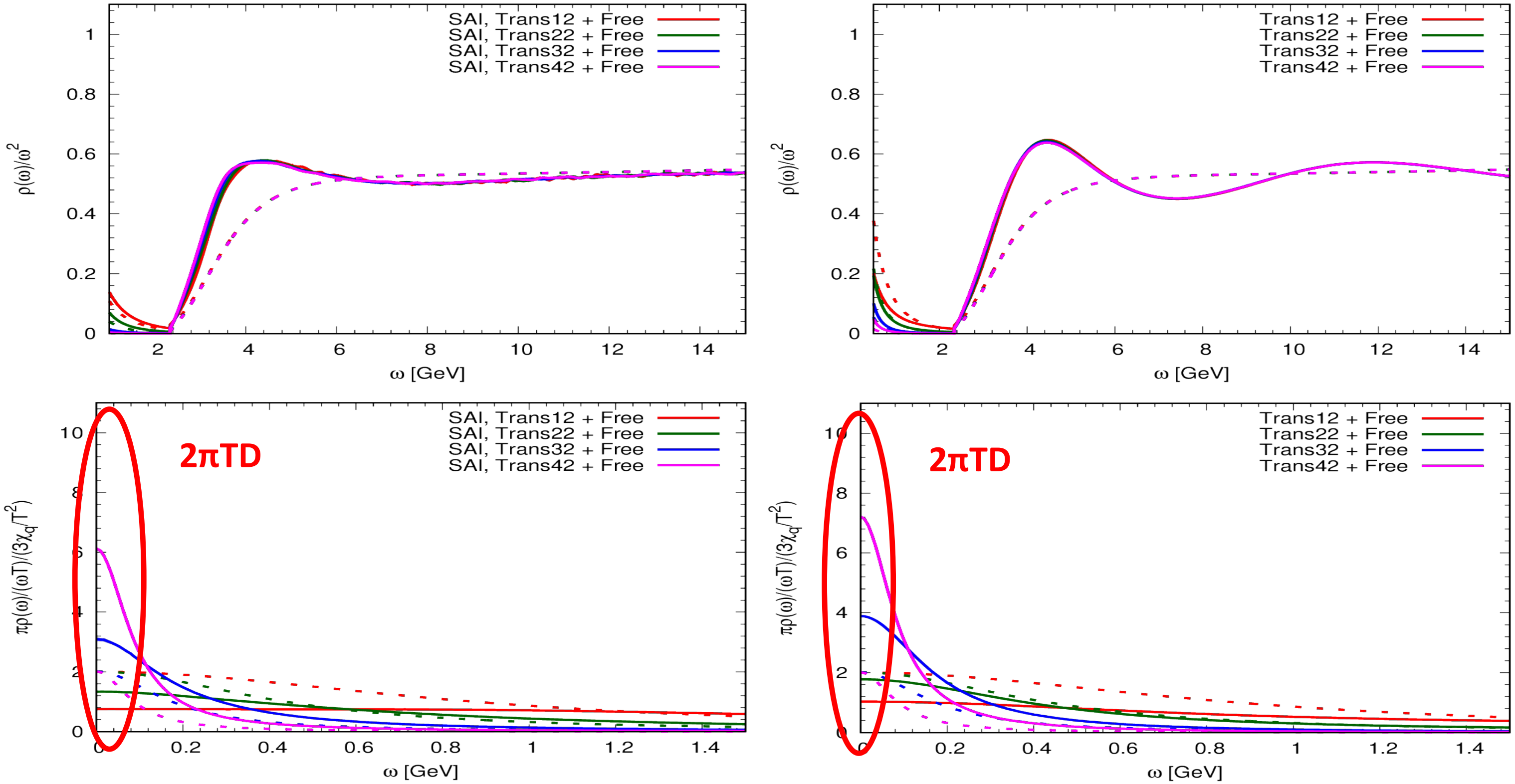
→ 2πTD ~ 1—2

DM dependence of the charmonium SPF at $1.5T_c$ (3)

Stochastic methods | MEM

High ω

Low ω



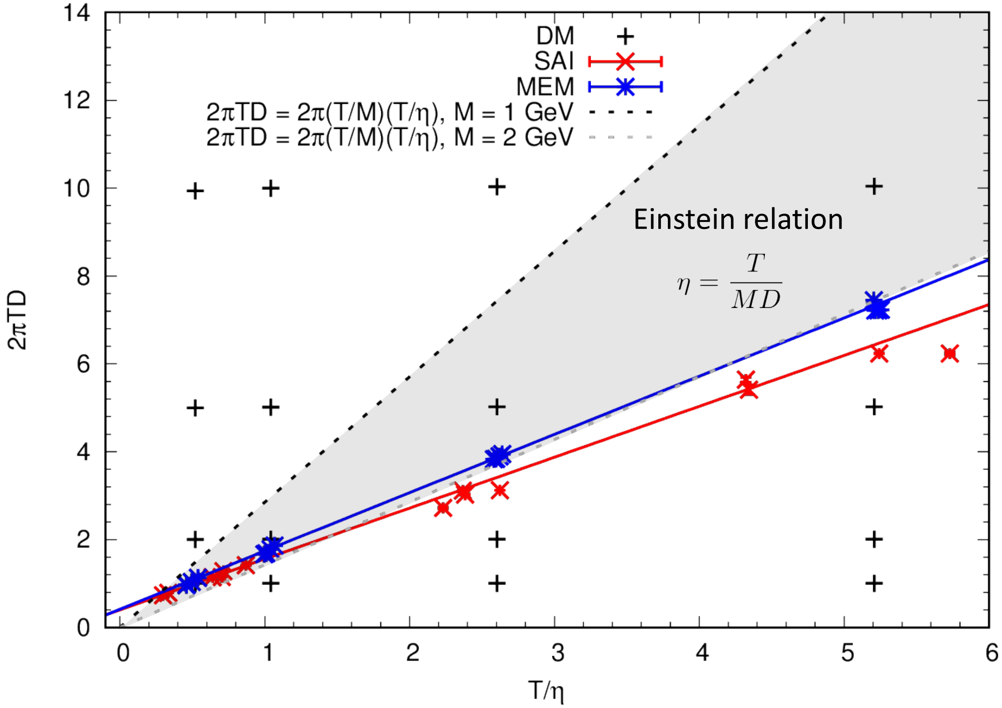
DM = Trans(12-42) + Free (height of the transport peak is fixed)

Trans12: η/T ~ 2, Trans22: η/T ~ 1, Trans32: η/T ~ 0.4, Trans42: η/T ~ 0.2

High frequency part has small DM dependence.

Low frequency part is sensitive to DMs. → 2πTD ~ 1－7

DM dependence of the charm quark diffusion coefficient at 1.5*T*c



Output 2πTD varies as DM η/T changes.
Output η/T is almost the same to DM η/T for MEM
It seems that D･η is always fixed.

$$G(\tau) = \int \frac{d\omega}{2\pi} c \frac{\omega\eta}{\omega^2+\eta^2} \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)}$$

$$\underset{\omega << T}{\propto c}$$

$$D = \frac{1}{6\chi_q} \lim_{\omega\to 0} c \frac{\eta}{\omega^2+\eta^2} \Rightarrow c \propto D\eta$$

Both SAI and MEM can only determine the coefficient c or D･η

→ The diffusion coefficient cannot be determined unless η/T is fixed.

2πTD = 1.16(4) T/η + 0.40(2) for SAI
2πTD = 1.33(4) T/η + 0.42(2) for MEM

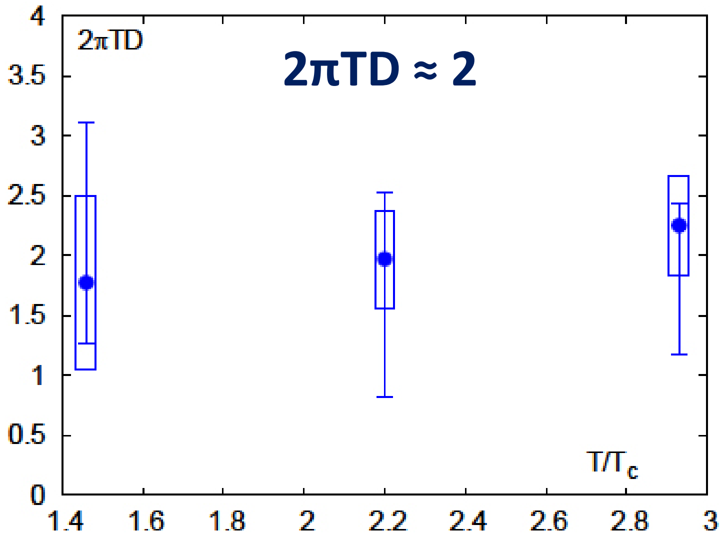
for T/η = 1—5 →

2πTD ~ 1.6—6.2 for SAI
2πTD ~ 1.8— 7.0 for MEM

The Einstein relation suggests 2πTD $\lesssim$ 6.

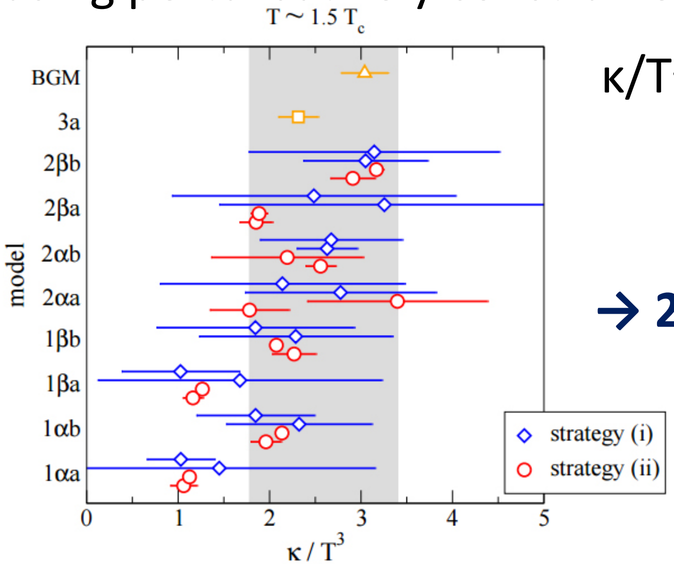
Comparison with recent lattice results

- Quenched QCD, using MEM



H.-T. Ding *et al.*, PRD 86 (2012) 014509

- Heavy quark effective theory, continuum limit, using perturbatively constrained fits



$\kappa/T^3 = 1.8 - 3.4$

$$D = \frac{2T^2}{\kappa}$$

→ **2πTD ≈ 3.7—7.0**

A. Francis, O. Kaczmarek, M. Laine, T. Neuhaus and HO, PRD 92 (2015) no.11, 116003

Our results are consistent with other lattice results.

- Perturbative estimate
  - 2πDT ≈ 71.2 in LO
    Moore and Teaney, PRD 71 (2005) 064904
  - 2πDT ≈ 8.4 in NLO
    Caron-Hout and Moore, PRL 100 (2008) 052301
- Strong coupling limit
  - 2πDT ≈ 1
    Kovtun, Son and Starinets, JHEP 0310 (2004) 064

Summary & outlook

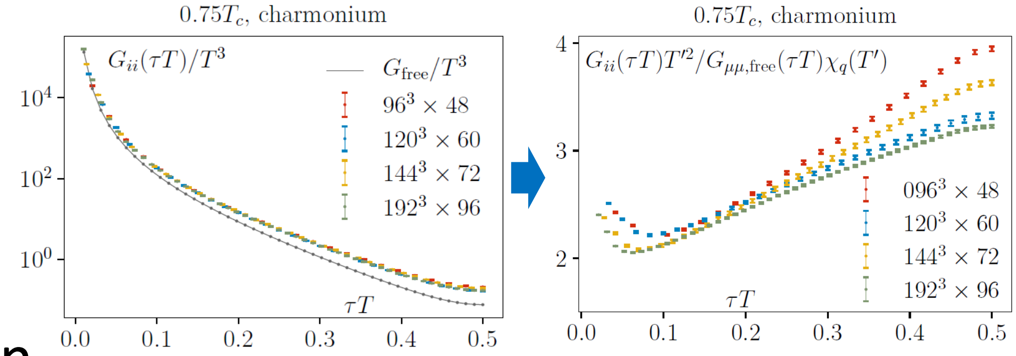
- We investigated vector charmonium and bottomonium SPFs on very large and fine quenched lattices in both indirect and direct ways.
- Charmonium SPFs for the vector channel at $0.75T_c$ and $1.5T_c$ were reconstructed with both MEM and stochastic methods.
  - Both MEM and the stochastic methods gave almost DM-independent stable SPFs having a clear J/Ψ peak at $0.75T_c$.
  - Most of the results may suggest that J/Ψ might melt already $T < 1.5T_c$
  - So far we observed a relation between $2\pi TD$ and T/η, which gives a range $1 \lesssim 2\pi TD \lesssim 7$ for $1 \lesssim T/\eta \lesssim 5$.

- More studies on SPF reconstruction are needed.
  - further checks of the DM-dependence and other systematic uncertainties
  - analysis of the temperature and quark mass dependence as well as other channels
  - **continuum extrapolation**

Continuum extrapolation of vector quarkonium correlators: Overview

1. Normalization

- Eliminating exponentially decreasing τ-dependence
  → free correlator
- Independent of renormalization
  → quark number susceptibility



2. Mass interpolation

- Precise tuning is necessary
- Using various quark masses in a range btw. charm and bottom

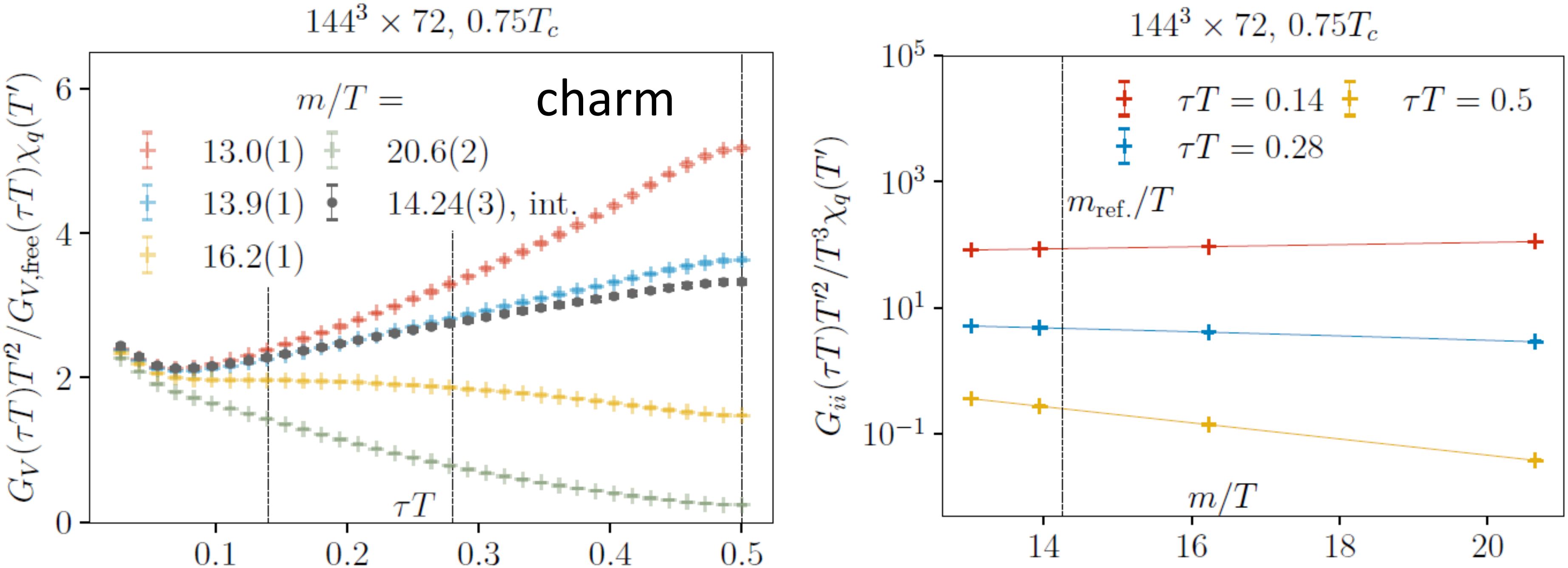
3. Continuum extrapolation

- Using 4 different lattice spacings

β	a[fm](a^{-1}[GeV])	N_σ	N_τ	T/T_c	#confs.
7.192	0.018 (11.19)	96	48	0.75	237
			32	1.1	476
			28	1.3	336
			24	1.5	336
			16	2.25	237
7.394	0.014 (14.24)	120	60	0.75	171
			40	1.1	141
			30	1.5	247
			20	2.25	226
7.544	0.012 (17.01)	144	72	0.75	221
			48	1.1	462
			42	1.3	660
			36	1.5	288
			24	2.25	237
7.793	0.009 (22.78)	192	96	0.75	224
			64	1.1	291
			56	1.3	291
			48	1.5	348
			32	2.25	235

Continuum extrapolation of vector quarkonium correlators: Mass interpolation

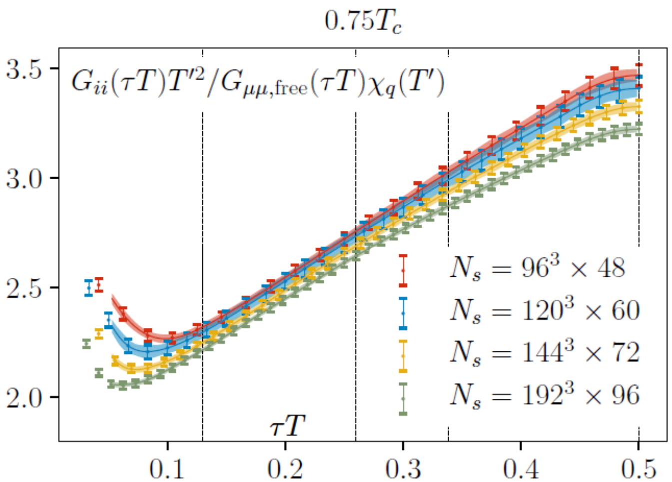
Fit ansatz: $$G\left(\frac{m}{T},\tau T\right)=\exp\left(a(\tau T)\left(\frac{m}{T}\right)^2+b(\tau T)\frac{m}{T}+c(\tau T)\right)$$



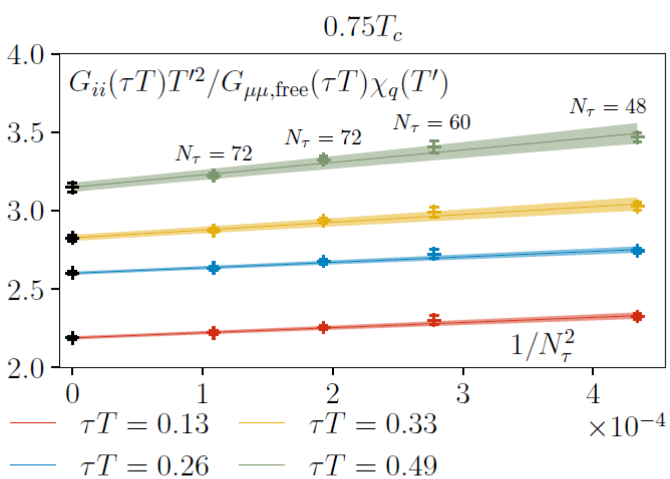
Correlators for the finest lattice is fixed as a reference.
Correlators for the other lattices are tuned to the same meson mass.
The mass at 0.75 T_c is used at higher temperatures.

Continuum extrapolation of vector quarkonium correlators: Continuum limit

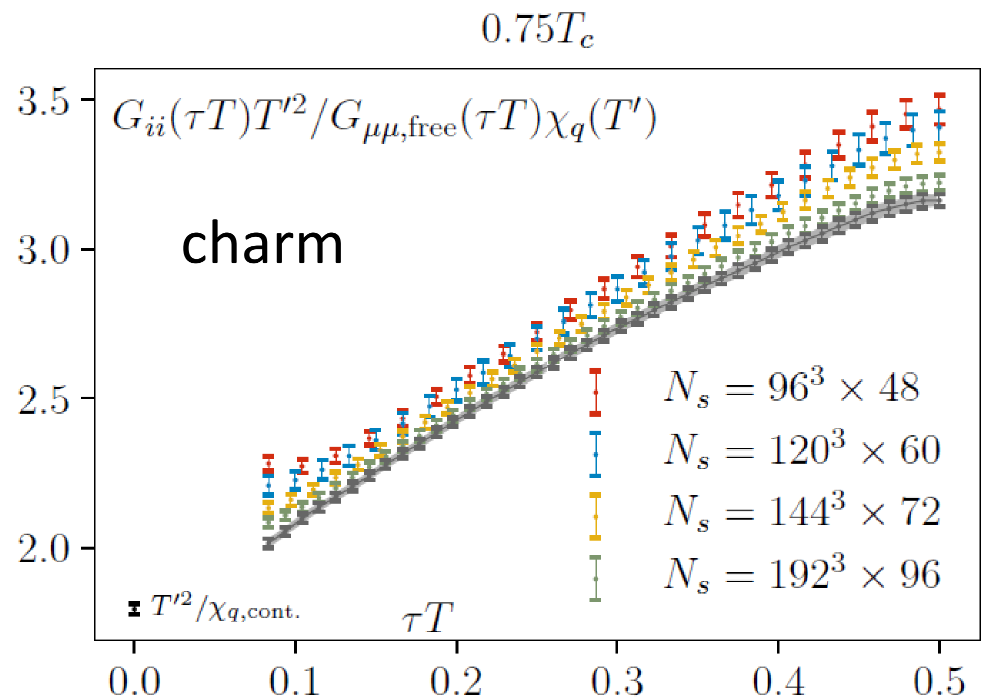
b-spline interpolation in τ



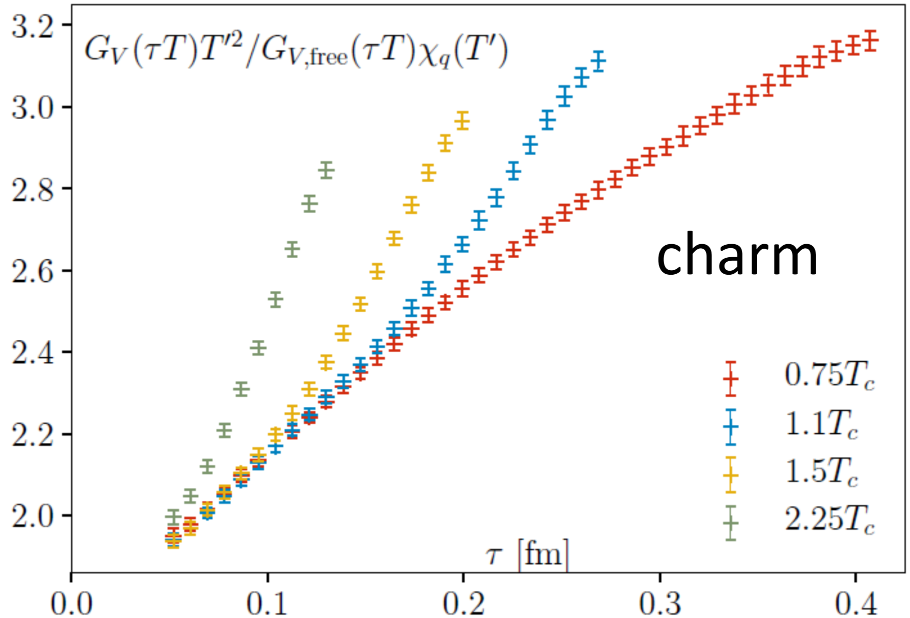
linear extrapolation in $1/N_\tau^2$

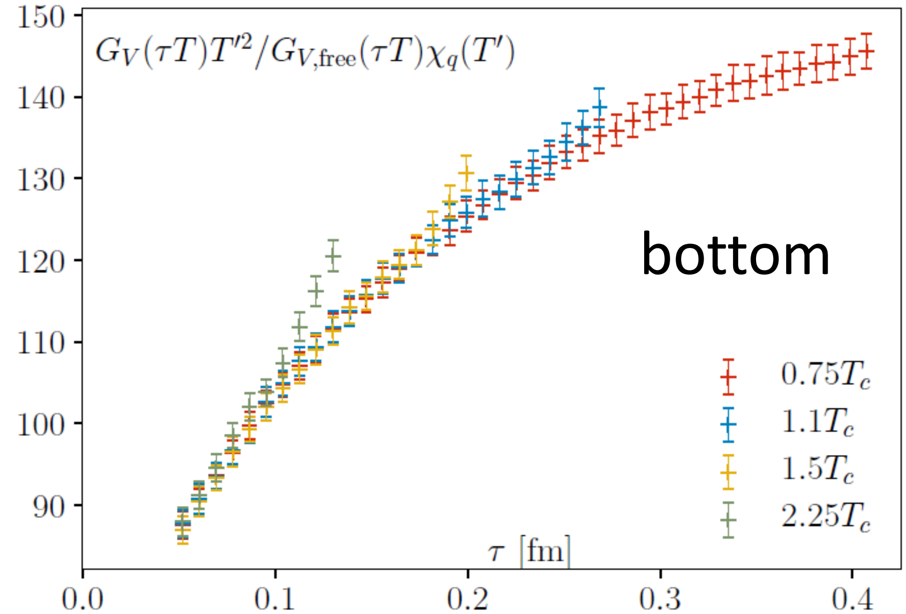


Large cutoff effect at small τ



Continuum extrapolation of vector quarkonium correlators: Summary





Temperature dependence for bottom is smaller than charm

Future plans:

- Other channels (pseudo-scalar, scalar, axial-vector)
- Reconstructing spectral functions

End

Backup slides

Reconstructed correlator

$$G_{\rm rec}(\tau, T; T') \equiv \int_0^\infty d\omega \rho(\omega, T') K(\omega, \tau, T)$$

$$\frac{G(\tau,T)}{G_{\rm rec}(\tau,T;T')} = \frac{\int_0^\infty d\omega \rho(\omega,T) K(\omega,\tau,T)}{\int_0^\infty d\omega \rho(\omega,T') K(\omega,\tau,T)}$$

equals to unity at all τ

different same

if the spectral function doesn't change at each temperature

S. Datta *et al.*, PRD 69 (2004) 094507

$$\frac{\cosh[\omega(\tau - N_\tau/2)]}{\sinh[\omega N_\tau/2]} = \sum_{\tau'=\tau;\Delta\tau'=N_\tau}^{N'_\tau - N_\tau + \tau} \frac{\cosh[\omega(\tau' - N'_\tau/2)]}{\sinh[\omega N'_\tau/2]}$$

$T = 1/(N_\tau a)$ $\quad N'_\tau = mN_\tau$ $\quad m = 1, 2, 3, \cdots$

$$G_{\rm rec}(\tau, T; T') = \sum_{\tau'=\tilde{\tau};\Delta\tau'=N_\tau}^{N'_\tau - N_\tau + \tau} G(\tau', T')$$

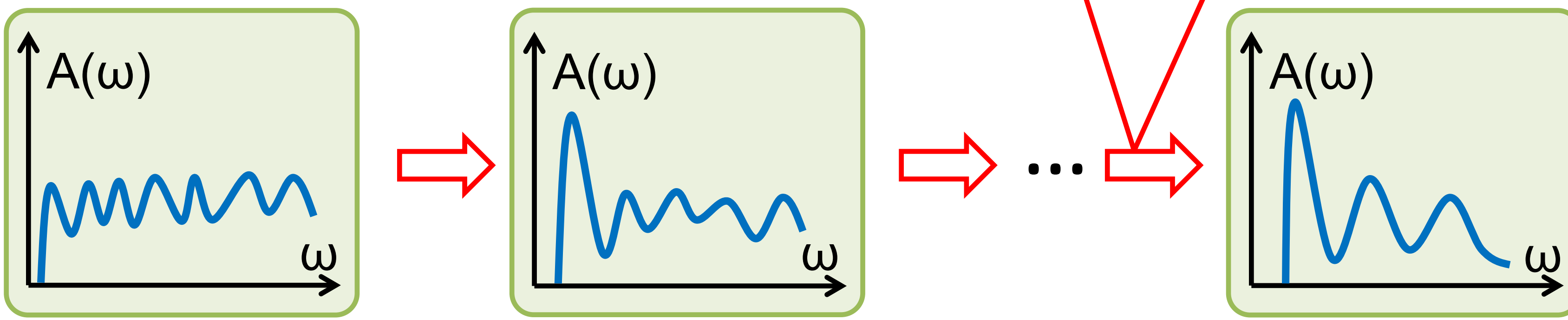
H.-T. Ding *et al.*, PRD 86 (2012) 014509

Stochastic method: basic idea

For given α (fictitious temperature, regularization parameter),

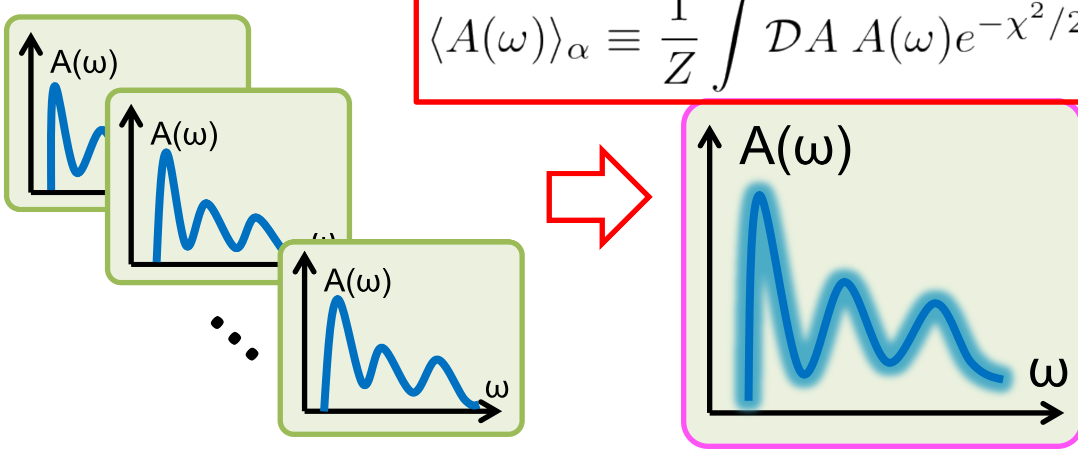
1. generate SPFs stochastically

$$P = \min\{1, e^{-\chi^2/2\alpha}\}$$



2. average over all possible spectra

$$\langle A(\omega)\rangle_\alpha \equiv \frac{1}{Z}\int \mathcal{D}A\; A(\omega) e^{-\chi^2/2\alpha} \qquad Z \equiv \int \mathcal{D}A\; e^{-\chi^2/2\alpha}$$



$$\chi^2 \equiv \sum_{\tau,\tau'} \Delta G(\tau) C^{-1}_{\tau,\tau'} \Delta G(\tau')$$

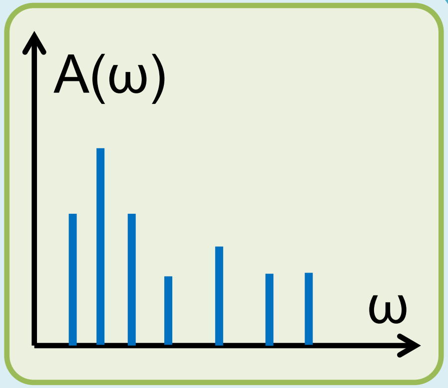
$C^{-1}_{\tau,\tau'}$: covariance matrix

$$\Delta G(\tau) \equiv \bar{G}(\tau) - \int d\omega\; A(\omega)\tilde{K}(\omega,\tau)$$

Stochastic method: basis

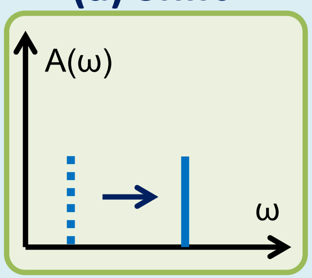
δ functions

$$A(\omega) = \sum_i r_i \delta(\omega - a_i)$$

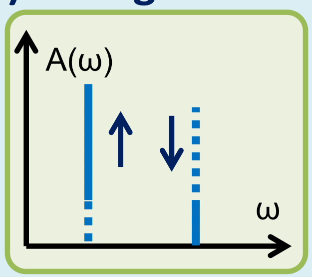


- Update schemes

(a) Shift



(b) Change residues



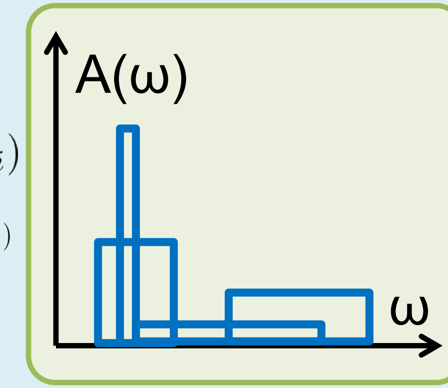
- Constraint

$$\sum_i r_i = \bar{G}(\tau_0)$$

Boxes

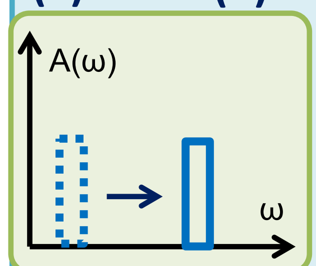
$$A(\omega) = \sum_i h_i R(\omega, a_i, w_i)$$

$$R(\omega, a_i, w_i) \equiv \begin{cases} 1 & (a_i - \frac{w_i}{2} \le \omega \le a_i + \frac{w_i}{2}) \\ 0 & (\text{otherwise}) \end{cases}$$

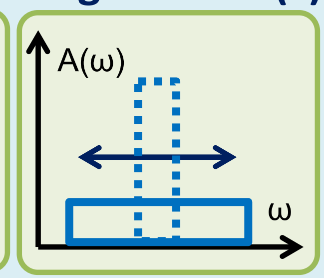


- Update schemes

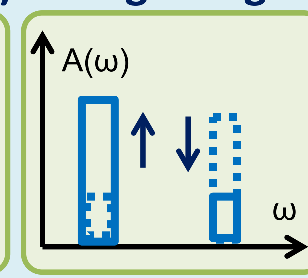
(a) Shift



(b) Change width



(c) Change heights



- Constraint

$$\sum_i h_i w_i = \bar{G}(\tau_0)$$

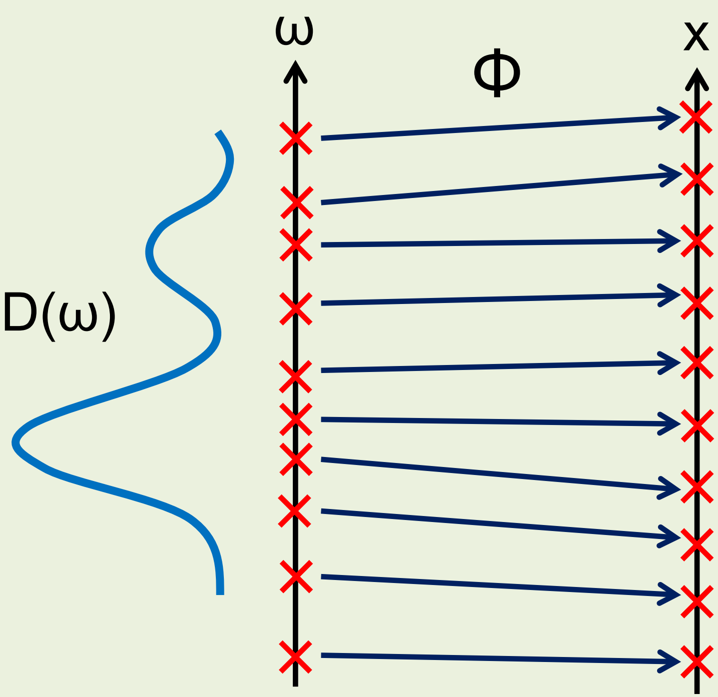
Update schemes which change the number of the basis are also possible.

Stochastic method: default model

K.S.D. Beach, arXiv:cond-mat/0403055

$$x \equiv \phi(\omega) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\omega} d\omega' D(\omega')$$

$D(\omega)$: Default model (prior information)



$$G(\tau) = \int d\omega\ A(\omega)\tilde{K}(\omega, \tau)$$

$$= \int dx\ n(x)\tilde{K}(\phi^{-1}(x), \tau)$$

$$n(x) \equiv \frac{A(\phi^{-1}(x))}{D(\phi^{-1}(x))}$$

$$\bar{G}(\tau_0) - \int dx\ n(x) = 0$$

$$\langle A(\omega)\rangle_\alpha = \langle n(\phi(\omega))\rangle_\alpha D(\omega)$$

Stochastic method: comparison with MEM(1)

Maximum entropy method (MEM)

$$F \equiv \chi^2/2 - \alpha S \qquad S \equiv -\int d\omega\; A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right)$$: entropy

Minimizing F

$$\left.\frac{\delta F}{\delta A}\right|_{A=\bar{A}} = 0 \quad \Rightarrow \quad \bar{A}(\omega)$$: the most likely solution

Stochastic method

K.S.D. Beach, arXiv:cond-mat/0403055

$$H[n] \equiv \chi^2 = \int dx\; \epsilon(x) n(x) + \frac{1}{2}\int dxdy V(x,y) n(x) n(y)$$: Hamiltonian

Mean filed treatment

$$(n(x) - \bar{n}(x))(n(y) - \bar{n}(y)) \approx 0 \quad \Rightarrow \quad \bar{A}(\omega) = \bar{n}(\phi(\omega)) D(\omega)$$

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Minimizing F

$$\left.\frac{\delta F}{\delta A}\right|_{A=\bar{A}} = 0 \Rightarrow \bar{A}(\omega)$$: the most likely solution

Equivalent!

Stochastic method

K.S.D. Beach, arXiv:cond-mat/0403055

$$H[n] \equiv \chi^2 = \int dx\, \epsilon(x) n(x) + \frac{1}{2}\int dx dy V(x-y) n(x) n(y)$$: Hamiltonian

Mean filed treatment

$$(n(x) - \bar{n}(x))(n(y) - \bar{n}(y)) \approx 0 \Rightarrow \bar{A}(\omega) = \bar{n}(\phi(\omega)) D(\omega)$$

Stochastic method: comparison with MEM(2)

S. Fuchs *et al.*, PRE81, 056701 (2010)

MEM

$$P[A|\bar{G}] = \frac{P[\bar{G}|A]P[A]}{P[\bar{G}]}$$

- Prior probability

$$P[A] \propto \exp(\alpha S)$$

- Likelihood function

$$P[\bar{G}|A] \propto \exp\left(-\chi^2/2\right)$$

- Posterior probability

$$P[A|\bar{G}] \propto e^{-F}$$

$$\Rightarrow \max P[A|\bar{G}] \leftrightarrow \min F \leftrightarrow \frac{\delta F}{\delta A} = 0$$

Stochastic method

$$P[n|\bar{G}] = \frac{P[\bar{G}|n]P[n]}{P[\bar{G}]}$$

- Prior probability

$$P[n] \propto \Theta[n]\delta\left(\int dx\, n(x) - \bar{G}(\tau_0)\right)$$

- Likelihood function

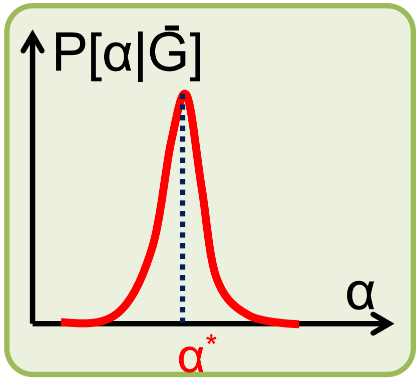
$$P[\bar{G}|n] \propto \exp\left(-\chi^2/2\alpha\right)$$

- Posterior probability

$$P[n|\bar{G}] = \Theta[n]\delta\left(\int dx\, n(x) - \bar{G}(\tau_0)\right) e^{-\chi^2/2\alpha}$$

$$\Rightarrow \langle n(x)\rangle_\alpha = \int \mathcal{D}n\; n(x)P[n|\bar{G}]$$

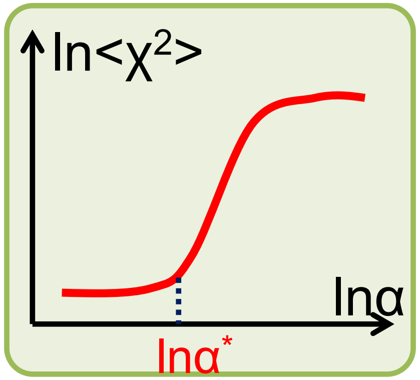
Stochastic method: eliminating α



(a) By using the posterior probability P[α|Ḡ]

$$P[\alpha|\overline{G}] \propto P[\alpha] \int \mathcal{D}A \, e^{-\chi^2/2\alpha} \quad P[\alpha] = 1, \; 1/\alpha$$

Choosing α at the peak location of P[α|Ḡ] or
Taking average $\langle\langle A(\omega)\rangle\rangle \equiv \int d\alpha \, \langle A(\omega)\rangle_\alpha P[\alpha|\bar{G}]$



(b) By using the log-log plot of α vs $\langle\chi^2\rangle$

Flat region at large α: default model dominant
Crossover region: both χ^2-fitting and the default model are important
Flat region at small α: χ^2-fitting dominant, overfitting

Choosing α at the kink of ln$\langle\chi^2\rangle$

K.S.D. Beach, arXiv:cond-mat/0403055

Stochastic method : summary

MEM

- Most likely solution

$$\max P[A|\bar{G}] \leftrightarrow \min F \leftrightarrow \frac{\delta F}{\delta A} = 0$$

where

$$P[A|\bar{G}] \propto e^{-F} \quad F \equiv \chi^2/2 - \alpha S$$

- Prior information

$$S \equiv -\int d\omega\ A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right)$$

- Eliminating α

$$P[\alpha|\bar{G}] \propto P[\alpha] \int \mathcal{D}A\ e^{-F[A]}$$

where

$$P[\alpha] = 1,\ 1/\alpha$$

➡

$$\langle A(\omega)\rangle = \int d\alpha A_\alpha(\omega) P[\alpha|\bar{G}]$$

Stochastic methods

- Most likely solution

$$\mathcal{D}'n \equiv \mathcal{D}n\ \Theta[n]\delta\left(\int dx\ n(x) - \bar{G}(\tau_0)\right)$$

Stochastically evaluate

$$\langle n(x)\rangle_\alpha = \int \mathcal{D}'n\ n(x) e^{-\chi^2/2\alpha} \quad \text{for } n(x) \equiv \frac{A(\phi^{-1}(x))}{D(\phi^{-1}(x))}$$

SAI (Stochastic Analytical Inference)

S. Fuchs *et al.*, PRE81, 056701 (2010)
HO, PoS LATTICE 2015, 175 (2016)

- Prior information

$$x \equiv \phi(\omega) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\omega} d\omega' D(\omega')$$

- Basis

δ functions

- Eliminating α

$$P[n|\bar{G}] \propto P[\alpha] \int \mathcal{D}'n\ e^{-\chi^2[n]/2\alpha}$$

➡ $\langle\langle n(x)\rangle\rangle = \int d\alpha \langle n(x)\rangle_\alpha P[n|\bar{G}]$

SOM (Stochastic Optimization Method)

A. S. Mishchenko *et al.*, PRB62, 6317 (2000)
H.-T. Shu *et al*, PoS LATTICE 2015, 180 (2016)

- Prior information

None ($x = \omega$)

dose not rely on DM!

- Basis

Boxes

- Eliminating α

Choosing α
at a critical point of $\langle\chi^2\rangle_\alpha$

Default models

- Free SPF

$$\rho(\omega) = \frac{N_c}{L^3} \sum_{\boldsymbol{k}} \sinh\left(\frac{\omega}{2T}\right) \left[b^{(1)} - b^{(2)} \frac{\sum_{i=1}^{3} \sin^2 k_i}{\sinh^2 E_{\boldsymbol{k}}(m)} \right] \times \frac{\delta(\omega - 2E_{\boldsymbol{k}}(m))}{(1+\mathcal{M}_{\boldsymbol{k}}(m))^2 \cosh^2 E_{\boldsymbol{k}}(m)}$$

$$\cosh E_{\boldsymbol{k}}(m) = 1 + \frac{\mathcal{K}_{\boldsymbol{k}}^2 + \mathcal{M}_{\boldsymbol{k}}^2(m)}{2(1+\mathcal{M}_{\boldsymbol{k}}(m))}$$

$$\mathcal{K}_{\boldsymbol{k}} = \sum_{i=1}^{3} \gamma_i \sin k_i$$

$$\mathcal{M}_{\boldsymbol{k}}(m) = \sum_{i=1}^{3} (1 - \cos k_i) + m$$

$b^{(1)} = 3$, $b^{(2)} = 1$ for the V channel

- A resonance peak

$$\rho(\omega) = \frac{\Gamma M}{(\omega^2 - M^2)^2 + M^2 \Gamma^2} \frac{\omega^2}{\pi}$$

- A transport peak

$$\rho(\omega) = \frac{\omega\eta}{\omega^2 + \eta^2}$$