Probing renormalized perturbation theory with data from lattice QCD at high energies

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ALPHA Collaboration

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Outline

- Status of $\alpha_s(m_Z)$
- The ALPHA collaboration project & results
- Non-perturbative finite volume couplings & SF couplings
- The Λ parameter or how to test perturbation theory
- Step-scaling functions and continum extrapolations & some results
- Alternative route: passing via the $\overline{\mathrm{MS}}$ scheme
- Conclusions & outlook

References:

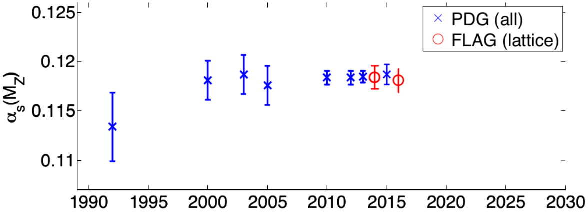
- Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Sint, Rainer Sommer [ALPHA Collaboration],
"Determination of the QCD Λ-parameter and the accuracy of perturbation theory at high energies,"
Phys. Rev. Lett. **117**, no. 18, 182001 (2016) arXiv:1604.06193 [hep-ph].
- Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Sint, Rainer Sommer [ALPHA Collaboration],
"Slow running of the Gradient Flow coupling from 200 MeV to 4 GeV in $N_\mathrm{f} = 3$ QCD,"
Phys. Rev. D **95**, no. 1, 014507 (2017), arXiv:1607.06423 [hep-lat].
- Mattia Bruno, Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Schaefer, Stefan Sint, Hubert Simma Rainer Sommer [ALPHA Collaboration],
"QCD Coupling from a Nonperturbative Determination of the Three-Flavor Λ Parameter,"
Phys. Rev. Lett. **119**, no. 10, 102001 (2017), arXiv:1706.03821 [hep-lat].
- A very nice summary has been given by Tomasz Korzec at Lattice '17:

  Tomasz Korzec, "Determination of the Strong Coupling Constant by the ALPHA Collaboration,"
arXiv:1711.01084 [hep-lat], to appear in the proceedings

Status of $\alpha_s(m_Z)$

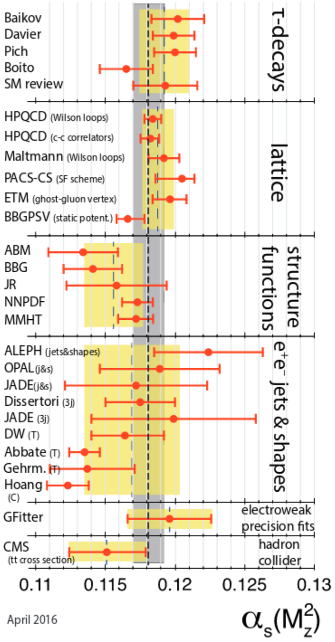
- $\alpha_s(m_Z)$ is a fundamental parameter of the Standard Model;
- Important input for LHC physics: accuracy $< 1\%$ is required!

Current status & world averages:



$$\alpha_s(m_Z) = \begin{cases} 0.1174(16) & \text{PDG 2016 (phenomenology)} \\ 0.1188(11) & \text{PDG 2016 (lattice)} \\ 0.1184(12) & \text{FLAG2 2013 (lattice)} \\ 0.1182(12) & \text{FLAG3 2016 (lattice)} \end{cases}$$

Reducing the error to $\Delta\alpha_s(m_Z) \approx 0.0006$ (ie. 0.5%)is a challenge!


τ-decays
Baikov
Davier
Pich
Boito
SM review
lattice
HPQCD (Wilson loops)
HPQCD (c-c correlators)
Maltmann (Wilson loops)
PACS-CS (SF scheme)
ETM (ghost-gluon vertex)
BBGPSV (static potent.)
structure functions
ABM
BBG
JR
NNPDF
MMHT
e+e- jets & shapes
ALEPH (jets&shapes)
OPAL (j&s)
JADE (j&s)
Dissertori (3j)
JADE (3j)
DW (T)
Abbate (T)
Gehrm. (T)
Hoang (C)
electroweak precision fits
GFitter
hadron collider
CMS (tt cross section)
0.11
0.115
0.12
0.125
0.13
April 2016
αs(Mz2)

ALPHA collaboration project

Build on CLS effort [Bruno et al, JHEP 1502 (2015) 043]:

- $\mathrm{N}_f = 2+1$ state of the art lattice QCD simulations
- nonperturbatively O(a) improved Wilson quarks & Lüscher-Weisz gauge action;
- open boundary conditions (avoids topology freezing)

Use 3 input parameters from experiment, e.g.

$$F_K, m_\pi, m_K \qquad \Rightarrow \qquad m_u = m_d, m_s, g_0$$

$\Rightarrow$ everything else becomes a prediction, for instance

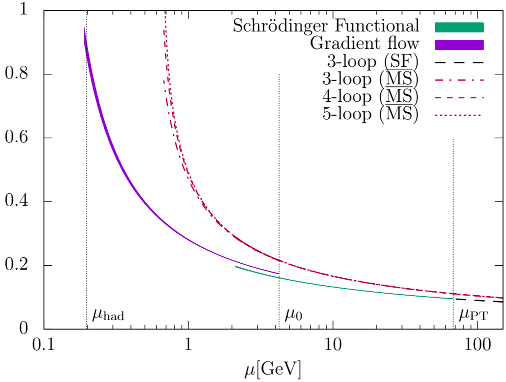
$$\alpha_s^{(\mathrm{N}_f=3)}(1000 \times F_K) \quad \text{(in any renormalization scheme)}$$

Final goal: $\alpha_s^{(\mathrm{N}_f=5)}(m_Z)$ in the $\overline{\mathrm{MS}}$-scheme

- Controlled systematics: avoid the use of perturbation theory except at high energies of O(m_Z)!
- Solve the problem of large scale differences using recursive finite volume techniques ("step-scaling").
- Has been accomplished in $\mathrm{N}_f = 3$ QCD (s. next page)

$\Rightarrow$ $\alpha_s(m_Z)$ currently still requires perturbative matching from $\mathrm{N}_f = 3$ to $\mathrm{N}_f = 5$ across the charm and bottom thresholds!

Result for $\alpha_s(m_Z)$ by the ALPHA collaboration



Scale setting:
$f_{\pi K} = \frac{1}{3}(2f_K + f_\pi) = 147.6$ (PDG), evaluated on CLS gauge configurations in terms of t_0 [Bruno, Korzec, Schaefer (2017)]

- $\Lambda^{(3)}_{\overline{\text{MS}}} = 341(12)$ MeV

$\Rightarrow$ $\alpha_s(m_Z) = 0.1185(8)(3)$

- perturbation theory (PT) is only used at high energies μ_{PT}
- 2 couplings are traced non-perturbatively:
  - GF ("gradient flow") from hadronic to intermediate energies ($\mu_0 \approx 4$ GeV);
  - SF ("Schrödinger functional") from μ_0 to high energies μ_{PT}

This talk will focus on the high energy running in the SF scheme!

Non-perturbatively defined finite volume couplings

Wanted: QCD observables, O, which ...

- are gauge invariant & non-perturbatively defined through the (Euclidean) QCD path integral:

$$\langle O \rangle = \mathcal{Z}^{-1} \int D[A,\psi,\overline{\psi}] O[A,\psi,\overline{\psi}] \exp\{-S\}$$

- depend on a single scale $\mu = 1/L$, with L^4 the space-time volume. Other dimensionful parameters (momenta, distances,..) are scaled with L or set to zero (quark masses);

- can be expanded perturbatively in $\alpha_s(\mu) = \bar{g}^2(L)/(4\pi)$:

$$\langle O \rangle = c_0 + c_1 \alpha_s(\mu) + c_2 \alpha_s^2(\mu) + \dots$$

$\Rightarrow$ give rise to *non-perturbatively defined couplings*:

$$\alpha_O(\mu) \stackrel{\text{def}}{=} \frac{\langle O \rangle - c_0}{c_1} = \alpha_s(\mu) + c_1' \alpha_s^2(\mu) + c_2' \alpha_s^3(\mu) + \dots$$

Example: a family of SF couplings

- Dirichlet b.c.'s in Euclidean time, abelian boundary values C_k, C'_k:

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu), \qquad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$\Rightarrow$ induce family of abelian, spatially constant background fields B_μ with parameters η, ν ($\rightarrow$ 2 abelian generators of SU(3)):

$$B_k(x) = C_k(\eta,\nu) + \frac{x_0}{L}\left(C'_k(\eta,\nu) - C_k(\eta,\nu)\right), \qquad B_0 = 0.$$

- Induced background field is unique up to gauge equivalence
- Effective action

$$\mathrm{e}^{-\Gamma[B]} = \int D[A,\psi,\overline{\psi}]\mathrm{e}^{-S[A,\psi,\overline{\psi}]}, \qquad \Gamma[B] = \tfrac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \mathrm{O}(g_0^2)$$

- Define family of SF couplings, parameter ν:

$$\frac{1}{\bar{g}_\nu^2(L)} \stackrel{\mathrm{def}}{=} \left.\frac{\partial_\eta \Gamma[B]}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \left.\frac{\langle \partial_\eta S\rangle}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \frac{1}{\bar{g}^2(L)} - \nu \bar{v}(L)$$

$\Rightarrow$ response of the system to a change of a colour electric background field. [Narayanan et al. '92]

Testing perturbation theory: use the Λ-parameter I

- Non-perturbatively defined coupling $\bar{g}^2(L)$ implies non-perturbative definition of β-function:

$$\beta(\bar{g}) \stackrel{\text{def}}{=} -L\frac{\partial \bar{g}(L)}{\partial L}, \qquad \beta(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

with universal coefficients b_0, b_1 (i.e. b_i, $i \geq 2$ scheme dependent)

$$b_0 = (11 - \tfrac{2}{3}\mathrm{N}_f)/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}\mathrm{N}_f)/(4\pi)^4 \,.$$

- *Exact* solution of Callan-Symanzik equation $\left[L\partial/\partial L + \beta(\bar{g})\partial/\partial\bar{g}\right] L\Lambda = 0$

$$L\Lambda = \varphi\left(\bar{g}(L)\right)$$

$$\varphi\left(\bar{g}\right) = \left[b_0\bar{g}^2\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2}} \exp\left\{-\int_0^{\bar{g}} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- Scheme dependence of Λ almost trivial:

$$g_{\mathrm{X}}^2(\mu) = g_{\mathrm{Y}}^2(\mu) + c_{\mathrm{XY}} g_{\mathrm{Y}}^4(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_{\mathrm{X}}}{\Lambda_{\mathrm{Y}}} = \mathrm{e}^{c_{\mathrm{XY}}/2b_0}$$

$\Rightarrow$ use $\Lambda = \Lambda_{\mathsf{SF},\nu=0}$ as reference.

Note: $\Lambda_{\overline{\mathrm{MS}}}$ is now non-perturbatively defined by $\Lambda = 0.3829 \times \Lambda_{\overline{\mathrm{MS}}}$ (for $\mathrm{N}_f = 3$)

Testing perturbation theory: use the Λ-parameter II

- Introduce a reference scale $1/L_0$ through:

$$\bar{g}^2(L_0) = 2.012 \quad \Rightarrow \quad \frac{1}{\bar{g}_\nu^2(L_0)} = \frac{1}{2.012} - \nu \times 0.1199(10) \quad \text{(s. later)}$$

- Consider

$$L_0\Lambda = \underbrace{L_0/L}_{\text{known}} \quad \times \quad \underbrace{\Lambda/\Lambda_\nu}_{\exp(-\nu \times 1.25516)} \quad \times \quad \varphi_\nu\left(\bar{g}_\nu(L)\right)$$

- Non-perturbative results for $1/L_0 \leq \mu \leq 1/L$ (s. below)
- Perturbation theory for $\mu > 1/L$ by replacing $\beta_\nu(g) \to \beta_{\nu,\text{3-loop}}(g)$ in:

$$\varphi_\nu\left(\bar{g}_\nu(L)\right) \quad \propto \quad \exp\left\{-\int_0^{\bar{g}_\nu(L)} dg \left[\frac{1}{\beta_\nu(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

$$\beta_{\nu,\text{3-loop}}(g) \quad = \quad -b_0 g^3 - b_1 g^5 - b_{2,\nu}\, g^7,$$

$$b_{2,\nu} \quad = \quad (-0.06(3) - \nu \times 1.26)/(4\pi)^3 \qquad \text{[Bode, Weisz, Wolff '99]}$$

- N.B.: $L_0\Lambda$ must be independent of L and ν $\Rightarrow$ excellent test of PT!

Non-perturbative running in steps and determination of L_0/L

Vary scale by factor 2, define step-scaling function [Lüscher, Weisz, Wolff '91]:

$$\sigma(u) = \bar{g}^2(2L)\Big|_{u=\bar{g}^2(L)},$$

- Connection to β-function:

$$\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{dg}{\beta(g)} = -\ln 2$$

- $\sigma(u)$ can be constructed as the continuum limit of lattice approximants (s. below)
- Once $\sigma(u)$ is available for a range of values $u \in [u_{\text{min}}, u_0]$

$\Rightarrow$ iteratively step up the energy scale:

$$u_0 = \bar{g}^2(L_0), \quad u_n = \sigma(u_{n+1}) = \bar{g}^2(L_n) = \bar{g}^2(2^{-n}L_0), \quad n = 0, 1, \ldots$$

$\Rightarrow$ scale ratios are $L_0/L_n = 2^n$, where n is the number of steps.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (defines value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

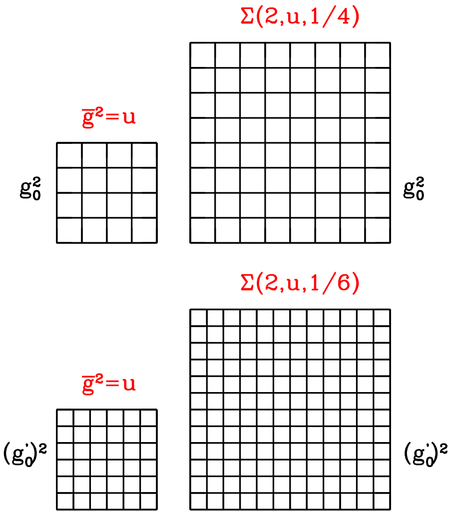
$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- ...

$$\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$$

.

- change u and repeat...



Step scaling function for $\nu = 0$

$$\Sigma(u, a/L) = \bar{g}^2(2L)|_{\bar{g}^2(L)=u}, \qquad \sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$$

- Simulate for a range of u-values $\in [1, 2.012]$ on lattices with $L/a = 4, 6, 8, 12$.
- Double lattice size and measure $\Sigma(u, a/L) = \bar{g}^2(2L)$
- analyze $\Sigma(u, a/L)$ directly
- Alternatively, reduce cutoff effects perturbatively up to 2-loop order:

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)} = \delta_1(L/a)u + \delta_2(L/a)u^2 + \mathrm{O}(u^3)$$

$\delta_{1,2}(L/a)$ are known [Bode, Weisz & Wolff '99]

$\Rightarrow$ cutoff effects in

$$\Sigma'(u, a/L) = \frac{\Sigma(u, a/L)}{1 + \delta_1(L/a)u + \delta_2(L/a)u^2}$$

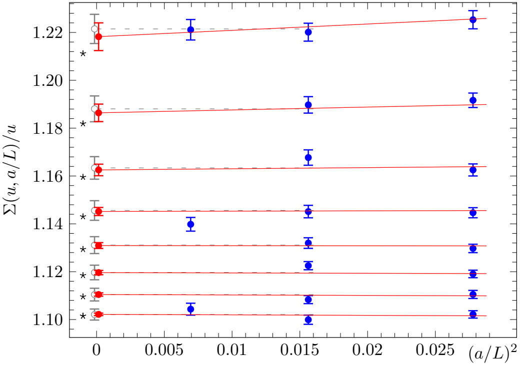
start at order u^4!

Continuum extrapolation of $\Sigma(u, a/L)$

Example for global fit ansatz:

$$\Sigma(u, a/L) = u + s_0 u^2 + s_1 u^3 + c_1 u^4 + c_2 u^5 + \rho_1 u^4 \frac{a^2}{L^2} + \rho_2 u^5 \frac{a^2}{L^2}$$

- s_0, s_1 fixed to perturbative values: $s_0 = 2b_0 \ln 2, \quad s_1 = s_0^2 + 2b_1 \ln 2$
- 4 parameters: c_1, c_2, ρ_1, ρ_2; 19 data points, χ^2/d.o.f. ≈ 1

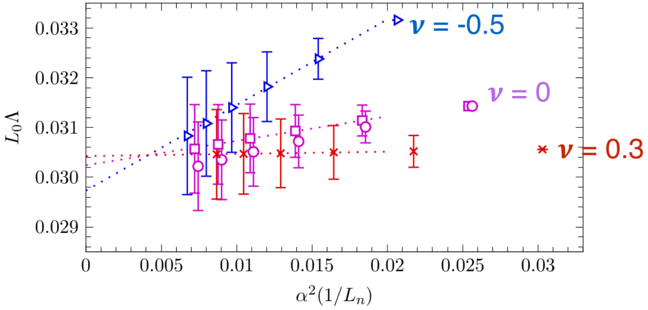


Treatment of lattice effects linear in a

Main disadvantage of SF boundary conditions:

- cutoff effects linear in a generated by the boundaries.
- Counterterms are $\mathrm{tr}(F_{0k}F_{0k})$ and $\overline{\psi}D_0\psi$, localized at the boundaries $x_0 = 0, L$
- can be cancelled by tuning the counterterm coefficients c_t and $\tilde{c}_\mathrm{t}$.
- In PT: c_t known to 2-loops and $\tilde{c}_\mathrm{t}$ to one-loop order
- To avoid terms linear in a in the continuum extrapolations, we
  - measure the sensitivity at the larger couplings to a variation of c_t and $\tilde{c}_\mathrm{t}$;
  - interpolate with the perturbative behaviour $\Rightarrow$ model for sensitivity
  - estimate the effect of imperfect tuning by shifting all data in either direction with Δc_t and $\Delta\tilde{c}_\mathrm{t}$ taken to be the last known order in PT.
  - carry out the continuum limit with O(a^2) terms only and add the differences in the central values in quadrature to the error.
  - the error is subdominant in all cases.

Result for $L_0\Lambda$



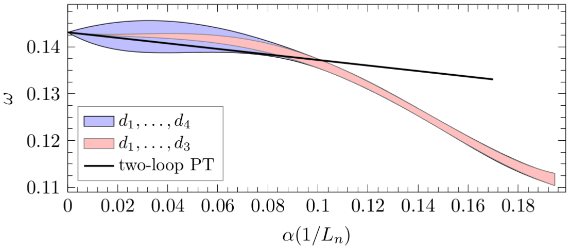
- All results agree at $\alpha = 0.1$, we quote

$$L_0\Lambda = 0.0303(8) \qquad \text{error} < 3\% \text{ !}$$

- For $\nu = 0.3$ this result could be inferred from larger values of α, but not for $\nu = -0.5$!
- As expected, all results have corrections $\propto \alpha^2$; effective coefficients can vary dramatically

⇒ Some luck is required to pick a "good scheme", i.e. with small higher order corrections.

Continuum results $\bar{v} = \omega(u) = v_1 + u \times v_2 + \dots$



- Continuum extrapolation analogous to $\sigma(u)$, but much more data between $L/a = 6$ to $L/a = 24$ covering a factor 4 in resolution!
- consider 2 continuum parameterizations (v_1, v_2 are known from PT):

$$\omega(u) = v_1 + v_2 u + d_1 u^2 + d_2 u^3 + d_3 u^4$$
$$\omega(u) = v_1 + d_1 u + d_2 u^2 + d_3 u^3 + d_4 u^4$$

- $L_0\Lambda$ calculation for $\nu \neq 0$ requires $\bar{v}(L_0) = \omega(2.012) = 0.1199(10)$ ($u = 2.012 \Leftrightarrow \alpha = 0.16$)
- Observe large deviation from perturbation theory at $\alpha = 0.19$:

$$\left(\omega(\bar{g}^2) - v_1 - v_2\bar{g}^2\right)/v_1 = -3.7(2)\alpha^2$$

- The coefficient is too large for PT to be trustworthy at these couplings!

Alternative test via the $\overline{\mathrm{MS}}$-scheme I

Idea: First match the SF coupling to the $\overline{\mathrm{MS}}$-scheme then evaluate the Λ-parameter using up to 5-loop order PT available for this scheme.

- Relation between couplings, allowing for a scale factor s:

$$4\pi\alpha_{\overline{\mathrm{MS}}}(s/L) = \bar{g}^2_{\overline{\mathrm{MS}}}(L/s) = \bar{g}^2_\nu(L) + p_1^\nu(s)\bar{g}^4_\nu(L) + p_2^\nu(s)\bar{g}^6_\nu(L) + \mathrm{O}(\bar{g}^8)$$

- Same as earlier, except now in the $\overline{\mathrm{MS}}$ scheme:

$$\Lambda_{\overline{\mathrm{MS}}} L_0 = \tfrac{sL_0}{L}\varphi_{\overline{\mathrm{MS}}}\left[\bar{g}_{\overline{\mathrm{MS}}}(L/s)\right] = s\,2^n\varphi_{\overline{\mathrm{MS}}}\left[\sqrt{\bar{g}^2_\nu(L) + p_1^\nu(s)\bar{g}^4_\nu(L) + p_2^\nu(s)\bar{g}^6_\nu(L)}\right],$$

- expect to see independence of the number of steps n, scale factor s and parameter ν.
- Look at $\nu = 0$, depdendence on n and s.
- Note: The neglected order for Λ:

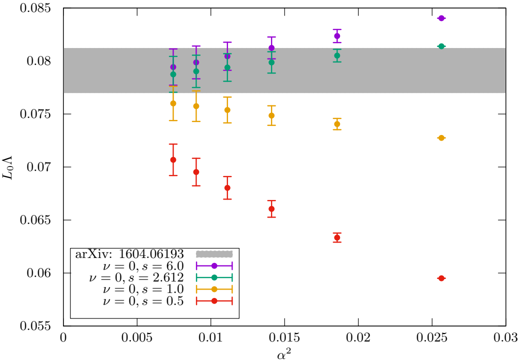
$$\Delta g^2 \tfrac{d\varphi}{dg^2} \propto \Delta g^2 \left\{g\beta(g)\right\}^{-1} = \Delta g^2 \times \mathrm{O}(g^{-4})$$

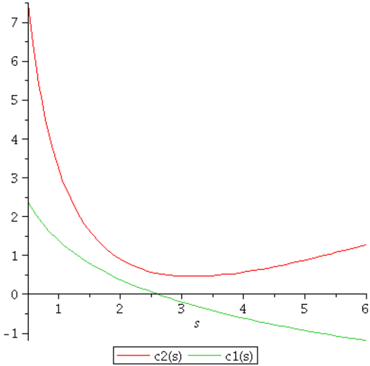
$\Rightarrow$ truncation error: $\mathrm{O}(g^8) \times \mathrm{O}(g^{-4}) = \mathrm{O}(g^4) = \mathrm{O}(\alpha^2)$.

Alternative test via the $\overline{\text{MS}}$-scheme II

$$\alpha(sq) = \alpha_\nu(q) + c_1^\nu(s)\alpha_\nu^2 + c_2^\nu(s)\alpha_\nu^3(q) + \ldots, \qquad p_i^\nu = c_i^\nu/(4\pi)^i$$

PRELIMINARY





- Choice of scale factor is important, coefficients can get large.
- "fastest apparent convergence" principle: $c_1(s^*) = 0$ which means $s^* = \Lambda_{\overline{\text{MS}}}/\Lambda \approx 2.612$ seems like a good idea.

Conclusions and outlook

- Perturbative calculations are subject to errors which are difficult to estimate
- Lattice QCD provides a testing ground for perturbation theory at *high* energies/small volumes
  - finite volume becomes essential part of observables
  - $\Rightarrow$ infinite volume calculations are of limited use;
  - PT remains feasible with careful choice of b.c.'s.
  - finite volume protects against infrared/renormalon problems; for SF coupling find a secondary minimum with an action gap of $5\pi/(6\alpha) \approx 2.62/\alpha$
  - $\Rightarrow$ negligible here, e.g. $\exp(-2.62/0.2 \approx 2 \times 10^{-6}$.
- Similar studies are possible with quark masses (cf. David Preti's talk)
- Gradient flow couplings:
  - requires perturbative calculations
  - $\Rightarrow$ could be done using NSPT [Dalla Brida & Hesse '13, Dalla Brida & Luscher '17]
- BUT: Perturbation theory at low scales often can and should be avoided by using recursive finite size scaling techniques!

Thank you!