Non-perturbative Renormalization and Running of Quark Masses in $N_f = 3$ QCD

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in collaboration with

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ALPHA Collaboration

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Introduction

The goal of this project is to compute the **NP renormalization and running of the quark masses with Nf=3 QCD** with a crucial control on systematics and high accuracy in a large range of scales: from the EW scale down to an hadronic scale to make contact with large volume simulations.

Since this work is a joint project with the one of the **running coupling** by the ALPHA Collaboration we follow the **same strategy** they have been using.

M.Bruno et al.
Phys.Rev.Lett. 119 (2017)
no.10, 102001

M.Dalla Brida et al.
Phys.Rev.Lett. 117 (2016)
no.18, 182001

RG Equations

Renormalization Group functions for the coupling and mass are given by

$$\mu \frac{d\bar{g}}{d\mu} = \beta(\bar{g})$$

$$\mu \frac{d\bar{m}}{d\mu} = \tau(\bar{g})\bar{m}$$

they admit a perturbative expansion as

$$\beta(g) = -g^3(b_0 + b_1 g^2 + b_2 g^4 + \dots) \qquad \tau(g) = -g^2(d_0 + d_1 g^2 + d_2 g^4 + \dots)$$

where d_0, b_0, b_1 are the only scheme independent coefficients

We than introduce the **Renormalization Group Invariant** (RGI) quantities, formal solution of the RG equations as

$$\Lambda = (b_0 \bar{g}^2)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2(\mu))} \exp\left\{ -\int_0^{\bar{g}(\mu)} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\} \mu$$

$$M = (2b_0 \bar{g}^2(\mu))^{-d_0/(2b_0)} \exp\left\{ -\int_0^{\bar{g}(\mu)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x} \right] \right\} \bar{m}(\mu)$$

The **RG evolution between two scales** $\mu, \mu/s$ is then

$$-\ln(s) = \int_{\sqrt{\bar{g}^2(\mu)}}^{\sqrt{\bar{g}^2(\mu/s)}} \frac{dg}{\beta(g)}$$

$$\frac{\bar{m}(\mu)}{\bar{m}(\mu/s)} = \exp\left\{ -\int_{\sqrt{\bar{g}^2(\mu)}}^{\sqrt{\bar{g}^2(\mu/s)}} \frac{\tau(g')}{\beta(g')} dg' \right\}$$

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$$M = (2b_0\bar{g}^2(\mu))^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}(\mu)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\}\bar{m}(\mu)$$

for $s = 2$ we have the "usual" definition of the **SSFs** (in the continuum)

$$-\ln(2) = \int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{dg}{\beta(g)} \qquad\qquad \sigma_P(u) = \frac{\bar{m}(\mu)}{\bar{m}(\mu/2)} = \exp\left\{-\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{\tau(g')}{\beta(g')} dg'\right\}$$

Non-perturbative Renormalization

There are several ways to compute non-perturbatively renormalization constants on the lattice.

$$\bar{m}(\mu) = \lim_{a \to 0} \boxed{Z_m(a\mu, g_0)} m(g_0)$$

In general, it is possible to distinguish between:

Infinite Volume e.g. like RI-MOM [Martinelli et al. Nucl.Phys. B445 (1995)]

Finite Volume

The scale is naturally identified by $\boxed{\mu = 1/L}$ no need to inject extern momenta.
Possibility to **built a recursion** to cover large range of scales $\mathcal{O}(\Lambda_{\rm QCD}) \to \mathcal{O}(M_W)$

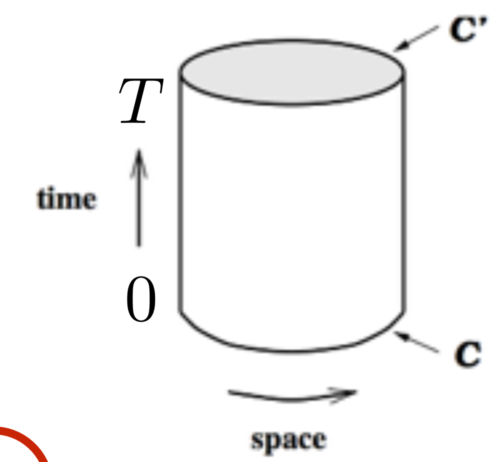
Schrödinger Functional (SF) [Lüscher et al. 1992] [Sint 1993]

Schrödinger Functional (SF)

[Lüscher et al. 1992]
[Sint 1993]

The SF is the functional integral on a hyper cylinder with **PBC in spatial directions** and **Dirichlet conditions in time**

$$\mathcal{Z}[C',\bar{\rho}',\rho';C,\bar{\rho},\rho] \int D[U]D[\psi]D[\bar{\psi}]e^{-S[U,\bar{\psi},\psi]}$$



For the **gauge potential** we have

$$A_k(x)|_{x_0=T} = C'_k \qquad \text{at} \quad \mathrm{x}_0 = \mathrm{T}$$
$$A_k(x)|_{x_0=0} = C_k \qquad \text{at} \quad \mathrm{x}_0 = 0$$

BG field →

$$\bar{g}^2_{\mathrm{SF}} = \frac{\partial_\eta \Gamma_0[B]|_{\eta=0}}{\partial_\eta \Gamma[B]|_{\eta=0}}$$

while for **fermions**

$$P_-\psi(x)|_{x_0=T} = \rho' \qquad P_+\psi(x)|_{x_0=0} = \rho \qquad \psi(x+L\hat{k}) = e^{i\theta_k}\psi(x)$$
$$\bar{\psi}(x)P_+|_{x_0=T} = \bar{\rho}' \qquad \bar{\psi}(x)P_-|_{x_0=0} = \bar{\rho} \qquad \bar{\psi}(x+L\hat{k}) = \bar{\psi}(x)e^{-i\theta_k}$$

for more details SEE TALK BY: S.Sint (TOMORROW @09:40)

Expectation values can be evaluated

$$\langle \mathcal{O} \rangle = \left\{ \frac{1}{\mathcal{Z}} \int D[U]D[\psi]D[\bar{\psi}]\mathcal{O}e^{-S[U,\bar{\psi},\psi]} \right\}_{\bar{\rho}'=\rho'=\bar{\rho}=\rho=0}$$

Fermionic correlation functions are usually computed without BG field.

The observable may contain "**boundary fields**"

$$\zeta(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}(\mathbf{x})} \qquad \bar{\zeta}'(\mathbf{x}) = -\frac{\delta}{\delta\rho'(\mathbf{x})}$$

$$\bar{\zeta}(\mathbf{x}) = -\frac{\delta}{\delta\rho(\mathbf{x})} \qquad \zeta'(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}'(\mathbf{x})}$$

SF Renormalization Scheme

M.Luscher et al.
Nucl.Phys. B384 (1992)

Taking the (non-anomalous) Ward Identity

$$\partial_\mu (A_R)^{ij}_\mu = (\bar{m}_i + \bar{m}_j) P^{ij}_R$$

with

$$(A_R)^{ij}_\mu(x) = Z_A \bar{\psi}_i(x)\gamma_\mu\gamma_5\psi_j(x)$$
$$P^{ij}_R(x) = Z_P \bar{\psi}_i(x)\gamma_5\psi_j(x)$$

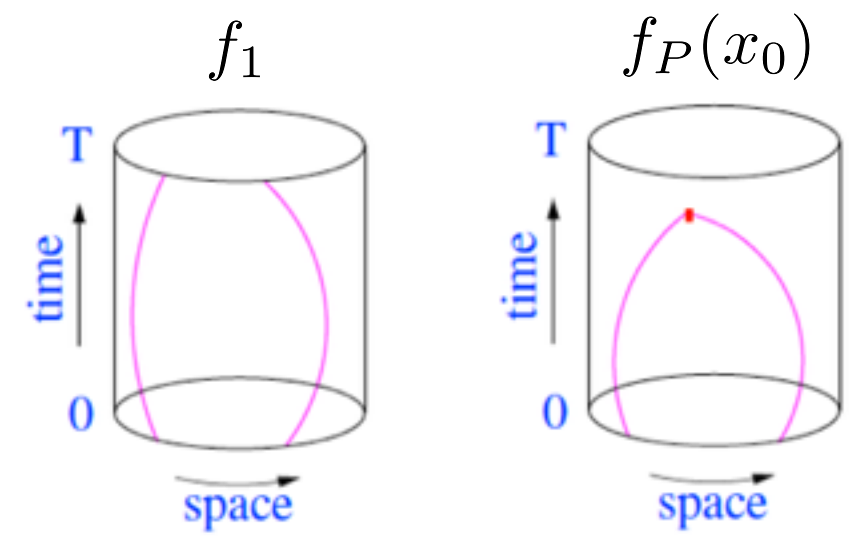
Z_A is finite, all the scale dependence is encoded in $Z_P \longrightarrow \tau = -\gamma_P$

The SF renormalization condition is imposed at **vanishing quark mass**

$$Z_P(g_0, L/a)\frac{f_P(L/2)}{\sqrt{3 f_1}}\Bigg|^{\theta=0.5}_{m=0} = c_3(\theta, a/L)$$

M.Luscher et al.
Nucl.Phys. B582 (2000)

M.Della Morte et al.
Nucl.Phys. B729 (2005)



The lattice version of the SSF is than defined as the ratio of renormalization constants at L and $2L$ identifying $\mu = L^{-1}$ and for $s = 2$

$$\Sigma_P(u, g_0, L/a) = \frac{Z_P(g_0, 2L/a)}{Z_P(g_0, L/a)}\Bigg|_{u=\bar{g}^2(L)}$$

$$\sigma_P(u) = \lim_{a\to 0} \Sigma_P(u, g_0, L/a)$$

the continuum limit is taken by keeping

$$u = \bar{g}^2(L) = \text{const}$$

Two Schemes, Two Regions, More fun

The computational cost of measuring the SF coupling grows fast at low energies and in particular towards the continuum limit. Thus it is challenging to reach the low energy domain characteristic of hadronic physics, especially if one aims at maintaining the high precision.

for more details
SEE TALK BY:
S.Sint
(TOMORROW @09:40)

The GF coupling seems to be better suited for this task. The relative precision of the coupling in this scheme is typically high and shows a weak dependence on both the energy scale and the cutoff.

M.Bruno et al.
Phys.Rev.Lett. 119 (2017) no.10, 102001

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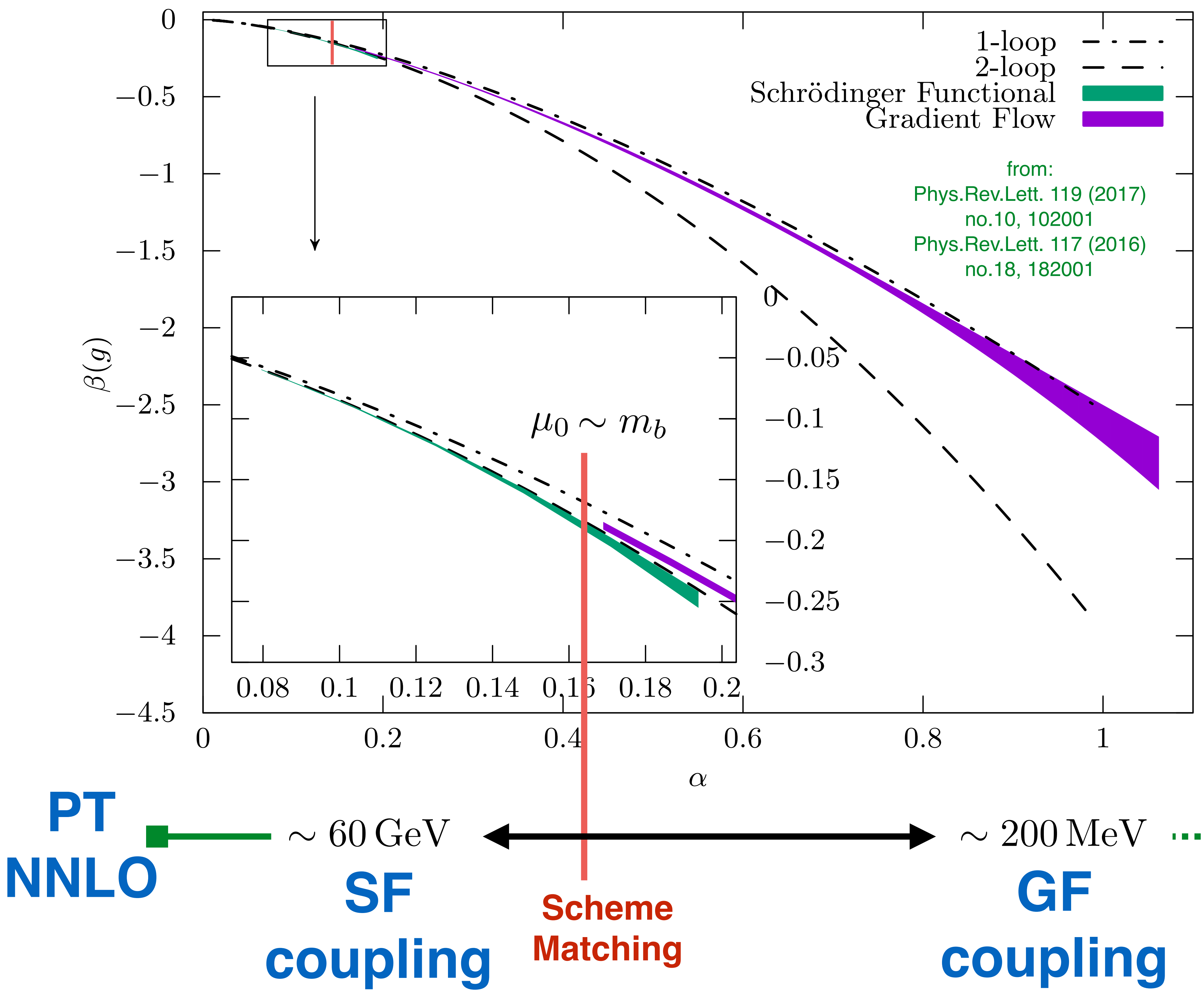


for more details SEE TALK BY: S.Sint (TOMORROW @09:40)

Continuum Limit SF-SF

$$\Sigma_P(u, a/L) = \sigma_P(u) + \rho\,(u)(a/L)^2$$

The 1-loop improved SSF are defined as

$$\Sigma_P^{(1)}(u, a/L) = \frac{\Sigma_P(u, a/L)}{1 + \delta(a/L)u}$$

where $\delta(a/L)$ are the 1-loop SSF cutoff effects.

[Sint and Weisz Nucl.Phys. B545 (1999) 529-542]

We consider lattices

$$L/a = [6, 8, 12]$$
$$2L/a = [12, 16, 24]$$

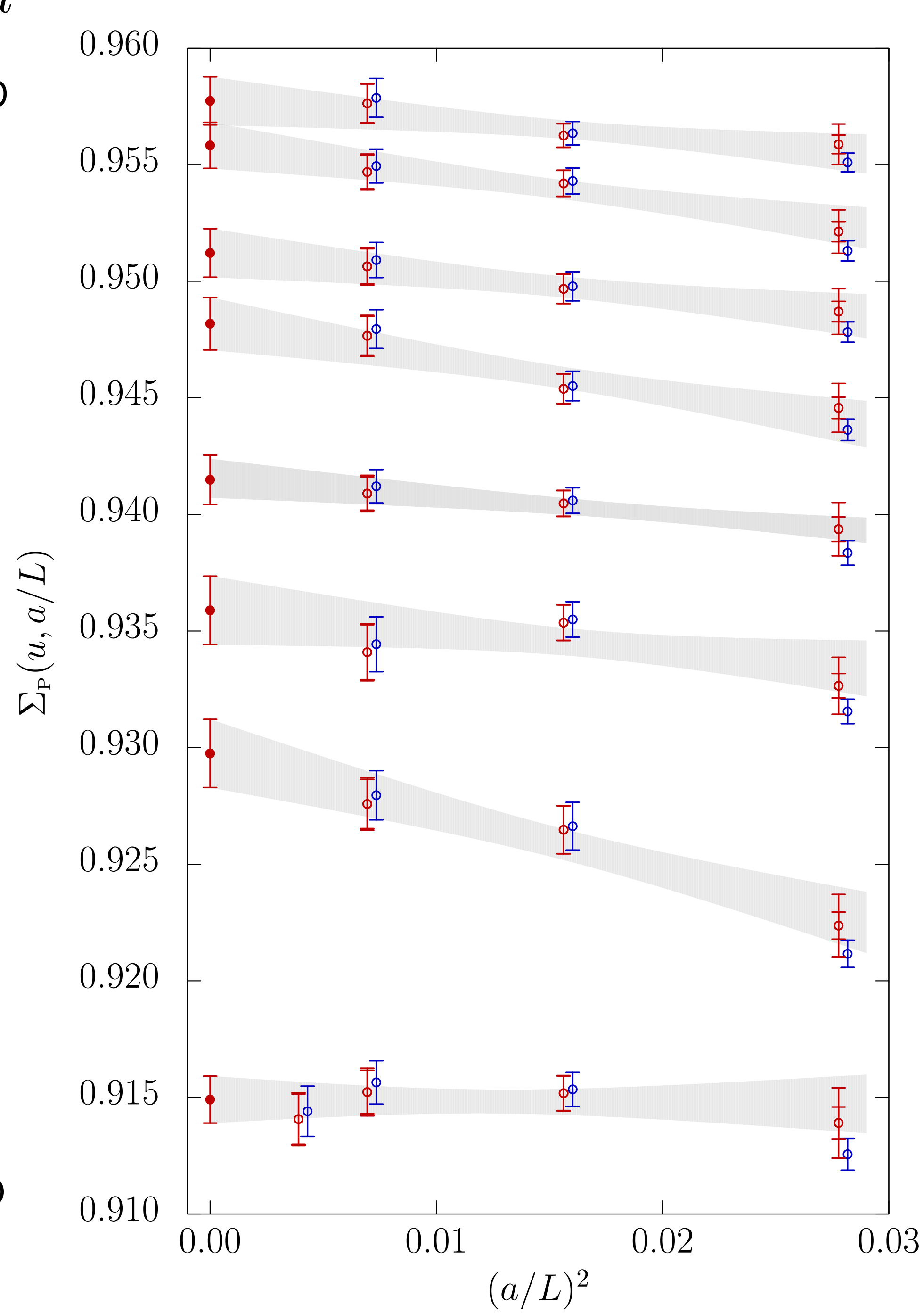
at the **switching scale**

$$\bar{g}^2_{\mathrm{SF}}(\mu_0) = 2.012$$

[Dalla Brida et al. PRL 117 (2016)]

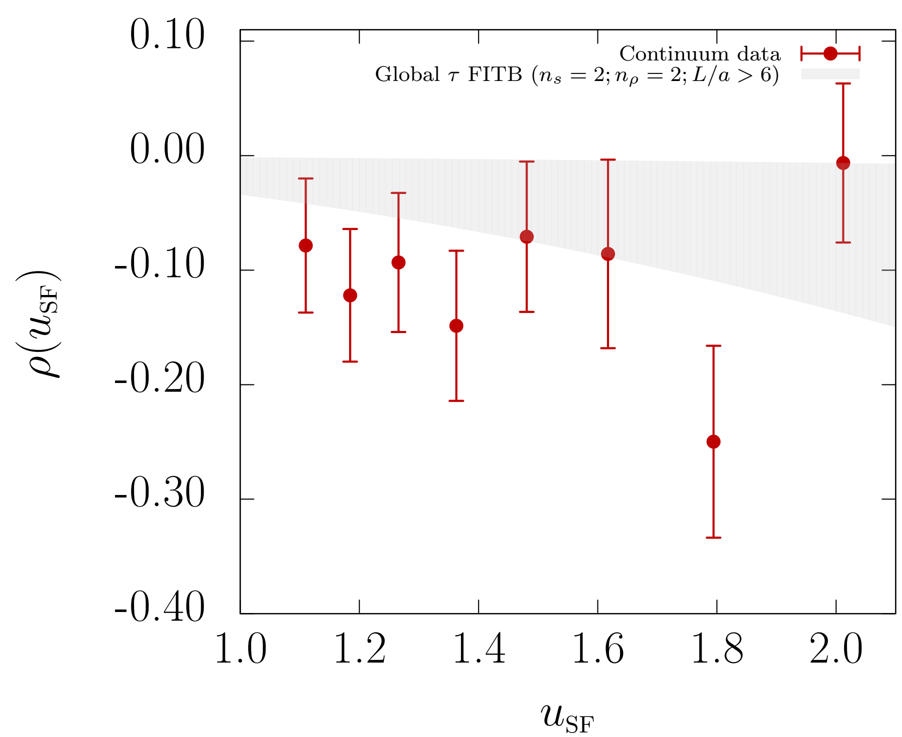
in order to have a better control on the continuum extrapolation we consider also the step $16 \to 32$

$$u_{SF} = [1.1100, 1.1844, 1.2565, 1.3627, 1.4808, 1.6173, 1.7943, 2.0120]$$



we produced the gauge ensemble with **Wilson plaquette** action and non-perturbatively O(a)-improved Nf=3 fermions

8 independent extrapolations



Continuum SSF (SF-SF)

Using

$$\log(\Sigma_P(u,a/L) - \rho(u)(a/L)^2) = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dx \left[\frac{\tau(x)}{\beta(x)}\right]$$

and with the knowledge of **non-perturbative β** parametrised as

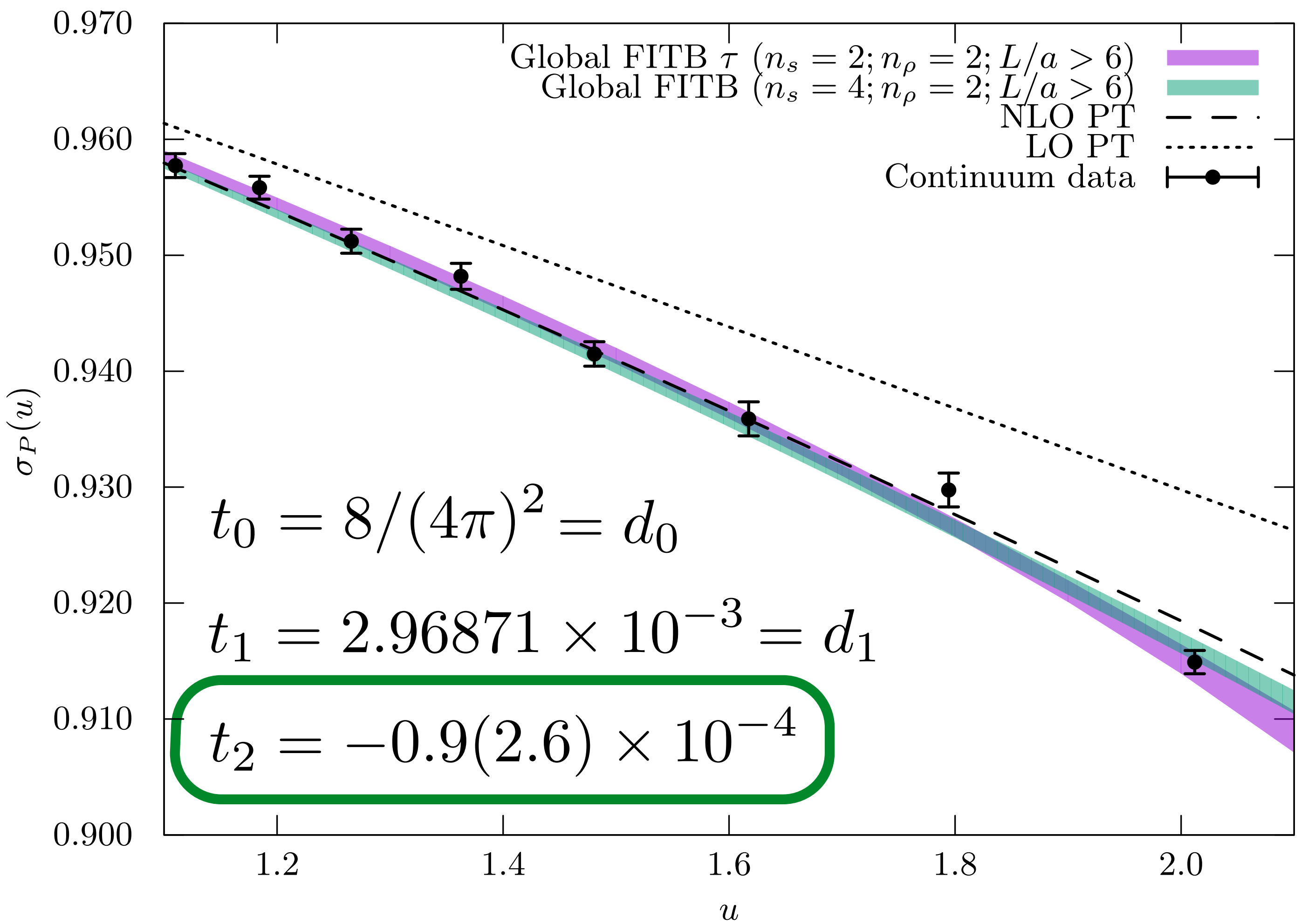
$$\beta(x) = -x^3(b_0 + b_1 x^2 + b_2 x^4 + b_3^{\mathrm{eff}} x^6)$$

$$(4\pi)^4 b_3^{\mathrm{eff}} = 4(3)$$

[Dalla Brida et al. Phys.Rev.Lett. 117 (2016)]

It is possible to extract from our SSF the **non-perturbative anomalous dimension** of the mass

$$\tau(x) = -x^2 \sum_{n=0}^{n_s} t_n x^{2n}$$



The SSF can be now “re-constructed” and compared with a usual polynomial fit

$$\sigma_P(u) = 1 + \sum_{n=1}^{n_s} c_n u^n$$

a few words about Gradient Flow coupling

As mentioned before, at $\mu_0/2 \sim 2\,\mathrm{GeV}$ **we change definition of the renormalized coupling** to the GF coupling.

In the continuum

$$B_\mu(x, t=0) = A_\mu(x)$$

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}$$

The analogous can be defined on the lattice in a way not to introduce any O(a) effect.

Moreover, it is possible to define it in the SF **finite volume**, fixing the smearing radius to be proportional to the volume

$$\bar{g}^2_{\mathrm{GF}}(L) = \mathcal{N}^{-1}(c)\frac{t^2}{4}\frac{\langle G^a_{ij}(x,t)G^a_{ij}(c,t)\delta_{Q,0}\rangle}{\langle\delta_{Q,0}\rangle}\Bigg|_{\sqrt{8t}=cL, x_0=T/2}$$

[Dalla Brida et al. Phys.Rev. D95 (2017)]

In the definition we are explicitly **projecting in the Q=0 topological sector**

Numerical evidence shows [Fritzsch P. and Ramos A. JHEP 1310 (2013)] that this coupling is **more accurate than the SF one in the deep non-perturbative energy region**, and then more suited for reaching betas from large volume simulations (CLS)

the switching scale in this other scheme is identified by $\bar{g}^2_{\mathrm{GF}}(\mu_0/2) = 2.6723(64)$

[Dalla Brida et al. Phys.Rev. D95 (2017)]

Continuum SSF (GF-SF)

$$u_{GF} = [2.1257, 2.3900, 2.7359, 3.2029, 3.8643, 4.4901, 5.3010]$$

As for the **high-energy region**, we proceed by a global analysis. Since we used the same gauge ensambles used for the running coupling project [Dalla Brida et al. arXiv:1607.06423 [hep-lat]] **the correlation is taken into account.**

$$L/a = [8, 12, 16]$$
$$2L/a = [16, 24, 32]$$

Nf=3 O(a)-imp fermions and **LW** gauge action same enables of

as before, using the SSF we are able to write

$$\log[\sigma_P(u, a/L) - \rho(u)(a/L)^2] = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dx f(x)$$

where, we parametrise the integrand of the evolution function as

$$f(x) = \frac{\tau(x)}{\beta(x)} = \frac{1}{x}\sum_{n=0}^{n_f} f_n x^{2n}$$

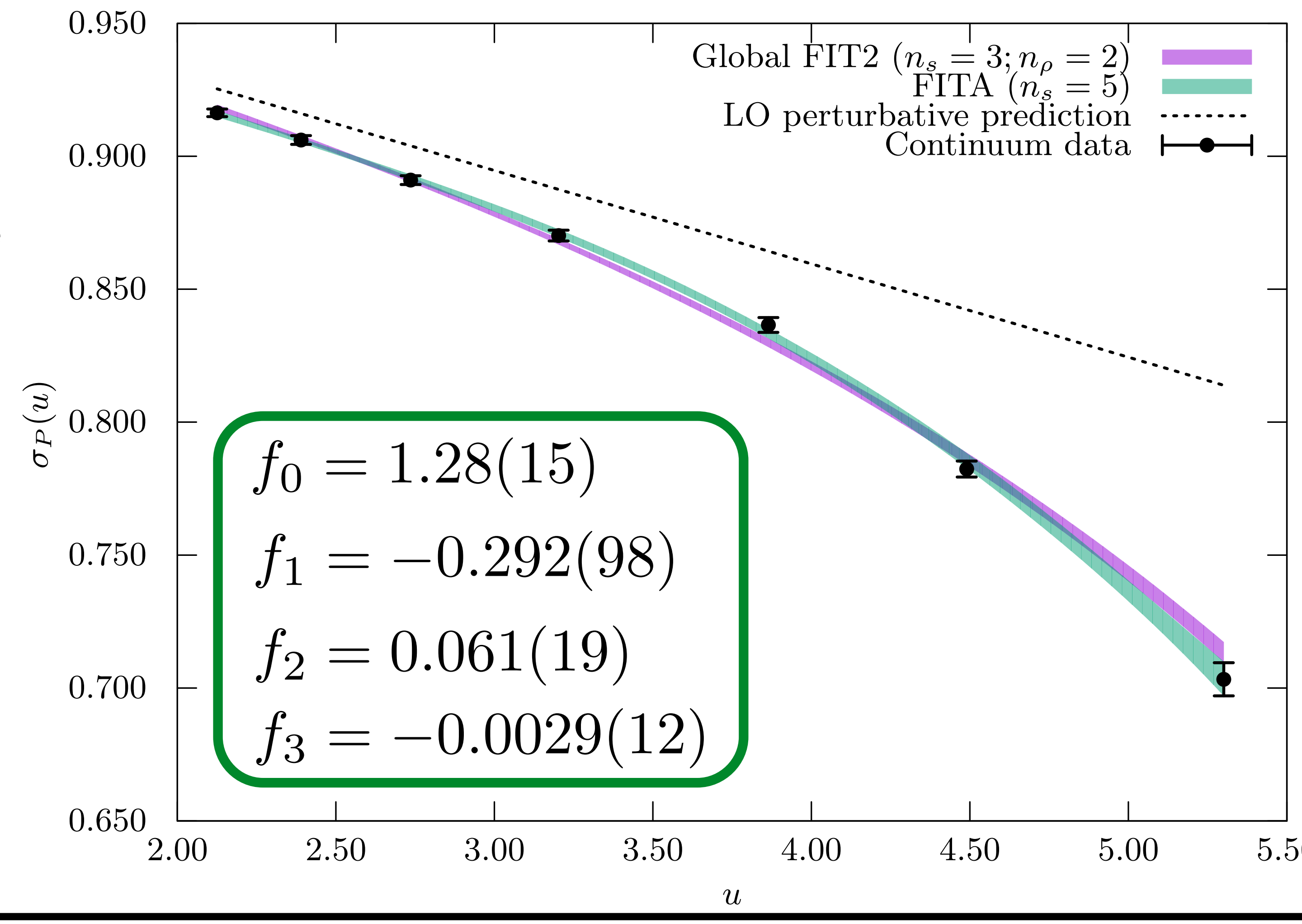
In order to isolate the anomalous dimension, we recomputed

$$\beta(x) = \frac{-x^3}{\sum_{k=0}^{k_t} p_k x^{2k}}$$

[Dalla Brida et al. Phys.Rev. D95 (2017)]

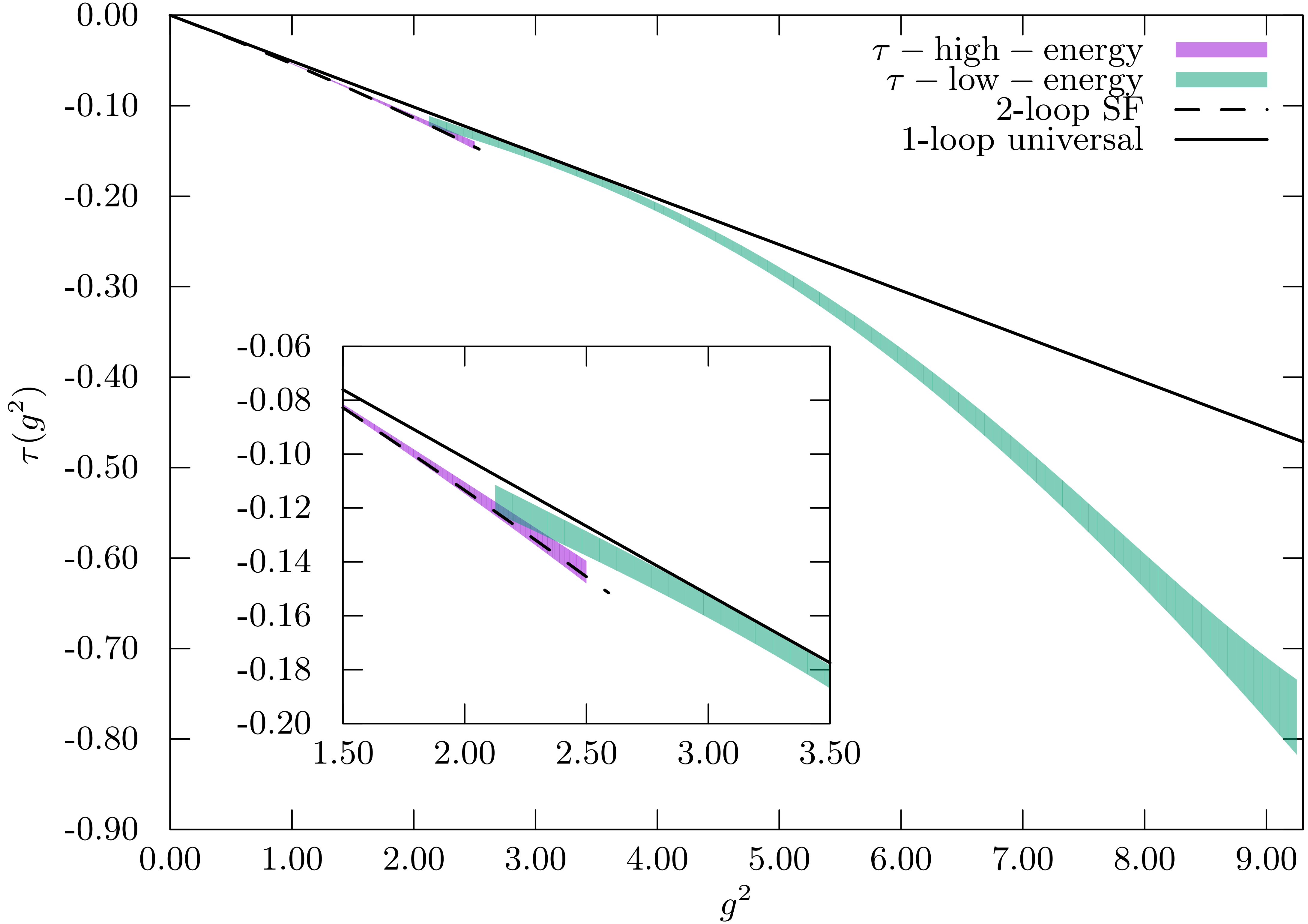
and then tau is finally given by

$$\tau(\bar{g}) = -\bar{g}^2 \frac{\sum_{n=0}^{n_f} f_n \bar{g}^{2n}}{\sum_{k=0}^{k_t} p_k \bar{g}^{2k}}$$



NP Mass Anomalous Dimension SF-SF & GF-SF

$$\tau(g) = -g^2(d_0 + d_1 g^2 + d_2 g^4 + \dots)$$



NP evolution in SF-SF & GF-SF

The **evolution between two scales** is given by

$$R^{(k)} = \exp\left\{-\int_{\sqrt{u_k}}^{\sqrt{u_{-1}}} dx\,\frac{\tau(x)}{\beta(x)}\right\} = \prod_{n=0}^{k} \sigma_P(u_n)$$

where for $\mu > \mu_0/2$

$$u_{-1} = \sigma(\bar{g}^2_{\rm SF}(\mu_0))$$

$$u_0 = \bar{g}^2_{\rm SF}(\mu_0) = 2.012$$

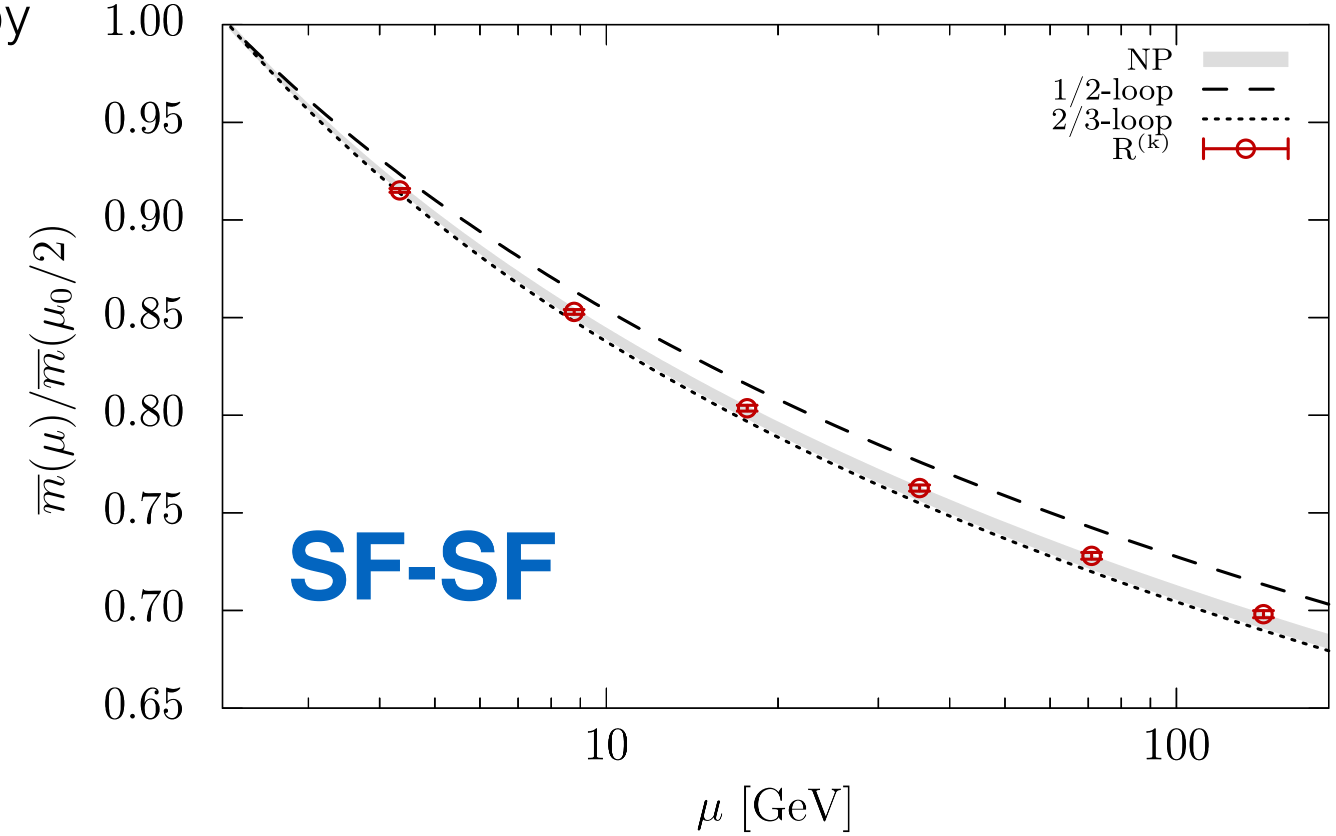
$$\vdots$$

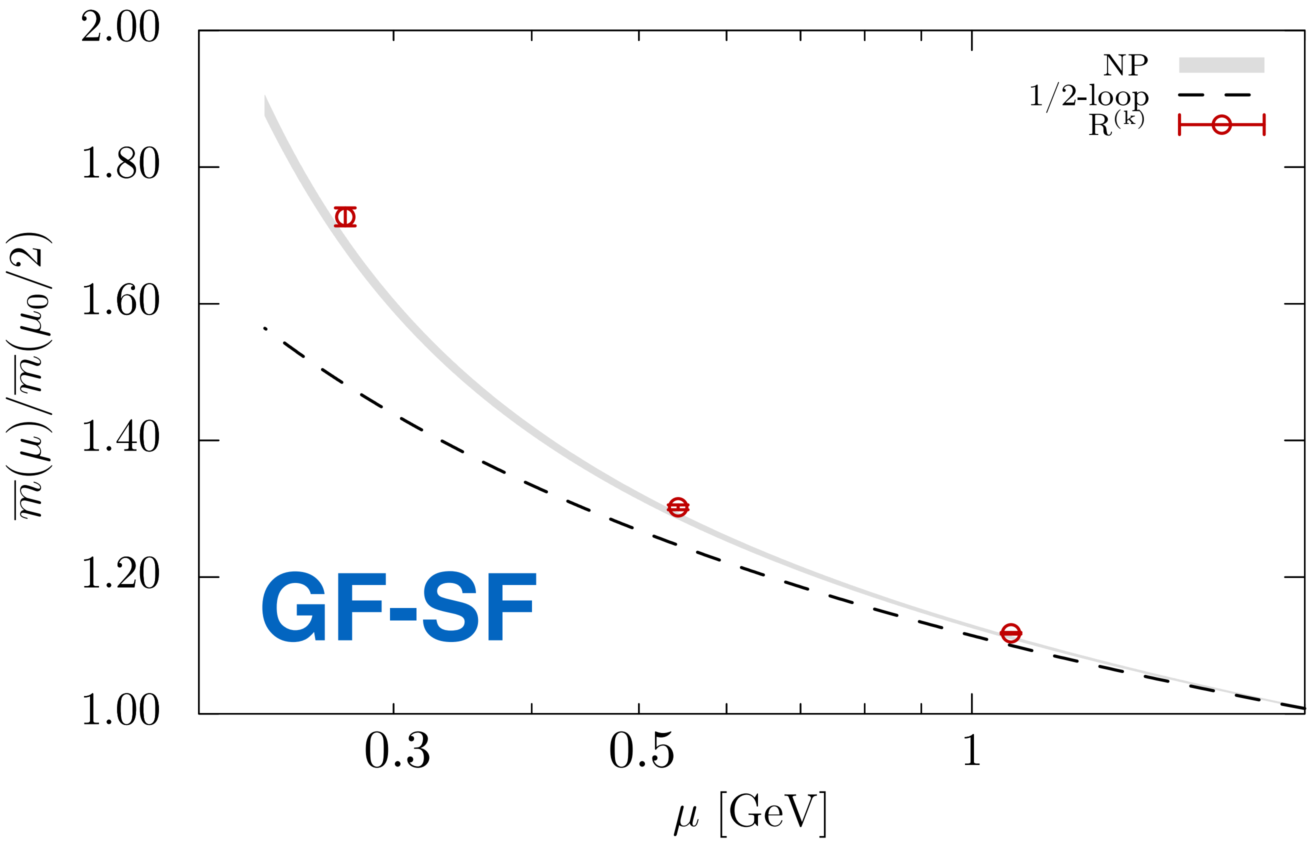
$$u_k = \bar{g}^2_{\rm SF}(2^k\mu_0)\ ,\ k \geq -1$$

the same can be computed for $\mu < \mu_0/2$ with

$$u_k = \bar{g}^2_{\rm GF}(2^k\mu_0)\ ,\ k \leq -1$$

$$R^{(k)} = \prod_n (\sigma_P(u_n))^{-1}$$





Perturbative Matching

Given the non-perturbative anomalous dimension, we checked the "functional approach" to u=0

$$M/\bar{m}(\mu) = (2b_0\bar{g}^2(\mu))^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}(\mu)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\}$$

can be rephrased as

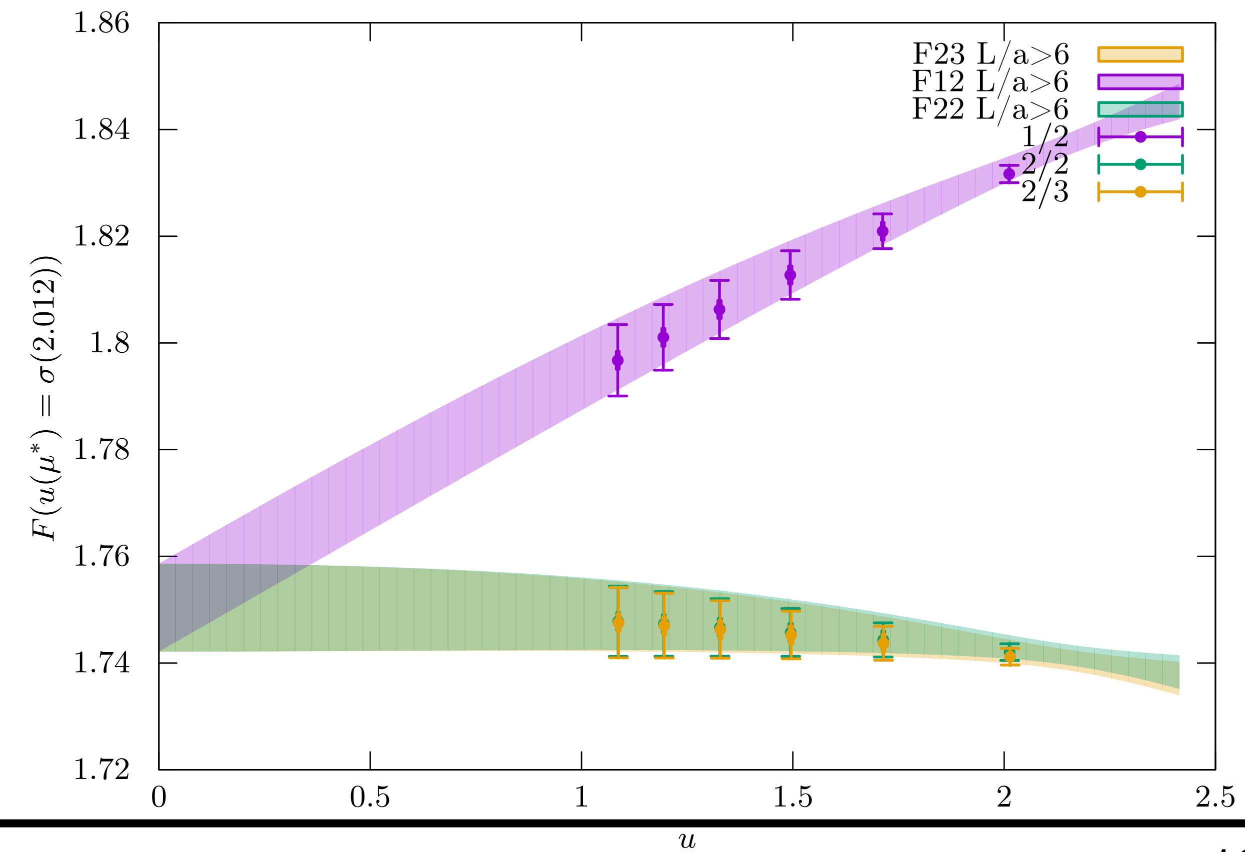
$$M/\bar{m}(\mu_0/2) = \left[2b_0\bar{g}^2_{\mathrm{SF}}(\mu_0/2)\right]^{-d_0/(2b_0)} \exp\left\{-\int_0^{g} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]^{\mathrm{PT}}\right\}$$

$$\times \exp\left\{-\int_g^{\bar{g}(\mu_0/2)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\}$$

We quote as a **final result**, the running from the lowest scale covered by our non-perturbative simulations obtained integrating the non-perturbative anomalous dimension

$$\frac{M}{\bar{m}(\mu_0/2)} = 1.7505(89)$$

Running from $\mu_0/2 \sim 2\,\mathrm{GeV}$



Hadronic Matching

In order **to make the contact with large volumes simulations** we spanned the range $\beta \in [3.4, 3.9]$ with several volumes i.e. 10,12,16, 20, 24 in order to keep the renormalized coupling constant at a value reachable by our non-perturbative running.

$$\frac{M}{\bar{m}(\mu_{\rm had})} = 0.9255(89) \qquad u_{\rm had} = 9.25 \quad \leftrightarrow \quad \mu_0/19.05(36) = \mu_{\rm had} = 221(4)\,\text{MeV}$$

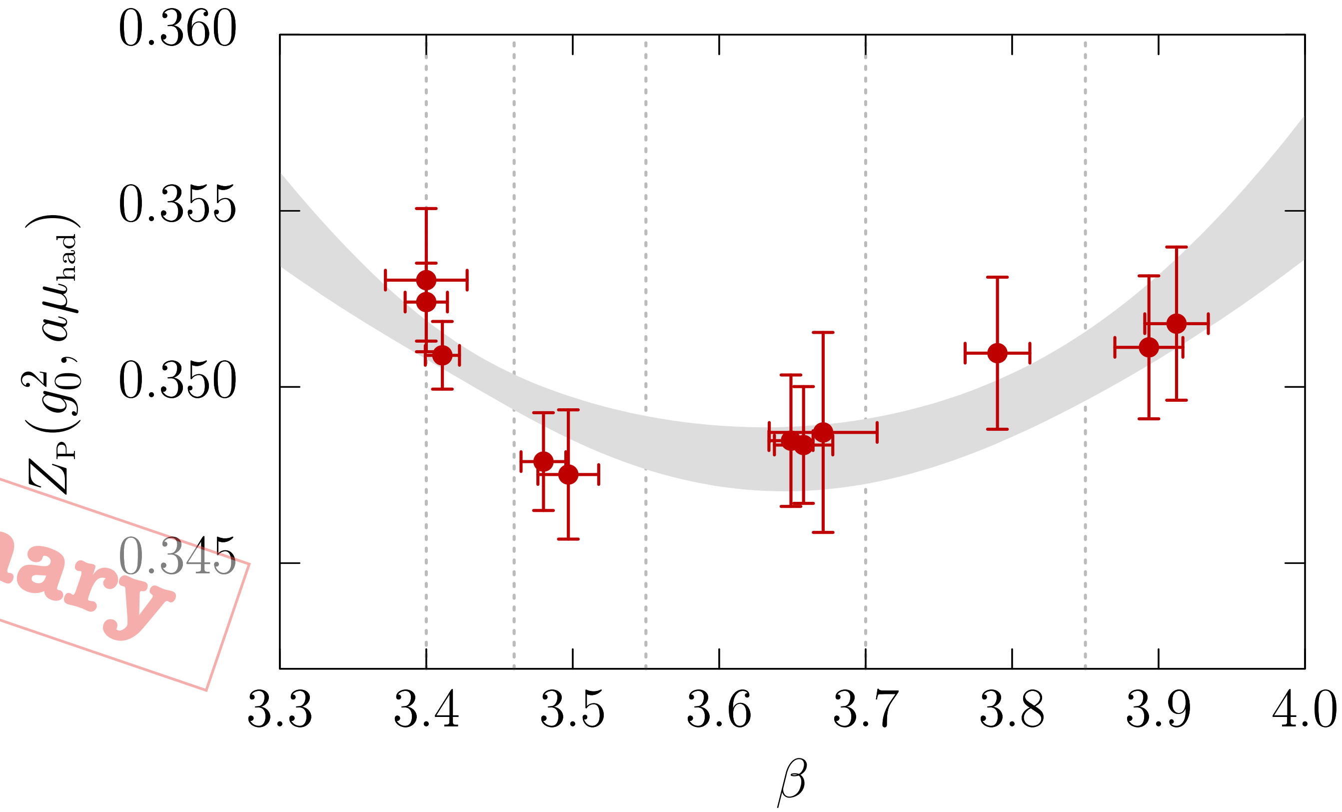
Notice, that since we are using the anomalous dimension, instead of an SSF recursion, we have much more flexibility in fixing the hadronic scale, which does not have to correspond to a scale which is proportional to the switching scale by an integer!

$$Z_M(g_0^2) = \frac{M}{\bar{m}(\mu_{\rm had})} \frac{Z_A(g_0^2)}{Z_P(g_0^2, a\mu_{\rm had})}$$

$\beta = 6/g_0^2$	Z_M
3.40	2.0311(39)(195)
3.46	2.0475(35)(197)
3.55	2.0690(47)(199)
3.70	2.0959(55)(202)
3.85	2.1120(58)(203)

Preliminary

$\sim 1\%$ relative (total) errors



Using $Z_A(g_0^2)$ from chiSF [Dalla Brida M., Korzec T. and Sint S.]

Matching with Large Volumes, and computation of light quark masses is ongoing!

Conclusions & Outlook

- We have computed the **NP running quark mass for Nf=3** between $\sim 200\,\mathrm{MeV}$ and $\sim M_W$ at high precision

- For the first time we dealt with **two schemes**, providing a strategy for a **NP matching** between them at the intermediate scale of $\sim\ 2\,\mathrm{GeV}$

- We are also providing for the first time an **"effective" NP anomalous dimension** for both SF and GF-based schemes allowing to chose μ_{had} in a broad range of values.

- The next point in the project is the **matching with large volume betas** required for the calculation of the (light) **quark masses**.

- Along with the mass project we have collected data for applying the same strategy to the **Tensor current** the only other bilinear with an independent anomalous dimension.

- Same is going to be applied on the **4-fermion operators ΔF=2**

Backup

more on Running and matching

The running from a generic scale μ is given by

$$M/\bar{m}(\mu) = (2b_0\bar{g}^2(\mu))^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}(\mu)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\}$$

In the specific case of an hadronic scale, it can be factorised as follows

$$M/\bar{m}(\mu_{\rm had}) = \boxed{(2b_0\bar{g}^2_{\rm SF}(\mu_0/2))^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}_{\rm SF}(\mu_0/2)} dx \left[\frac{\tau_{\rm SF-SF}(x)}{\beta_{\rm SF}(x)} - \frac{d_0}{b_0 x}\right]\right\}^{\rm NP}}$$

$$\boxed{\times \exp\left\{-\int_{\bar{g}_{\rm GF}(\mu_0/2)}^{\bar{g}_{\rm GF}(\mu_{\rm had})} dx \left[\frac{\tau_{\rm GF-SF}(x)}{\beta_{\rm GF}(x)}\right]\right\}^{\rm NP}}$$

$$= \left.\frac{M}{\bar{m}(\mu_0/2)}\right|_{\rm SF-SF} \left.\frac{\bar{m}(\mu_0/2)}{\bar{m}(\mu_{\rm had})}\right|_{\rm GF-SF}$$

With a usual SSF recursion it would be

$$M/\bar{m}(\mu_{\rm had}) = \boxed{\frac{M}{\bar{m}(2^N\mu_0)}} \boxed{\frac{\bar{m}(2^N\mu_0)}{\bar{m}(2^{N-1}\mu_0)} \cdots \frac{\bar{m}(\mu_0)}{\bar{m}(\mu_0/2)}} \boxed{\frac{\bar{m}(\mu_0/2)}{\bar{m}(\mu_0/2^2)} \cdots \frac{\bar{m}(2\mu_{\rm had})}{\bar{m}(\mu_{\rm had})}}$$

PT NLO | **NP SF-SF** | **NP GF-SF**

SF Renormalization Scheme

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Nucl.Phys. B384 (1992)

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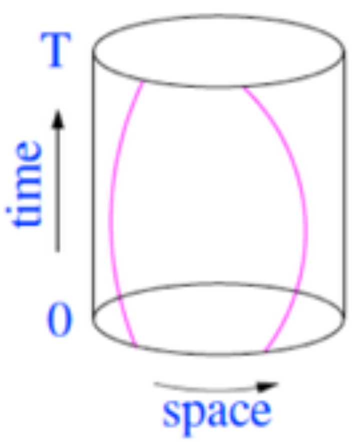
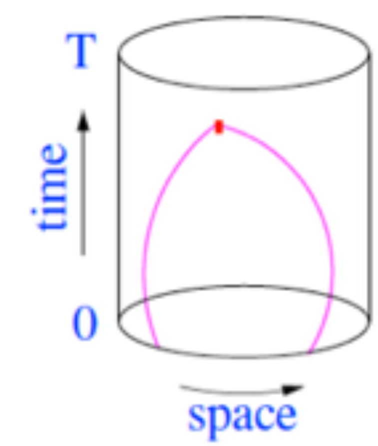
$$Z_P(g_0, L/a)\frac{f_P(L/2)}{\sqrt{3f_1}}\bigg|_{m=0}^{\theta} = c_3(\theta, a/L) \qquad \theta = 0.5$$

The correlation functions entering the definition above are given by



$$f_P(x_0) = -\frac{1}{3}\int d^3\mathbf{y}d^3\mathbf{z}\langle\bar\psi(x)\gamma_5\frac{1}{2}\tau^a\psi(x)\bar\zeta(\mathbf{y})\gamma_5\frac{1}{2}\tau^a\zeta(\mathbf{z})\rangle$$

$$f_1 = -\frac{1}{3L^6}\int d^3\mathbf{u}d^3\mathbf{v}d^3\mathbf{y}d^3\mathbf{z}\langle\bar\zeta(\mathbf{u})\gamma_5\frac{1}{2}\tau^a\zeta(\mathbf{v})\bar\zeta(\mathbf{y})\gamma_5\frac{1}{2}\tau^a\zeta(\mathbf{z})\rangle$$



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$$\Sigma_P(u, g_0, L/a) = \frac{Z_P(g_0, 2L/a)}{Z_P(g_0, L/a)}\bigg|_{u=\bar g^2(L)} \qquad \sigma_P(u) = \lim_{a\to 0}\Sigma_P(u, g_0, L/a)$$

$$u_{SF} = [1.1100, 1.1844, 1.2565, 1.3627, 1.4808, 1.6173, 1.7943, 2.0120]$$

$$u_{GF} = [2.1257, 2.3900, 2.7359, 3.2029, 3.8643, 4.4901, 5.3010, 5.8673, 6.5489]$$

Two Schemes, Two Regions, More fun

The peculiarity of this work is to consider two **different renormalization scheme for the couplings**

$$u_{SF} = [1.1100, 1.1844, 1.2565, 1.3627, 1.4808, 1.6173, 1.7943, 2.0120]$$ **High**

$$u_{GF} = [2.1257, 2.3900, 2.7359, 3.2029, 3.8643, 4.4901, 5.3010, 5.8673, 6.5489]$$ **Low**

but **same renormalization condition for the mass**!

A change of scheme for both coupling and mass can be written in terms of the differences of finite parts χ

$$g'_R = g_R\sqrt{\chi_g(g_R)} \qquad \beta'(g'_R) = \left\{\beta(g_R)\frac{\partial g'_R}{\partial g_R}\right\}$$

$$m'_R = m_R\chi_m(g_R) \qquad \tau'(g'_R) = \left\{\tau(g_R) + \beta(g_R)\frac{\partial}{\partial g_R}\ln\chi_m(g_R)\right\}_{g_R=g_R(g'_R)}$$

At 1-loop for instance one can easily see how the NLO anomalous dimension vary from one scheme to another due to a change of scheme in the renormalized coupling through the finite parts $\chi_g^{(1)}$

$$\chi(g_R) \overset{g_R\to 0}{\sim} 1 + \sum_{k=1}^{\infty}\chi^{(k)}g_R^{2k} \qquad d'_1 = d_1 + 2b_0\chi_m^{(1)} - d_0\chi_g^{(1)}$$

Sint and Weisz [Nucl.Phys. B545 (1999) 529-542]

Since we do not know the perturbative finite parts from GF and we do not want to rely on PT at $2L_0 \sim m_b/2$ we perform a **NP matching**

$N_f = 3$ u-by-u GF-SF

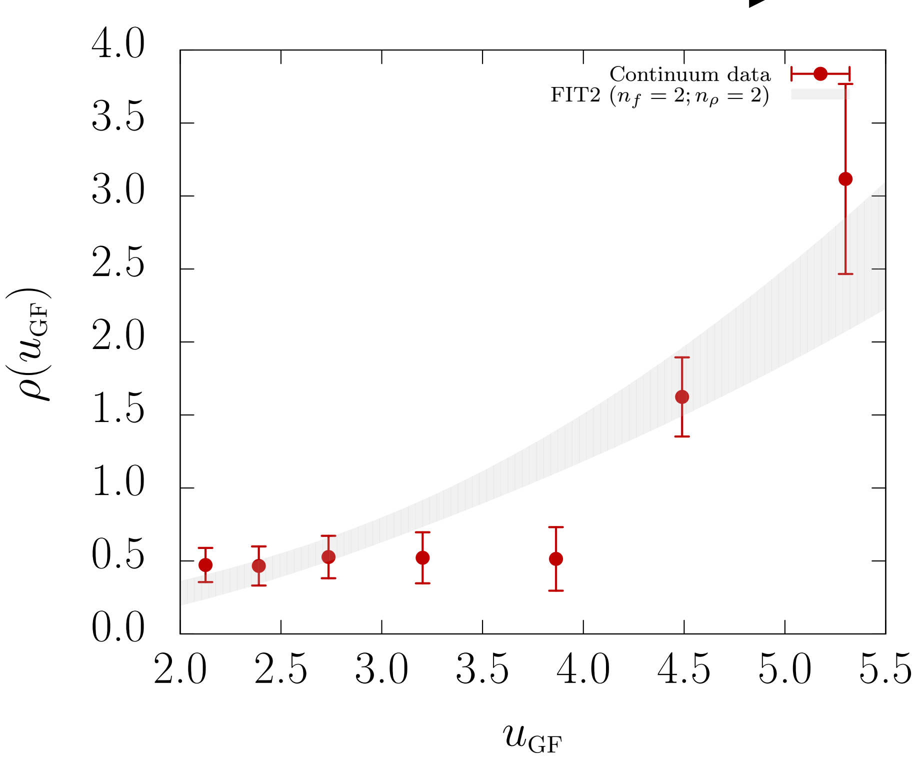
$$\Sigma_P(u, a/L) = \sigma_P(u) + \rho'(u)(a/L)^2$$

$$u_{GF} = [2.1257, 2.3900, 2.7359, 3.2029, 3.8643, 4.4901, 5.3010, 5.8673, 6.5489]$$

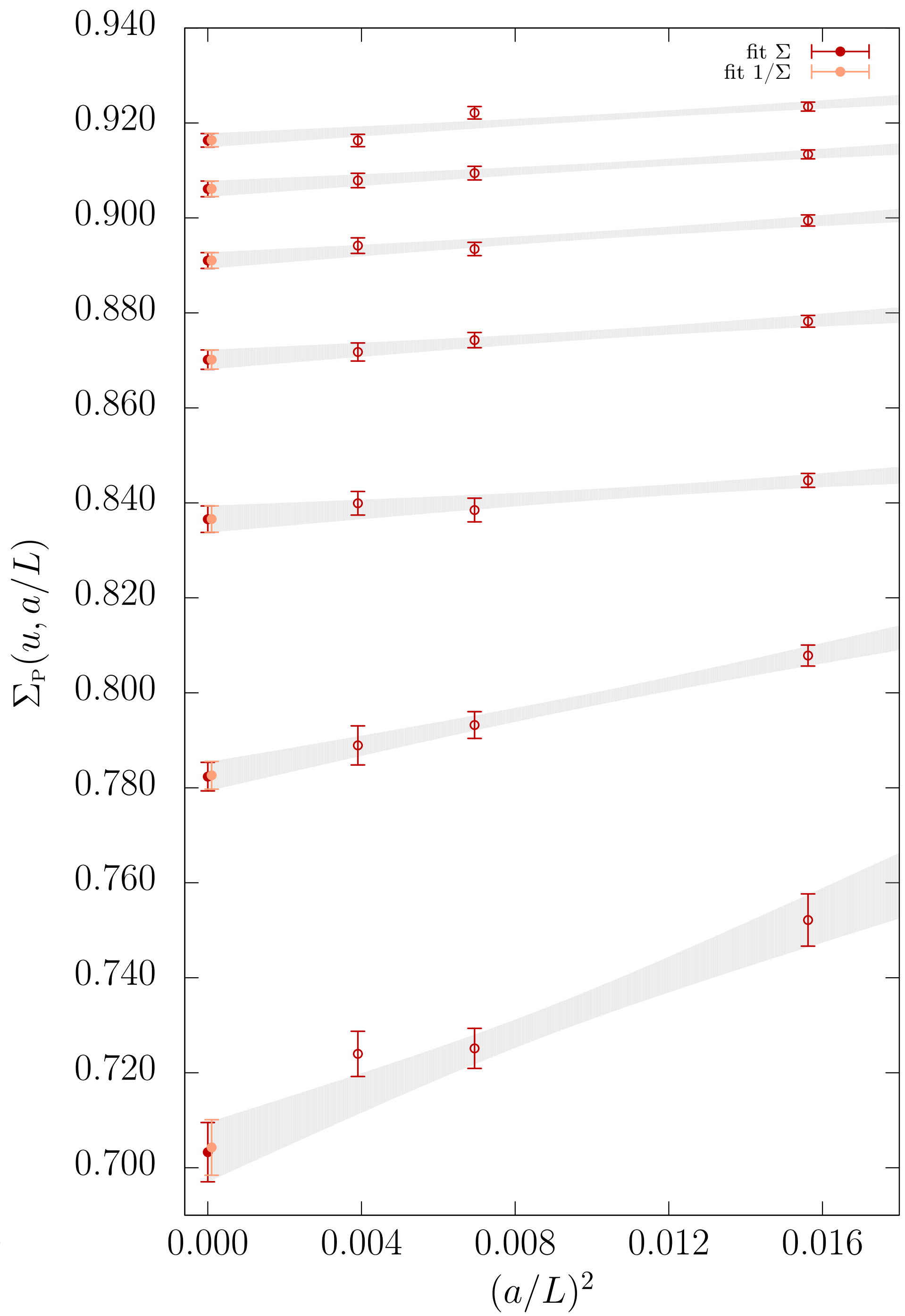
not included in our analysis

Nf=3 O(a)-imp fermions and **LW** gauge action same enables of [Dalla Brida et al. Phys.Rev. D95 (2017)]

points NOT exactly on LCP!
(only a check)



"u-by-u" extrapolations



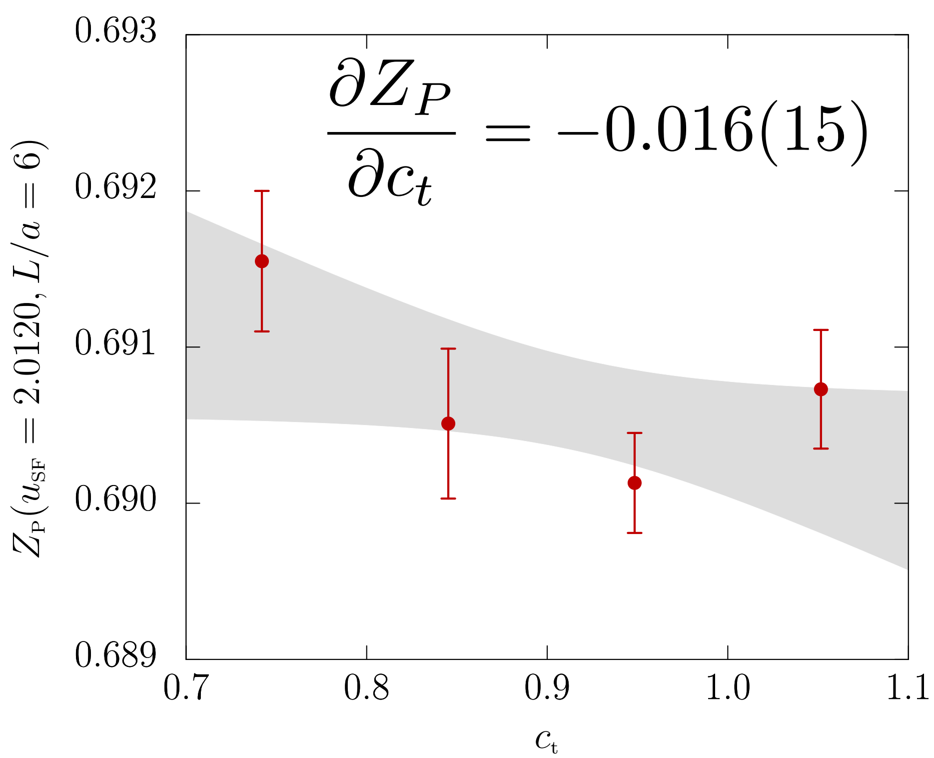
Due to the large cutoff effect induced by the GF coupling we use larger lattices respect to the ones used in SF

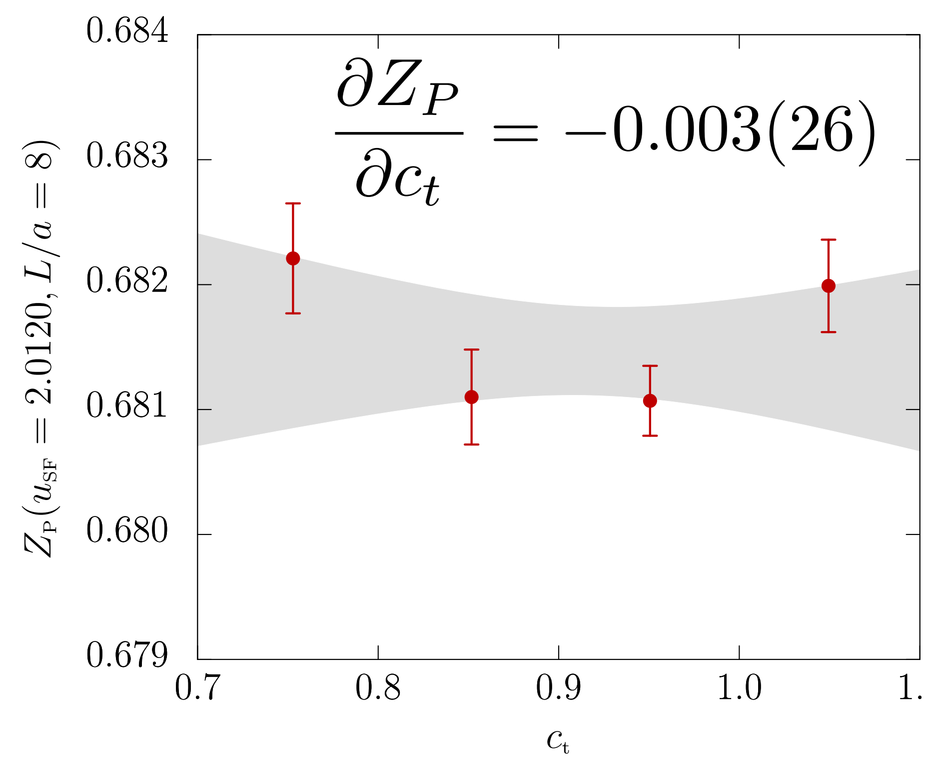
$$L/a = [8, 12, 16]$$

$$2L/a = [16, 24, 32]$$

computation of the 1-loop SSF cutoff effects is ongoing.

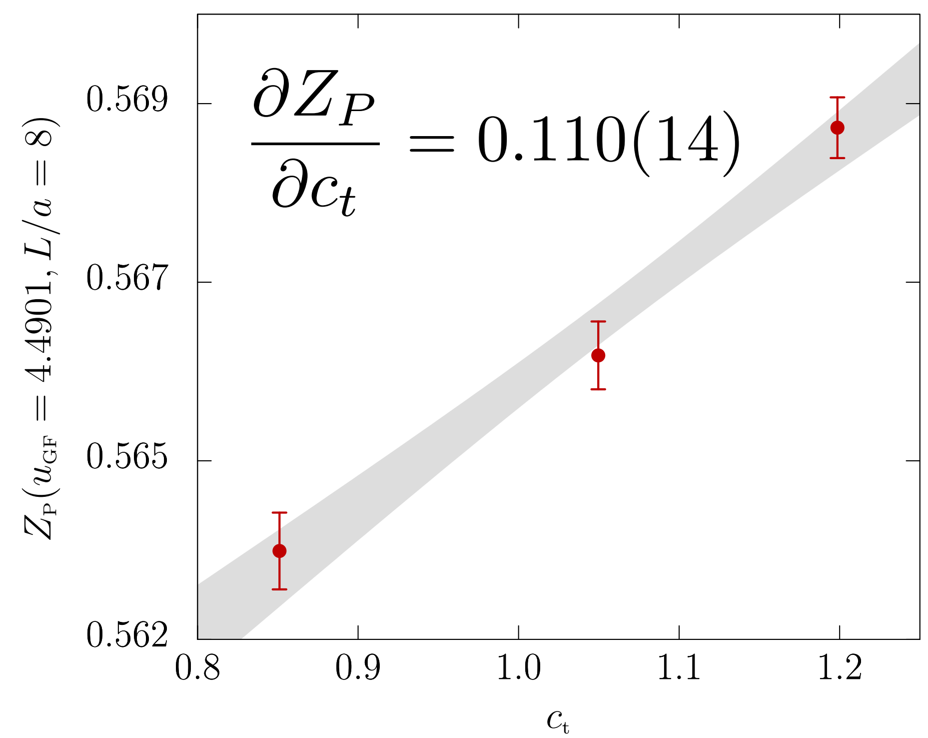
Systematics Effects - ct



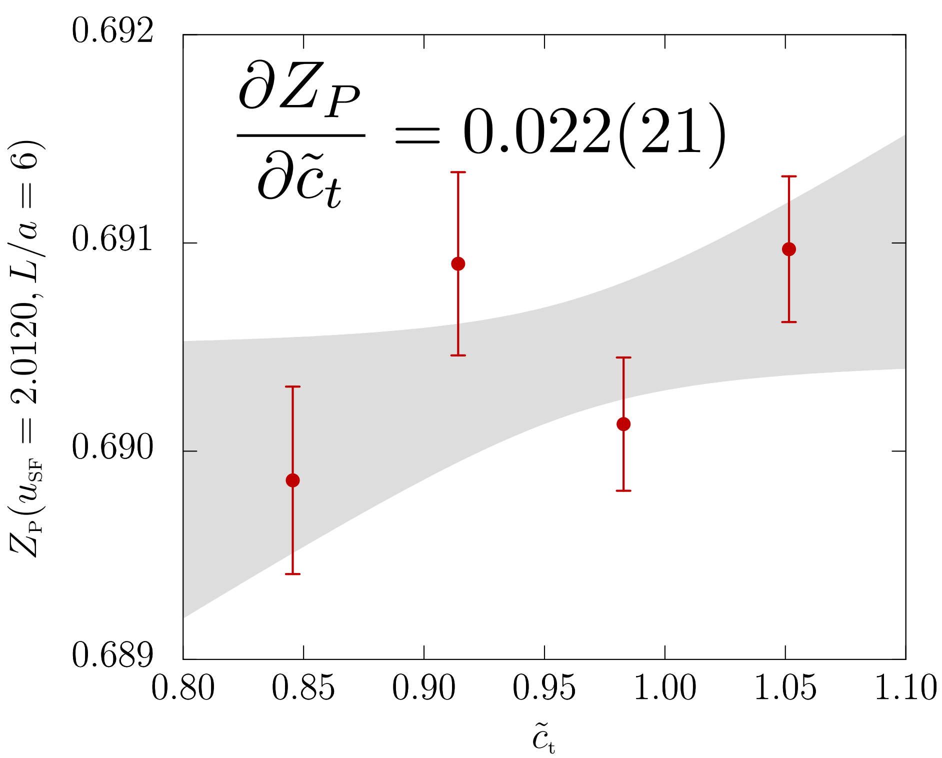


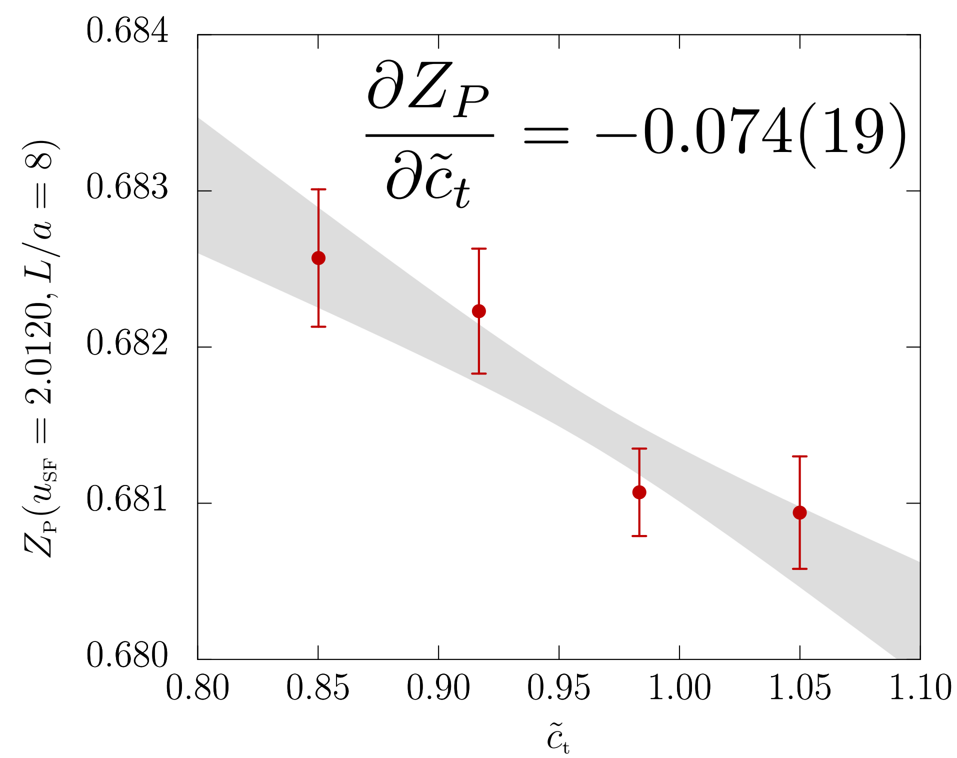
$$\delta_{c_t} Z_P \approx \left| \frac{\partial Z_P}{\partial c_t} \right| (a/L) \delta_{c_t}$$

$$\delta c_t = 1 - c_t^{\mathrm{pert}}$$



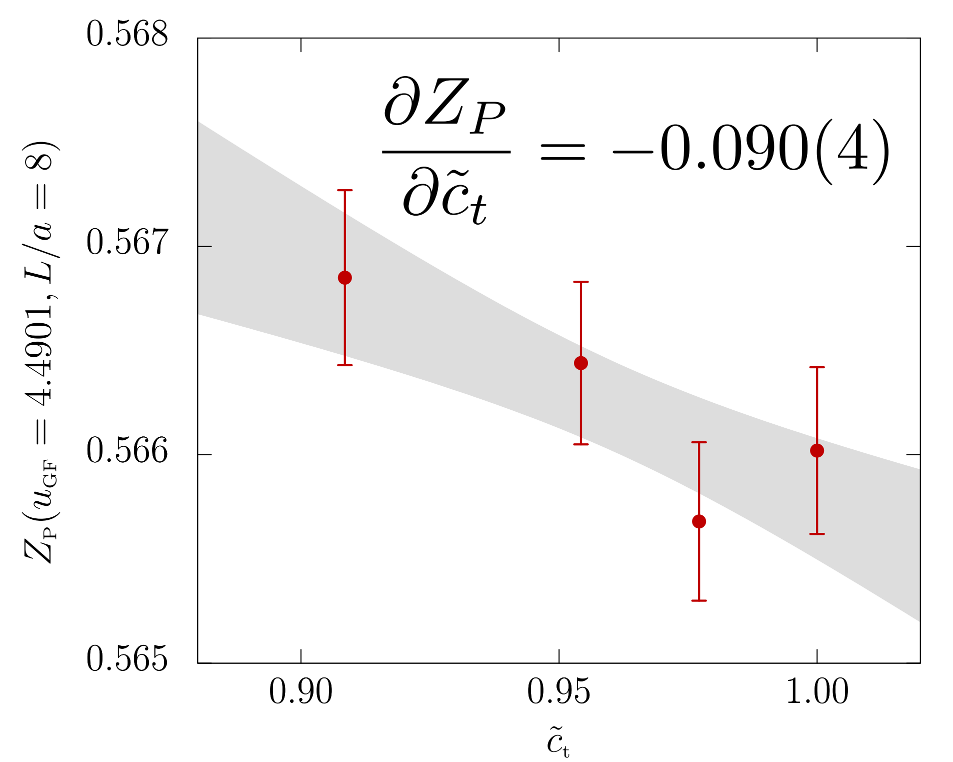
Systematics Effects - cttilde





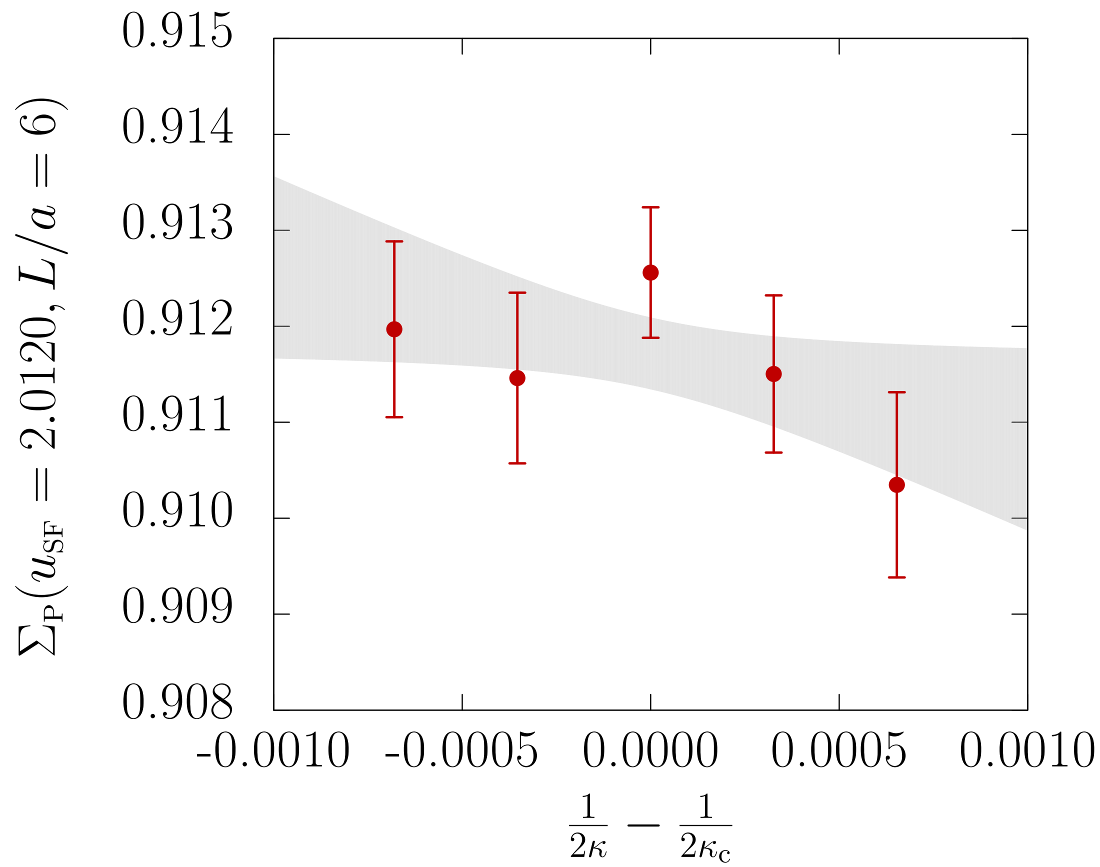
$$\delta_{\tilde{c}_t} Z_P \approx \left| \frac{\partial Z_P}{\partial \tilde{c}_t} \right| (a/L) \delta_{\tilde{c}_t}$$

$$\delta \tilde{c}_t = 1 - \tilde{c}_t^{\rm pert}$$



Systematics Effects - mcrit

$$\rho_{\kappa_c} = -0.15(15)$$



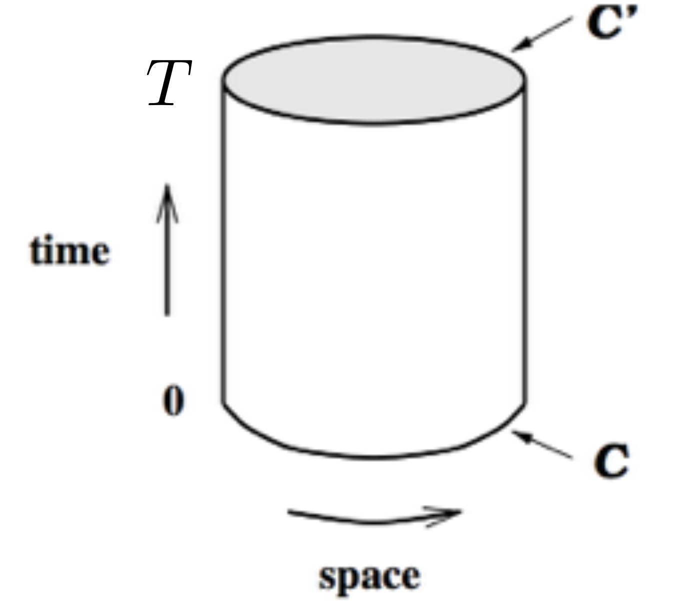
$$\rho_{\kappa_c} = \frac{1}{L} \frac{\partial \Sigma_P}{\partial m}\Big|_{u,L}$$

$$\rho_{\kappa_c} \Sigma_P = \rho_{\kappa_c} \mathrm{tol}(Lm) \qquad \mathrm{tol}(Lm) = 0.001$$

Schrödinger Functional (SF)

[Lüscher et al. 1992]
[Sint 1993]

The SF is the functional integral on a hyper cylinder with **PBC in spatial directions** and **Dirichlet conditions in time**



$$\mathcal{Z}[C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho] \int D[U]D[\psi]D[\bar{\psi}]e^{-S[U,\bar{\psi},\psi]}$$

For the **gauge potential** we have

$$A_k(x)|_{x_0=T} = C'_k \qquad \text{at} \quad \mathrm{x}_0 = \mathrm{T}$$

$$A_k(x)|_{x_0=0} = C_k \qquad \text{at} \quad \mathrm{x}_0 = 0$$

For some choices of C_k and C'_k it can be showed that the induced **background field** $B_\mu(x)$ is an absolute minimum of the action!

e.g.

$$C_k = \frac{i}{L}\begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} \qquad \begin{aligned} \phi_1 &= \eta - \pi/3 \\ \phi_2 &= -\eta/2 \\ \phi_3 &= -\eta/2 + \pi/3 \end{aligned}$$

Gauge action minumum!

$$C'_k = \frac{i}{L}\begin{pmatrix} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{pmatrix} \qquad \begin{aligned} \phi'_1 &= -\phi_1 - 4\pi/3 \\ \phi'_2 &= -\phi_3 + 2\pi/3 \\ \phi'_3 &= -\phi_2 + 2\pi/3 \end{aligned}$$

$$B_0 = 0$$

$$B_k = [x_0 C'_k + (L - x_0)C_k]/L$$

SF gauge coupling

[Lüscher et al. 1992]
[Sint 1993]

It is possible to define a coupling as a "response coefficient" to a variation of a constant colour electric field

$$G_{0k} = \partial_0 B_k = [C_k - C'_k]/L$$

By the definition of the effective action of the induced BG field

$$\Gamma[B] = -\ln \mathcal{Z}[C, C']$$

expanding in perturbation theory

$$\Gamma[B] \sim \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \mathcal{O}(g_0^2)$$

we finally have

$$\bar{g}^2_{\mathrm{SF}} = \left.\frac{\partial_\eta \Gamma_0[B]|_{\eta=0}}{\partial_\eta \Gamma[B]|_{\eta=0}}\right|_{m_q=0}$$

← Boundary conditions on fermion field allow to simulate really at vanishing quark masses

Non-perturbative definition of the **renormalized gauge coupling**, suitable for numerical simulations!