UNIVERSITÀ DEGLI STUDI DI BARI ALDO MORO

SM&FT 2017 - The XVII Workshop on Statistical Mechanics and non-pertubative Field Theory

INFN
Istituto Nazionale di Fisica Nucleare

PAOLA RUGGIERO

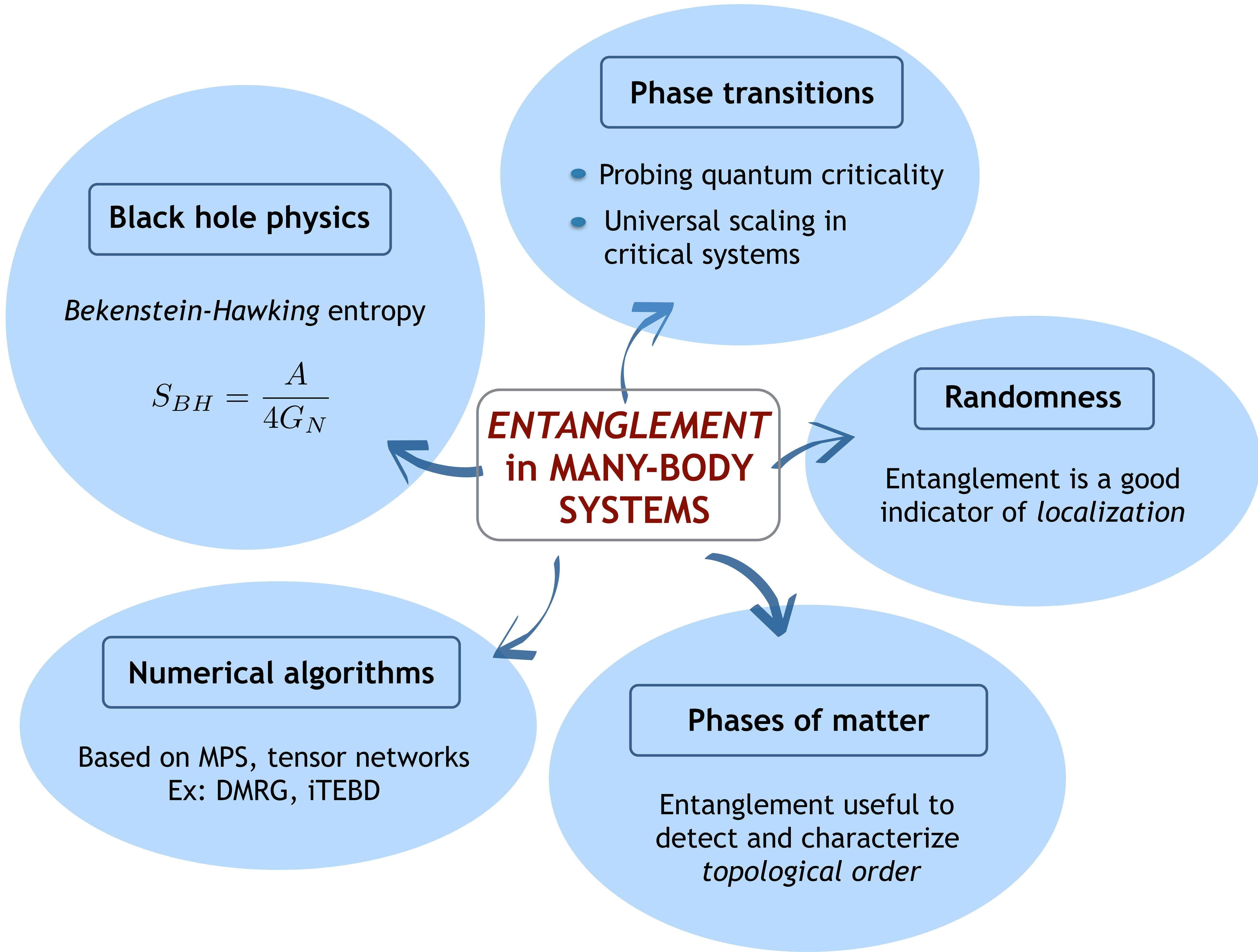
ENTANGLEMENT MEASURES IN MANY-BODY QUANTUM SYSTEMS: THE NEGATIVITY SPECTRUM

with P. CALABRESE, V. ALBA

PRB 94, 195121 (2016)

December 13th 2017

SISSA
ma per seguir virtute e canoscenza


ENTANGLEMENT in MANY-BODY SYSTEMS
Black hole physics
Bekenstein-Hawking entropy
$S_{BH} = \frac{A}{4G_N}$
Phase transitions
Probing quantum criticality
Universal scaling in critical systems
Randomness
Entanglement is a good indicator of *localization*
Numerical algorithms
Based on MPS, tensor networks
Ex: DMRG, iTEBD
Phases of matter
Entanglement useful to detect and characterize *topological order*

HOW DO WE MEASURE ENTANGLEMENT?

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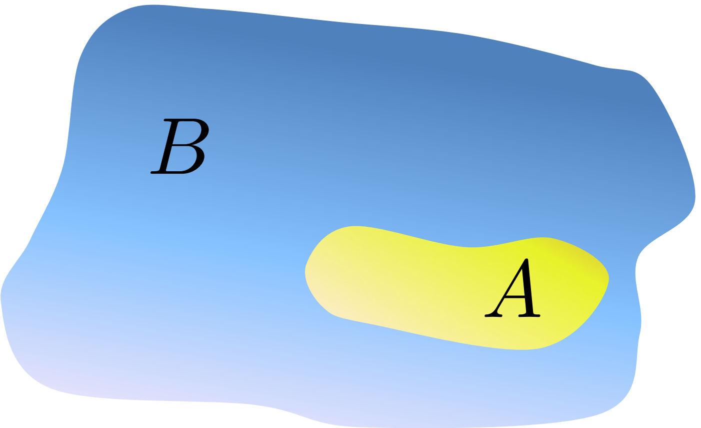
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \rho = |\psi\rangle\langle\psi| \qquad \rho_A = tr_B \rho$$



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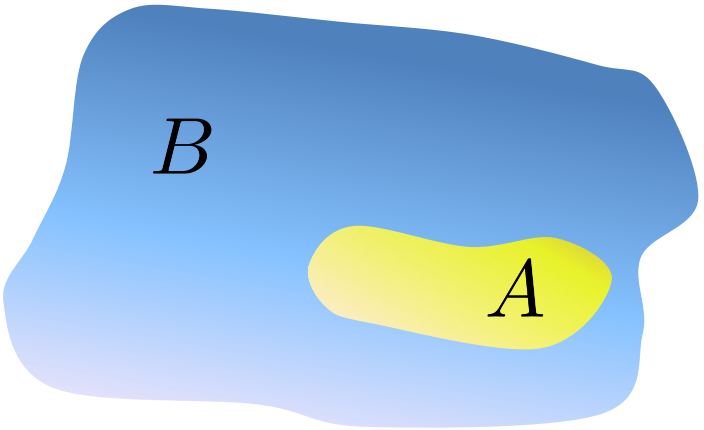
$$S_A = -tr(\rho_A \log \rho_A)$$



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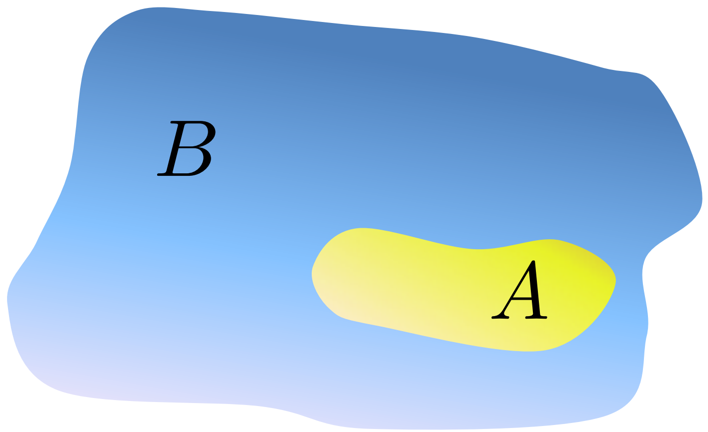
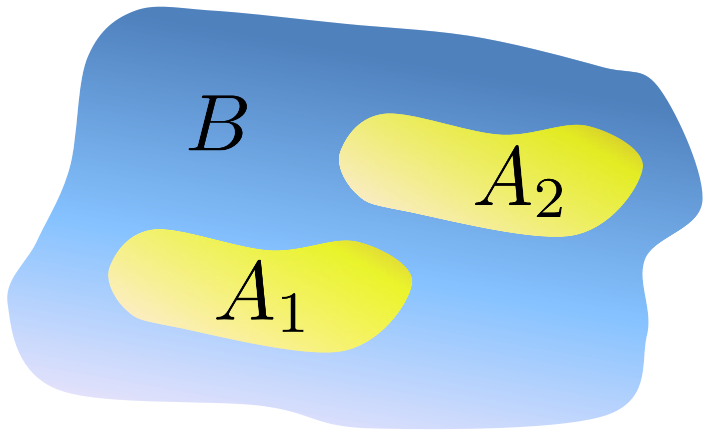


- **Entanglement negativity:** Mixed bipartite systems ⟷ Tripartite system

Partial transpose: $\langle e_i, e_j | \rho_A^{T_2} | e_k e_l \rangle = \langle e_i, e_l | \rho_A | e_k e_j \rangle$

Not positive definite!

$$\boxed{\mathcal{E}_{A_1,A_2} = \ln Tr|\rho_A^{T_2}|}$$



REPLICA APPROACH TO ENTANGLEMENT MEASURES

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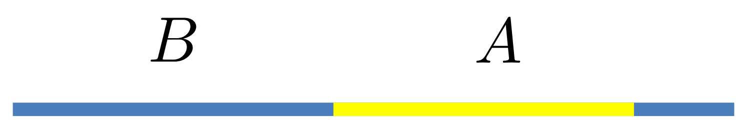

Entanglement entropy: $S_A = -tr(\rho_A \log \rho_A)$

$Tr\rho_A^n$ ⇒ Evaluated for $n \in \mathbb{N}$ and analytically continued

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} Tr\rho_A^n$$

[*Calabrese, Cardy, 2004*]

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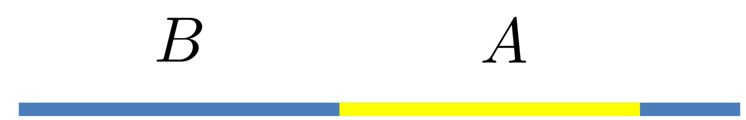

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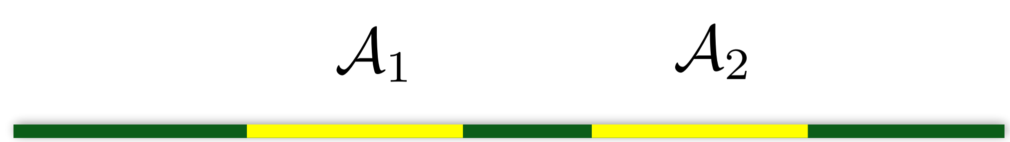


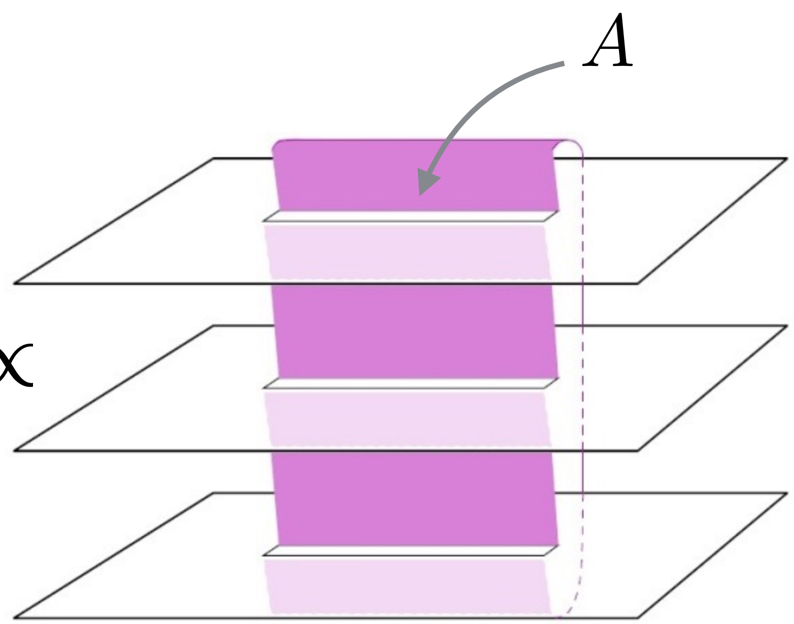
Entanglement negativity: $\mathcal{E} \equiv \ln Tr |\rho^{T_2}|$

$$\begin{cases} Tr(\rho^{T_2})^{n_e} = \sum_{\lambda_i > 0} |\lambda_i|^{n_e} + \sum_{\lambda_i < 0} |\lambda_i|^{n_e} \\ Tr(\rho^{T_2})^{n_o} = \sum_{\lambda_i > 0} |\lambda_i|^{n_o} - \sum_{\lambda_i < 0} |\lambda_i|^{n_o} \end{cases} \Rightarrow \mathcal{E} = \lim_{n_e \to 1} \ln Tr (\rho^{T_2})^{n_e}$$

[*Calabrese, Cardy, Tonni, 2011*]

QFT, PATH INTEGRAL FORMALISM

$Tr \rho_A^n \propto$



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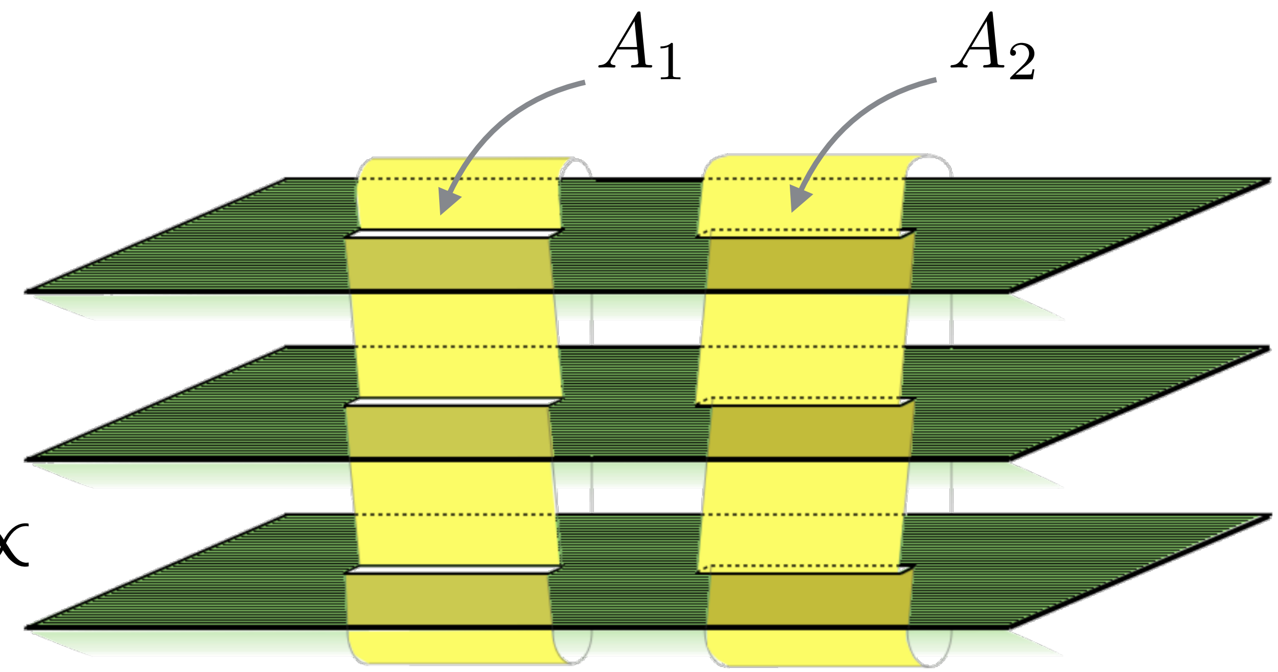

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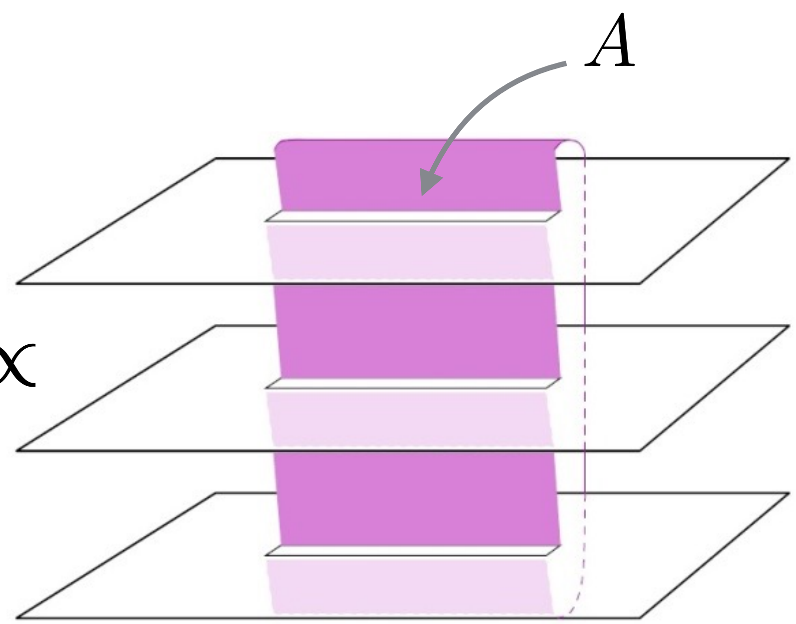

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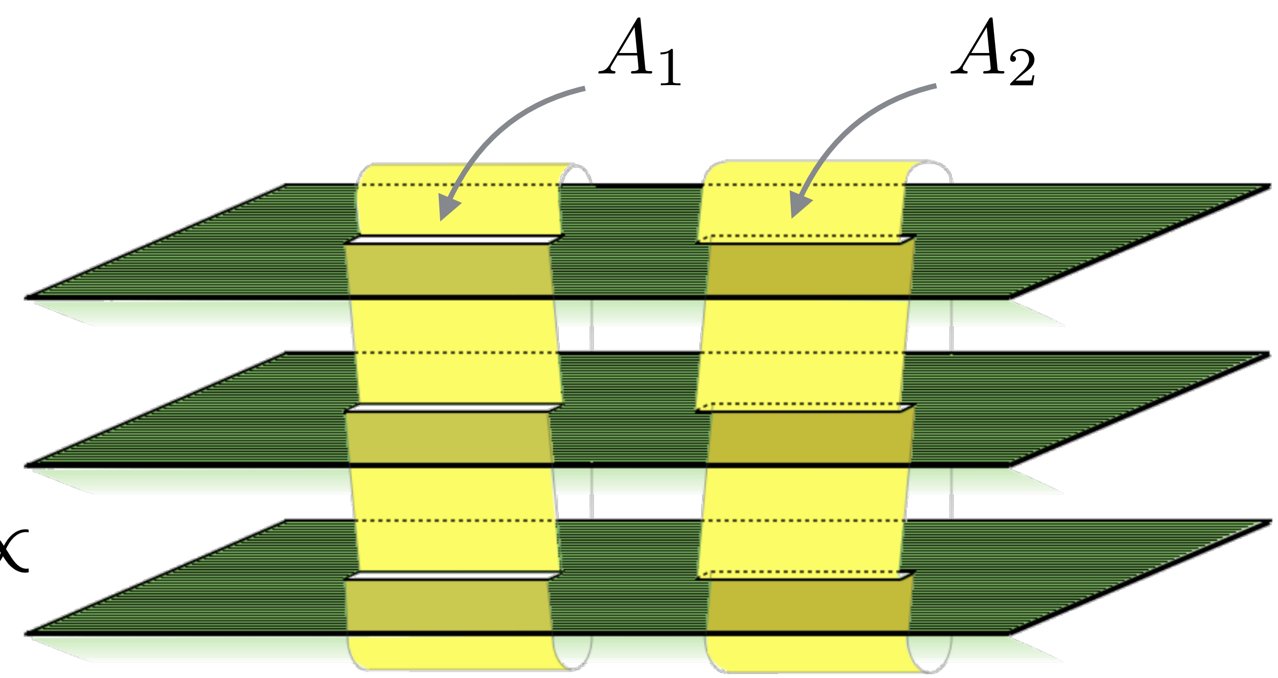

"BRANCH POINT" TWIST FIELDS: $\mathcal{T}, \tilde{\mathcal{T}}$

$$tr\rho_A^n = \mathcal{N}\langle \mathcal{T}(u_1,0)\tilde{\mathcal{T}}(v_1,0)\cdots\mathcal{T}(u_m,0)\tilde{\mathcal{T}}(v_m,0)\rangle$$

$$tr(\rho_A^{T_2})^n = \mathcal{N}\langle \mathcal{T}(u_1,0)\tilde{\mathcal{T}}(v_1,0)\tilde{\mathcal{T}}(u_2,0)\mathcal{T}(v_2,0)\rangle$$

Explicit results in CFTs!

NEGATIVITY SPECTRUM of ONE DIMENSIONAL CFTs

with V. Alba, P. Calabrese

Phys. Rev. B 94, 195121 (2016)

ENTANGLEMENT & NEGATIVITY SPECTRUM

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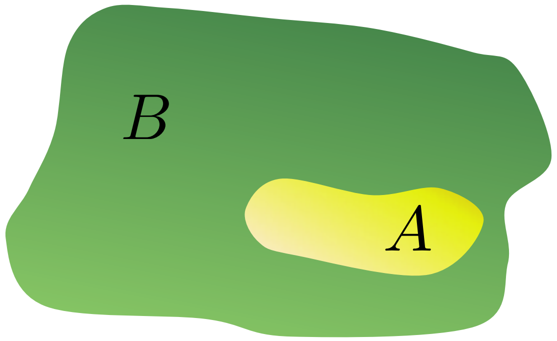
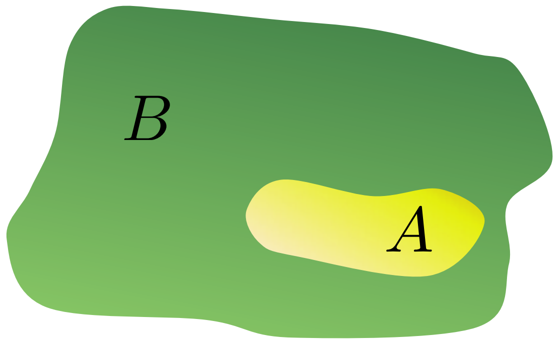


Reduced density matrix (RDM)

$$\rho_A = tr_B \rho$$

ENTANGLEMENT & NEGATIVITY SPECTRUM



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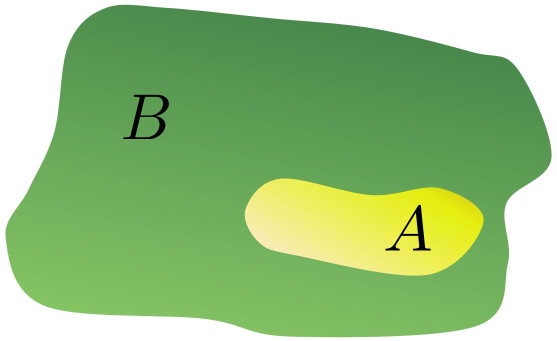

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- ***ENTANGLEMENT SPECTRUM:*** $\{\mu_i\}$

  [Li, Haldane, *2008*]

- Crucial to probe topological phases of matter
- [*Calabrese, Lefevre, 2008*]: Analytic derivation of eigenvalues distribution

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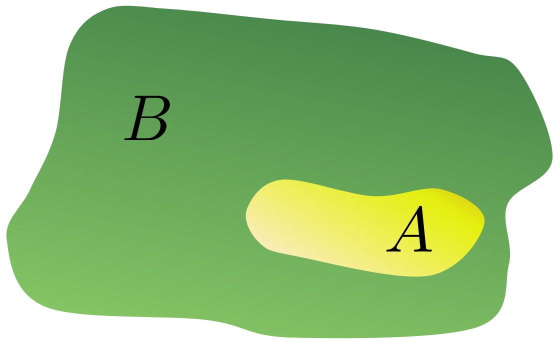

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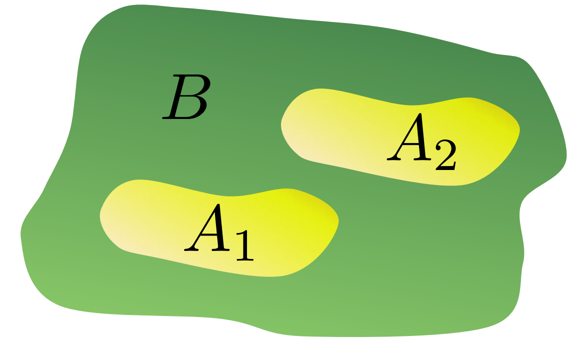

Partial transpose of the RDM

$$\langle e_i, e_j | \rho_A^{T_2} | e_k e_l \rangle = \langle e_i, e_l | \rho_A | e_k e_j \rangle$$

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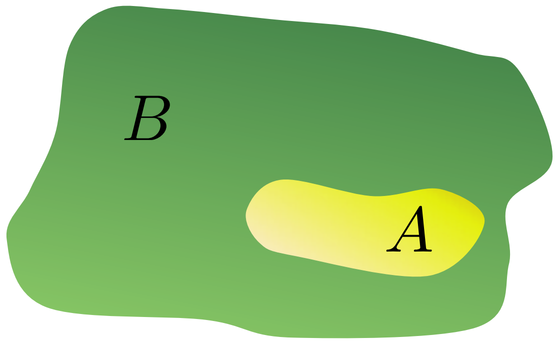

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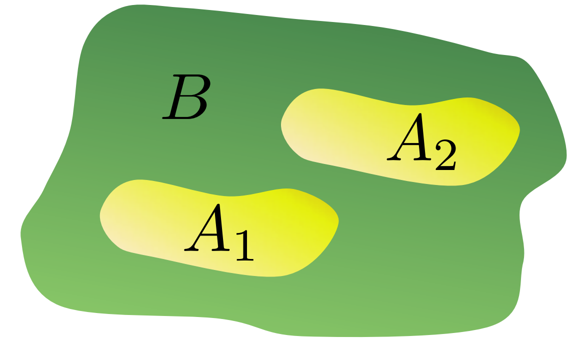

Partial transpose of the RDM

$$\langle e_i, e_j|\rho_A^{T_2}|e_k e_l\rangle = \langle e_i, e_l|\rho_A|e_k e_j\rangle$$

- **Logarithmic negativity:**

$$\mathcal{E}_{A_1,A_2} = \ln Tr|\rho_A^{T_2}| = \ln \sum_i |\lambda_i|$$

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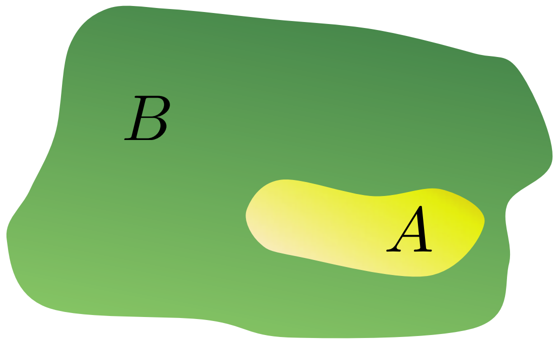

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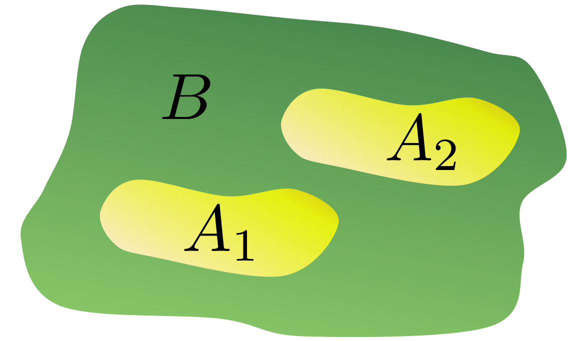

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- ***"NEGATIVITY SPECTRUM":*** $\{\lambda_i\}$

  [Ruggiero, Alba, Calabrese, *2016*]

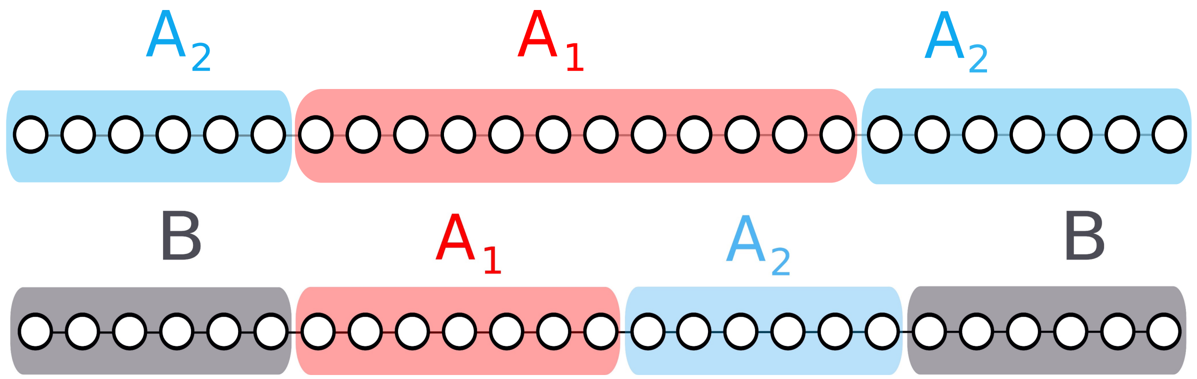
  - Can we spot new phases of matter?

    [Mbeng, Alba, Calabrese, *2016*]
  - Can we construct simpler entanglement measures for mixed states?

NEGATIVITY SPECTRUM

EIGENVALUES DISTRIBUTION

$$P(\lambda) \equiv \sum_i \delta(\lambda - \lambda_i)$$



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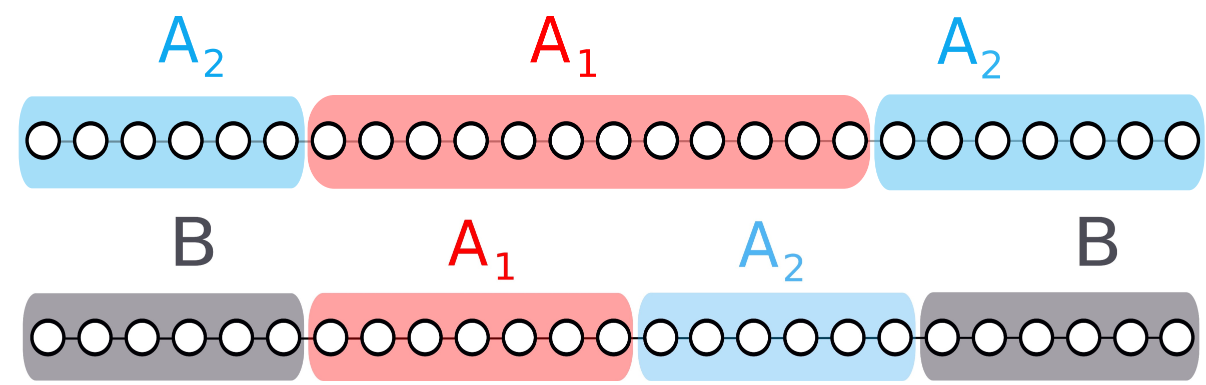

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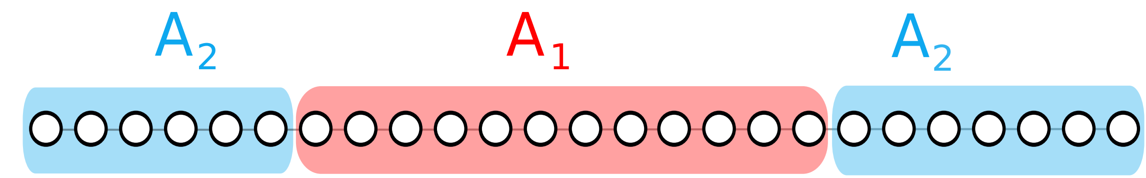
The moment problem:

- $$R_n^{T_2} \equiv Tr\left(\rho_A^{T_2}\right)^n = \sum_i \lambda_i^n = \int d\lambda\, \lambda^n P(\lambda)$$ **(moments)**
- $$f(z) \equiv \frac{1}{\pi}\int d\lambda \frac{\lambda\, P(\lambda)}{z-\lambda} = \frac{1}{\pi}\sum_{n=1}^{\infty} R_n^{T_2} z^{-n}$$ **(*Stieltjes transform*)**

$$P(\lambda) = \frac{1}{\lambda}\lim_{\epsilon\to 0} \mathrm{Im} f(\lambda - i\epsilon)$$

NEGATIVITY SPECTRUM in CFTs



Pure state:

$$R_n^{T_2} = \begin{cases} Tr\rho_{A_1}^{n_o} \\ (Tr\rho_{A_1}^{n_e/2})^2 \end{cases} \sim \begin{cases} e^{-b\left(n_o - \frac{1}{n_o}\right)} & n_o \text{ odd} \\ e^{-b\left(n_e - \frac{4}{n_e}\right)} & n_e \text{ even} \end{cases}$$

$$\xi \equiv \sqrt{b \ln(\lambda_M/\lambda)}$$

$$b \equiv -\lambda_M = \frac{c}{6} \ln \ell_1$$

$$P(\lambda) = \delta(\lambda_M - \lambda) + \frac{b\theta(\lambda_M - |\lambda|)}{|\lambda|\xi}\left[\frac{\text{sgn}(\lambda)}{2} I_1(2\xi) + I_1(4\xi)\right]$$

NEGATIVITY SPECTRUM in CFTs

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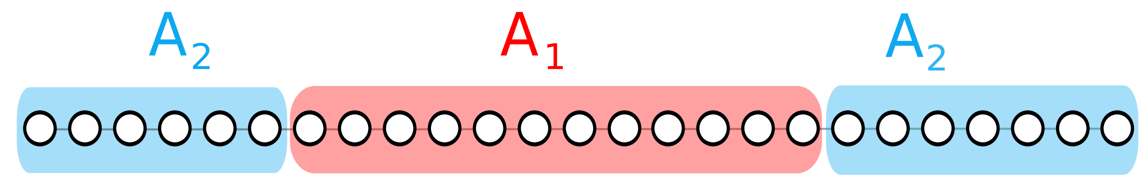

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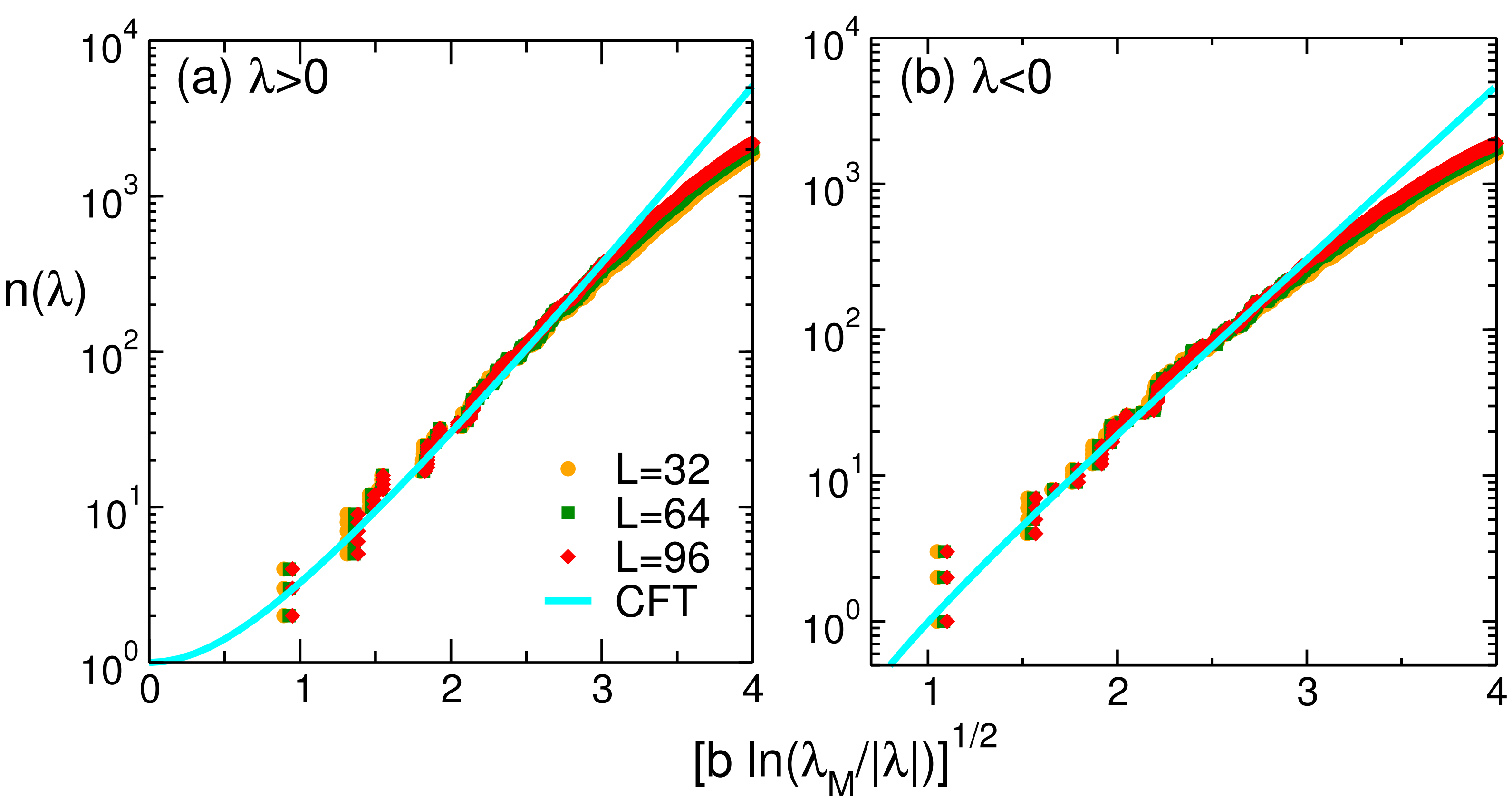

NUMBER DISTRIBUTION FUNCTION

$$n(\lambda) \equiv \begin{cases} \int_\lambda^{\lambda_M} d\lambda\, P(\lambda) & \lambda > 0 \\ \int_{\lambda_m}^{\lambda} d\lambda\, P(\lambda) & \lambda < 0 \end{cases}$$

NEGATIVITY SPECTRUM in CFTs

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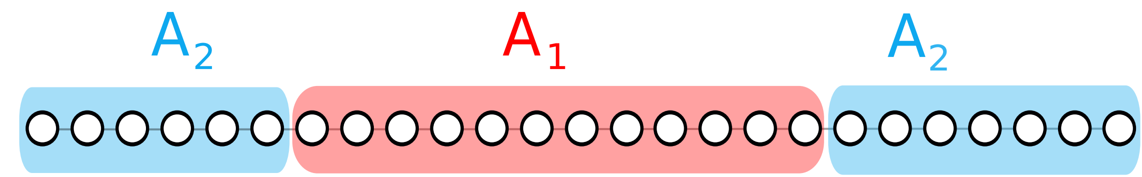

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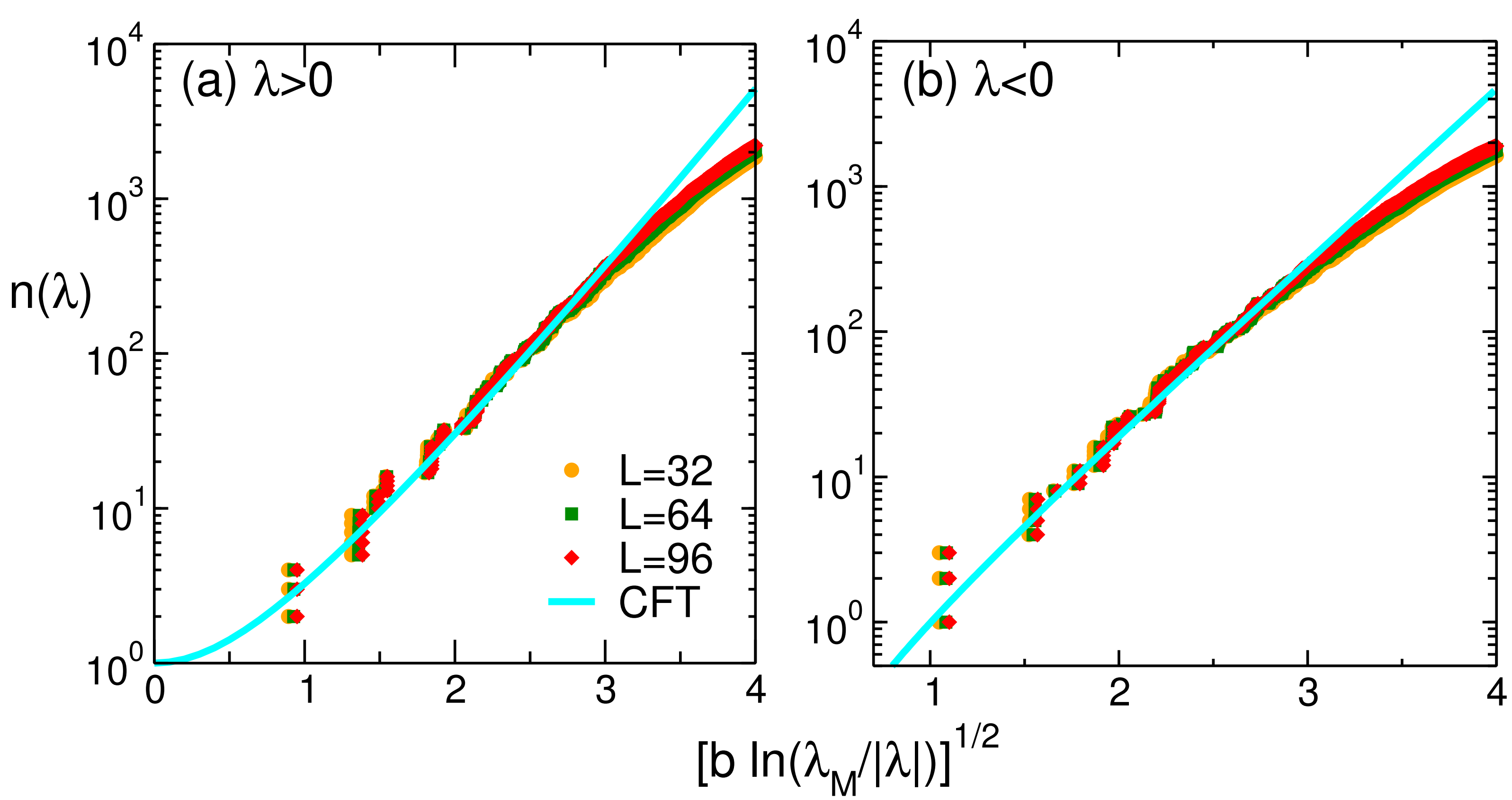

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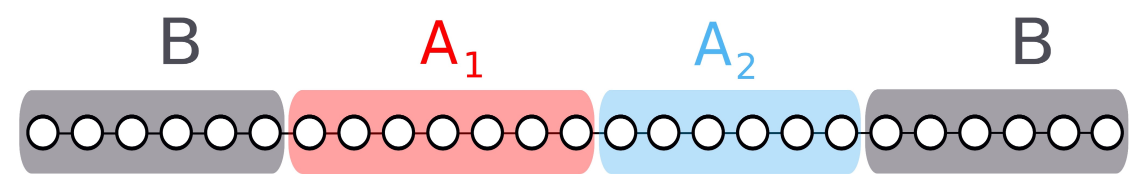
"**Superuniversal**"!
(in analogy with the entanglement spectrum)

NEGATIVITY SPECTRUM in CFTs

Mixed state:



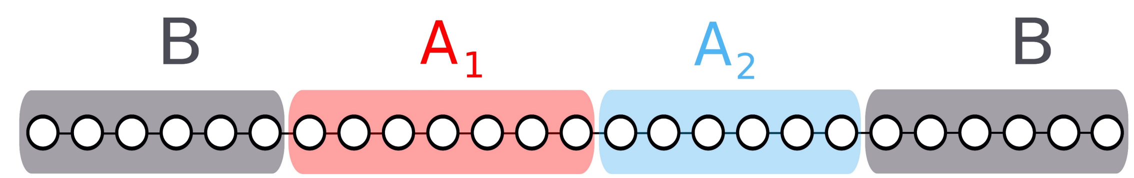
$$R_n^{T_2} \sim \begin{cases} (\ell_1 \ell_2)^{-\frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right)} (\ell_1 + \ell_2)^{-\frac{c}{6}\left(\frac{n_e}{2} + \frac{1}{n_e}\right)} \\ (\ell_1 \ell_2 (\ell_1 + \ell_2))^{-\frac{c}{12}\left(n_0 - \frac{1}{n_0}\right)} \end{cases}$$

$$\tilde{\xi} \equiv \sqrt{\tilde{b} \ln(\lambda_M / \lambda)} \qquad \tilde{b} \equiv 2b + \frac{c}{6} \ln \frac{\omega}{(1+\omega)^2} \qquad \omega \equiv \frac{\ell_2}{\ell_1}$$

$$P(\lambda) = \delta(\lambda_M - \lambda) + \frac{\theta(\lambda_M - |\lambda|)}{2|\lambda|} \left[\frac{b}{\xi} I_1(2\xi)\, \mathrm{sgn}(\lambda) + \frac{\tilde{b}}{\tilde{\xi}} I_1(2\tilde{\xi}) \right]$$

NEGATIVITY SPECTRUM in CFTs

Mixed state:



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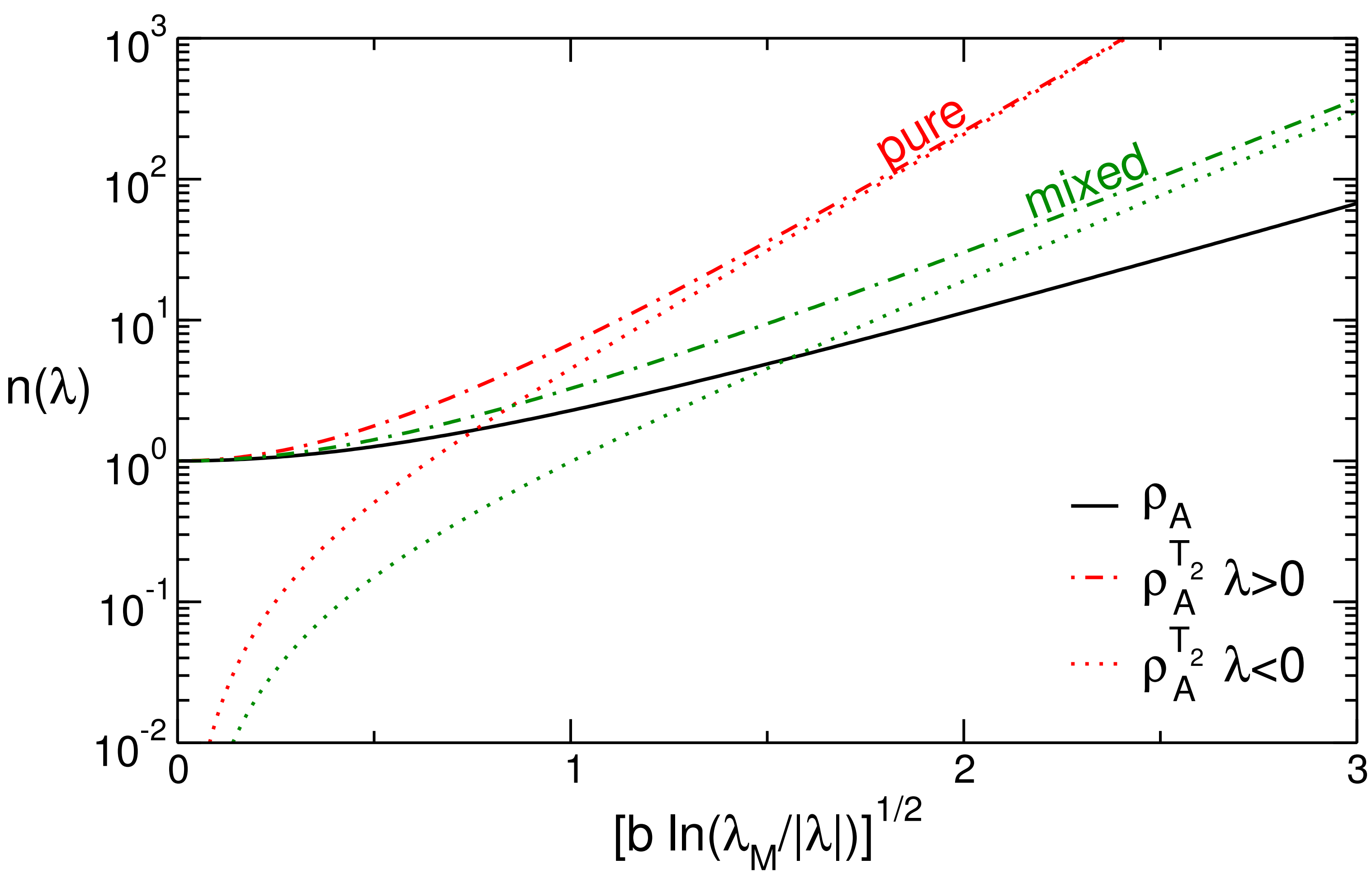
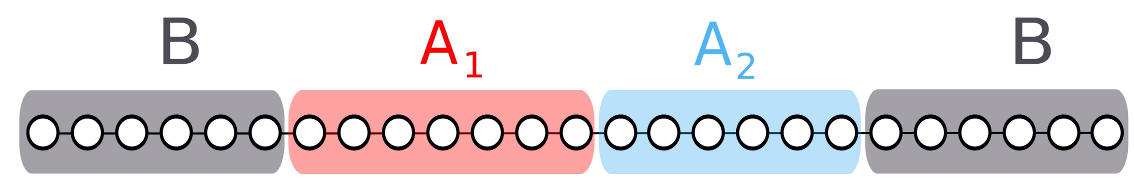


NUMBER DISTRIBUTION FUNCTION

$$n(\lambda; \xi, \tilde{\xi}) = n(\lambda; c, \omega)$$

NEGATIVITY SPECTRUM in CFTs

Mixed state:



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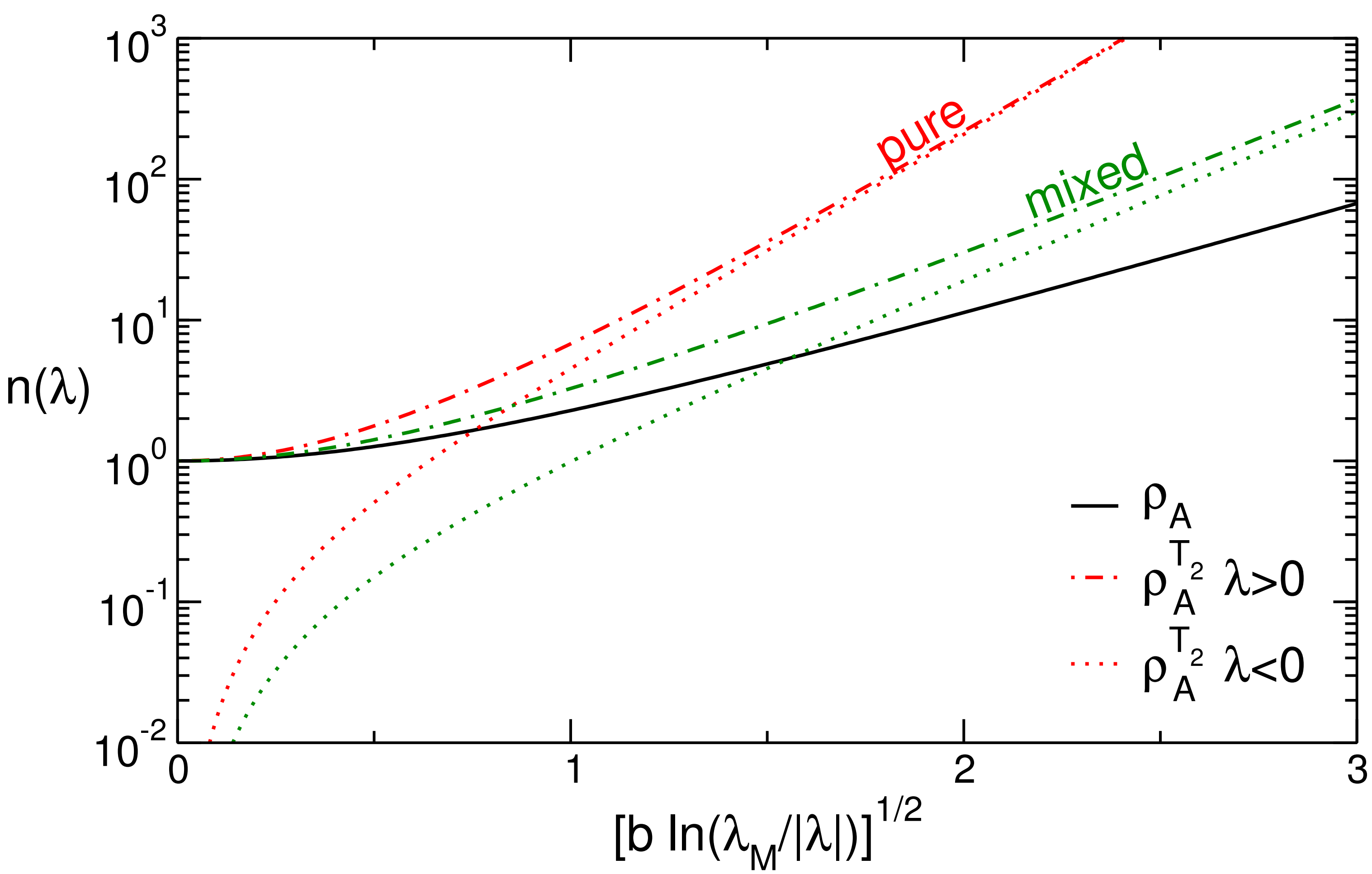

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$$n(\lambda;\xi,\tilde{\xi}) = n(\lambda;\cancel{\epsilon},\cancel{\omega})$$

Limit $\ell_1 \to \infty$:

- "**Superuniversal**"
- Independent on the geometry of the tripartition

Thank you for the attention.