Exact solution of the Polyakov loop models in the large-N limit

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I. Duals of lattice models

- Dual representations based on the plaquette formulation. Dual variables are introduced as variables conjugate to local Bianchi identities. The dual model is non-local due to the presence of connectors
- Dual representations based on 1) the character expansion of the Boltzmann weight and 2) the integration over link variables using Clebsch-Gordan expansion

$$Z = \sum_{r_p, r_l} \prod_p C_{r_p}(\beta_{\mu\nu}) \prod_x (6j \text{ links}) \prod_{\text{c}} (\text{6j cubes})$$

- In the strong coupling limit the model can be mapped onto monomer-dimer-closed baryon loop model for $SU(N)$
- Other approaches: n-link action, abelian colour cycles, $\cdots$
  Remark: abelian *vs* non-abelian dual representations

Finite-temperature effective model

Let $U(x) \in U(N), SU(N)$. The following spin model describes the Polyakov loop interaction in the finite-temperature LGT (simplest version)

$$S = \beta \sum_{x,n} \mathrm{ReTr} U(x) \mathrm{Tr} U^{\dagger}(x+e_n) + \sum_{x} \Big(h_r \mathrm{Tr} U(x) + h_i \mathrm{Tr} U^{\dagger}(x) \Big)$$

- $\beta = \beta(g^2)$; $h_r = h_r(m_q, \mu)$ and $h_i = h_i(m_q, \mu)$ are functions of quark mass m_q and baryon chemical potential μ. For one flavour

$$h_r = h(m_q)\, e^{\mu} \, , \; h_i = h(m_q)\, e^{-\mu}$$

- Global symmetries when $h_r = h_i = 0$: $U(1)$ for $U(N)$ and $Z(N)$ for $SU(N)$

- Phase transitions

- The effective action is complex if $h_r \neq h_i$ ($\mu \neq 0$)

II. Group integrals

The Taylor expansion of the Boltzmann weight of some non-abelian model leads to the problem of computing of integrals of the form

$$\mathcal{I}_N(r_1, r_2) \ = \ \int dU \prod_{k=1}^{r_1} U^{i_k j_k} \prod_{n=1}^{r_2} U^{m_n l_n *}$$

Only matrices in fundamental representation appear in the integrand. For $U(N)$ one finds (B. Collins, 2003, B.Collins et.al. 2003-2017)

$$\mathcal{I}_N(r_1, r_2) = \delta_{r_1, r_2} \sum_{\tau, \sigma \in S_r} Wg^N(\tau^{-1}\sigma) \prod_{k=1}^{r} \delta_{i_k, m_{\sigma(k)}} \delta_{j_k, l_{\tau(k)}} .$$

S_r - group of permutations of r elements. $Wg^N(\sigma)$ - Weingarten function which depends only on the length of cycles of σ. If λ is a partition of r, *i.e.* $\sum_{i=1}^N \lambda_i = r$ then

$$Wg^N(\sigma) = \frac{1}{(r!)^2} \sum_{\lambda} \frac{d^2(\lambda)}{s_\lambda(1^N)} \chi_\lambda(\sigma) \ , \ \lambda_1 \geq \cdots \geq \lambda_N \geq 0$$

$d(\lambda), \chi_\lambda(\sigma)$ - dimension and character of S_r, $s_\lambda(1^N)$ - the Schur function.

Integrals in the spin model

$$Q_N(s,\bar{s}) = \int dU\ (\mathrm{Tr}U)^s\ (\mathrm{Tr}U^*)^{\bar{s}}\ =\ \delta_{s,\bar{s}} \sum_{\lambda} d^2(\lambda)\ ,\ \text{for } U(N)$$

$$\begin{aligned} Q_N(s,\bar{s})\ &=\ \int dU\ (\mathrm{Tr}U)^s\ (\mathrm{Tr}U^*)^{\bar{s}} \\ &=\ \delta_{\bar{s}-s,kN} \sum_{\lambda} d(\lambda)\, d(\lambda+k)\ ,\ \text{for } SU(N) \end{aligned}$$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)$ is a partition of s.

If $l(\lambda)$ is the length of the partition λ, *i.e.*, the number of non-vanishing parts λ_i then

$$d(\lambda)\ =\ s!\ \frac{\prod_{1\leq i<j\leq l(\lambda)}(\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{l(\lambda)}(\lambda_i + l(\lambda) - i)!}\ .$$

1. Properties of $Wg^N(\sigma)$ - general form, recurrence relations, large-N asymptotic expansion, bounds - are known.

2. In abelian case one recovers the conventional dual model.

3. The constraints $s = \bar{s}$ and $\bar{s} - s = kN$ are essentialy abelian ones. One solves the constraints by introducing genuine dual variables, like in $U(1)$ and $Z(N)$ models. No other constraints are generated.

4. Dual theory is a theory with only local interaction.

5. The complex action problem is solved for all $U(N)$ and $SU(N)$ models in any dimension if the product $h_r h_i$ and the ratio h_r/h_i is non-negative.

III. Polyakov loop models and their duals

$$Z = \int \prod_x dU(x) \ \exp\left[S[\{U\}]\right] \ .$$

(for $SU(3)$ see: C. Gattringer, Nucl.Phys. B850 2011)

- Make Taylor expansion of the Boltzmann weight

- Perform group integration

- Introduce sources to calculate n-point correlations of the Polyakov loops, $\eta(x)$ and $\bar{\eta}(x)$

- After only few pages of manipulating with formulae one ends up with

The dual form of $SU(N)$ spin model (flux representation)

$$Z = \prod_l \left[\sum_{p(l)=-\infty}^{\infty} \sum_{q(l)=0}^{\infty} \left(\frac{\beta}{2}\right)^{|p(l)|+2q(l)} \frac{1}{(q(l)+|p(l)|)!q(l)!} \right]$$

$$\prod_x \left[\sum_{k(x)=-\infty}^{\infty} \sum_{t(x)=0}^{\infty} \frac{(h_r h_i)^{t(x)+\frac{1}{2}|m(x)|}}{t(x)!\,(t(x)+|m(x)|)!} \left(\frac{h_r}{h_i}\right)^{\frac{1}{2}m(x)} Q_N(s(x)) \right] ,$$

$$s(x) = \sum_{i=1}^{2d} \left(q(l_i) + \frac{1}{2}\,|p(l_i)| \right) + t(x) + \frac{1}{2} \sum_{n=1}^{d} (p_n(x) - p_n(x-e_n)) + \frac{1}{2}|m(x)| + \frac{1}{2} m(x) + \eta(x) ,$$

$$m(x) = \sum_{n=1}^{d} (p_n(x-n) - p_n(x)) - k(x)N + \bar{\eta}(x) - \eta(x) .$$

$U(N)$: only term $k(x) = 0$ contributes. Dependence on μ is cancelled from partition function and all invariant observables. Non-invariant ones depend on μ as $e^{-\mu \sum_x (\eta(x)-\bar{\eta}(x))}$. In $SU(N)$ invariants depend on μ as $e^{-\mu N \sum_x k(x)}$.

When external fields are vanishing $h_r = h_i = 0$, one gets $t(x) = m(x) = 0$. One performs duality transformations in any number of dimensions. *E.g.*, in $d = 2$ one finds the following expression on the dual lattice

$$Z = \sum_{r(x)=0}^{N-1} \sum_{k(l)=-\infty}^{\infty} \sum_{q(l)=0}^{\infty} \prod_p Q_N(s(p))$$
$$\prod_l \left[\frac{(\beta/2)^{|r(x)-r(x+n)+k(l)N|+2q(l)}}{(q(l)+|r(x)-r(x+n)+k(l)N|)!q(l)!} \right]$$

If $Q_N(s) = 1$, the remaining part is nothing but the dual of $Z(N)$ vector model. In three dimensions the original spin model is dual to an integer-valued gauge model.

IV. Asymptotics of group integrals in the large-N limit

The following relations hold for $U(N)$

$$\sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s} \frac{1}{(s!)^2}\, Q_N(s) = \mathrm{det} I_{i-j}(x)$$

and for $SU(N)$

$$\sum_{s,\bar{s}=0}^{\infty} \left(\frac{x}{2}\right)^{s+\bar{s}} e^{(s-\bar{s})\mu}\, \frac{1}{s!\bar{s}!}\, Q_N(s,\bar{s}) = \sum_{k=-\infty}^{\infty} e^{-kN\mu}\, \mathrm{det} I_{i-j+k}(x)$$

They can be used to derive various expansions of $Q_N(s)$ functions at large N and/or s.

$U(N)$: for any N and $s \leq N$

$$Q_N(s) \;=\; s!\;.$$

$SU(N)$: for any N and $s \leq N$, $\bar{s} = s + kN, k \geq 0$

$$Q_N(s,\bar{s}) \;\approx\; \frac{(s+kN)!}{(N!)^k}\;.$$

$SU(N)$: for large s and $\bar{s} = s + kN$, $k \geq 0$

$$Q_N(s,\bar{s}) \asymp \frac{G(N+1)}{(2\pi)^{(N-1)/2}} \frac{N^{2s+kN+N^2/2}}{\Gamma\left((N^2-1)/2\right)} B\left(2s+kN+3/2,(N^2-1)/2\right).$$

$k = 0$ gives leading term for $U(N)$. $G(m)$ is the Barnes function, $B(a,b)$ is the Beta-function.

V. Exact solution of Polyakov loop models

- Exact solution of the two-dimensional LGT in the large-N limit (D. Gross, E. Witten, Phys.Rev. D21 1980; S. Wadia, Phys.Lett. B93 1980) can be constructed if $\lambda = g^2 N$ is fixed ('t Hooft limit)

$$Z = \int dU \ \exp\left[\frac{1}{g^2}(\mathrm{Tr}U + \mathrm{Tr}U^{\dagger})\right] .$$

- Mean-field solution of $U(N)$ Polyakov loop model in the large-N limit at $\mu \neq 0$ (C. Christensen, Phys.Lett. B714 2012).

Third order phase transition has been found in both cases.

A. Strong-coupling phase: $d\beta < 1$

In this region the spin model can be exactly mapped onto the following Gaussian-like partition function ($h_\pm = h_r \pm h_i$)

$$Z = \int_{-\infty}^{\infty} \prod_x d\alpha_x d\sigma_x \ \exp\left[-\alpha_x G^{-1}_{xy} \alpha_y - \sigma_x G^{-1}_{xy} \sigma_y + \sum_x (h_+ \alpha_x - i h_- \sigma_x)\right]$$
$$\times \prod_x \left[1 + \frac{2}{N!} \, \mathrm{Re} \, (\alpha_x + i\sigma_x)^N\right]$$

with the Green function (in thermodynamic limit)

$$G_{xy} = \int_0^{2\pi} \left(\frac{d\phi}{2\pi}\right)^d \frac{e^{i\phi_n (x-y)_n}}{1 - \beta \sum_{n=1}^d \cos \phi_n}$$

The 1st line is the large-N limit of $U(N)$, the 2nd line presents first non-trivial correction from $SU(N)$.

Two different large-N limits can be constructed in this region.

Let $\tilde{h}_r = h_r/N$ and $\tilde{h}_i = h_i/N$ be fixed. Then, the free energy is calculated as

$$F = \lim_{N\to\infty} \lim_{L\to\infty} \frac{1}{N^2 L^d} \log Z = \frac{\tilde{h}_r \tilde{h}_i}{1 - d\beta}$$

This solution coincides with the mean-field solution for the $U(N)$ spin model.

$$F = \lim_{N\to\infty} \lim_{L\to\infty} \frac{1}{L^d} \log Z \;=\; \int_0^{2\pi} \left(\frac{d\phi}{2\pi}\right)^d \log[1 - \beta \sum_{n=1}^{d} \cos\phi_n] + \frac{h_r h_i}{1 - d\beta} + \frac{1}{N!} V_{SU(3)} \,,$$

$$V_{SU(3)} = \frac{h_r^N + h_i^N}{(1 - d\beta)^N} \,.$$

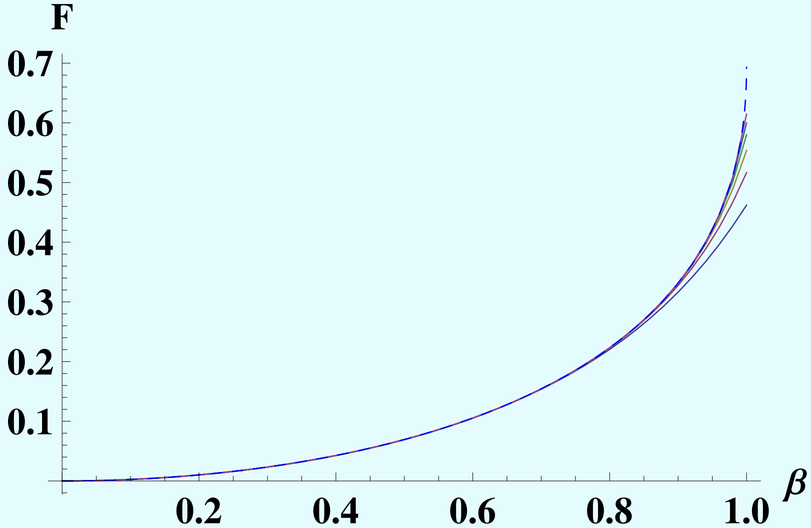
Here is a deconfinement phase transition at $d\beta = 1$ characterised by:

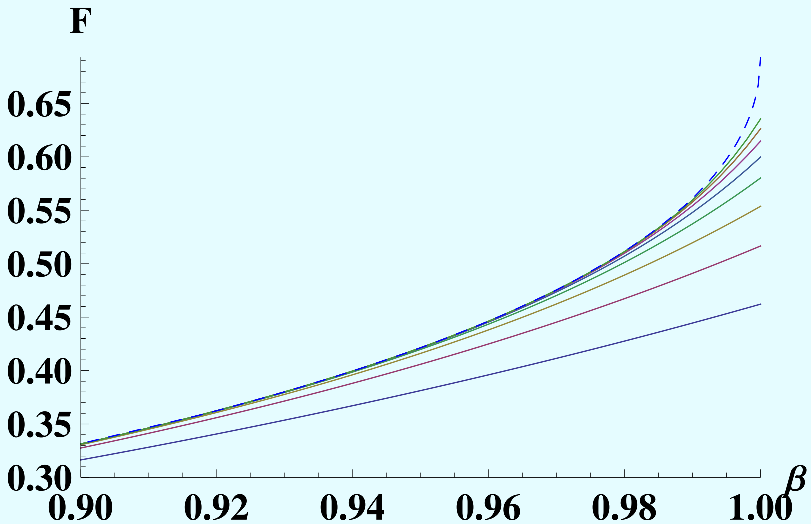
1. Vanishing mass gap

$$m(\beta) = (1 - d\beta)^{\nu} \;\;,\;\; \nu = \frac{1}{2}$$

2. Growing baryon density

$$B \;=\; \frac{\partial}{\partial\mu} F \;=\; \frac{1}{(N-1)!} \left(\frac{h}{1 - d\beta}\right)^N \sinh \mu N \,.$$





Exact $N = \infty$ solution *vs* finite $N = 3 - 10$ solutions for one-dimensional theory obtained via transfer matrix method. $\beta < 1$, $h_r = h_i = 0$

B. Weak-coupling phase: $d\beta > 1$

$$Z_N(\beta, h, \mu) = \prod_x \left[\frac{G(N+1)N^{N^2/2}}{\Gamma\left((N^2-1)/2\right)}\right] \sum_{k_x=-\infty}^{\infty} e^{-N\mu k_x}$$
$$\times \int_0^1 \prod_x dt_x \int_0^{2\pi} \prod_x \frac{d\phi_x}{2\pi} \; e^{N^2 S_{eff} + iNk_x\phi_x}$$

$$S_{eff} = \beta \sum_{x,n} t_x t_{x+n} \cos(\phi_x - \phi_{x+n}) + \sum_x \left(h t_x \cos\phi_x + \frac{1}{2}\log(1-t_x)\right)$$

- $U(N)$: S_{eff} is an action of the d-dimensional XY model with fluctuating positive coupling

- $SU(N)$: S_{eff} is an action of the d-dimensional $Z(N)$ vector model with fluctuating positive coupling

Free energy

$$F_N(\beta,h,\mu) = \lim_{L\to\infty} \frac{1}{L^d N^2} \log Z_N(\beta,h,\mu) = F_0 + \frac{1}{N^2} F_1 + F_{SU(N)}$$

Large-N limit is given by (t_s is a saddle-point solution)

$$F_0 = \beta d t_s^2 + h t_s + \frac{1}{2}\log(1-t_s) - \frac{1}{4} + \frac{1}{2}\log 2$$

Corrections due to fluctuations

$$F_1 = \frac{1}{2}(\log \mathrm{Det} M_{xy} \; + \log \mathrm{Det} G_{xy})$$

Corrections from $SU(N)$ (depend on chemical potential)

$$F_{SU(N)} = \frac{1}{L^d N^2} \log \left[\sum_{k_x} e^{-\frac{1}{4} k_x G_{xy} k_y - N\mu \sum_x k_x} \right] \sim \frac{h t_s \mu^2}{2}, \beta t_s >> 1$$

$$G_{xy} \; = \int_0^{2\pi} \left(\frac{d\phi}{2\pi}\right)^d \frac{e^{i\phi_n (x-y)_n}}{h t_s/2 + \beta t_s^2 (d - \sum_{n=1}^d \cos\phi_n)}$$

When $\beta = \mu = 0$, F_0 coincides with Gross-Witten and Wadia solution for the one-plaquette integral in the large-N limit.

Two-point correlation function

$$\Gamma_N(\beta,h,\mu) = t_s^2 \exp\left[\frac{1}{2t_s^2N^2}(M_0+M_R) - \frac{1}{2N^2}(G_0-G_R)\right]$$

The large-N limit is trivial: $\Gamma = t_s^2$. At large but finite N the properties of Γ depend on the dimension and presence/absence of external field h.

- M_R decays exponentially in any dimension: $M_R \sim e^{-mR}$

- $D_R = G_0 - G_R$ decays exponentially in any dimension if $h \neq 0$ and is bounded from above for $d \geq 3$ if $h = 0$. In these two cases the quark-antiquark potential is screened.

- If $d = 2$ and $h = 0$, $D(R) \sim \log R$. The whole correlation decays algebraically. This property hints on a BKT phase transition in the system.

VI. Conclusions and perspectives

- Dual formulation is constructed for all $U(N)$ and $SU(N)$ Polyakov loop models
- Dual Boltzmann weight is positive in the presence of baryonic chemical potential for all N
- Exact solution is given in the large-N limit
- Numerical simulations: three-quark potential
- Numerical simulations: liquid phase at finite-density (oscillatory behaviour of the Polyakov loop correlators)
- Extension to gauge models: see arXiv:1712.03064