Corner contributions to Entanglement Entropy in $AdS_4/BCFT_3$

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SISSA

based on *Domenico Seminara, J. S. and Erik Tonni [1708.05080] JHEP*

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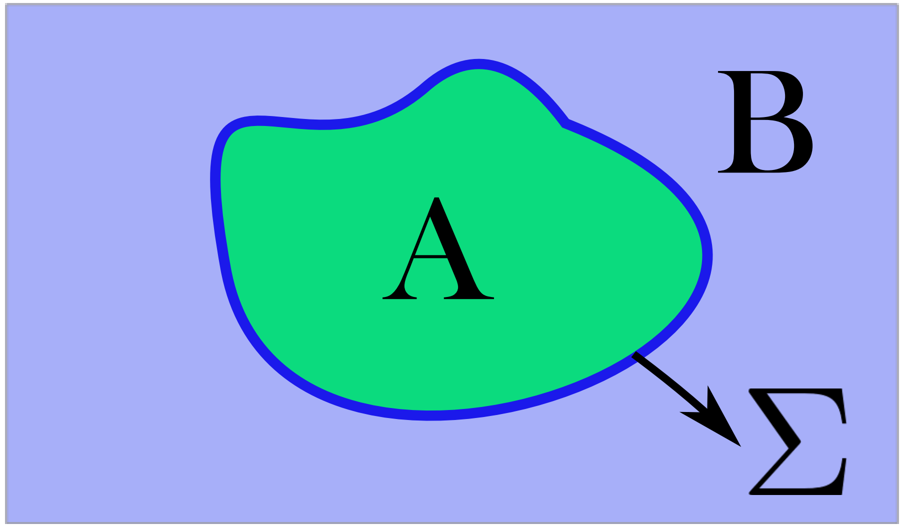
Outline

- Entanglement Entropy and the Logarithmic Terms in 2+1 Conformal Field theories
- Holographic Setup for AdS/BCFT
- Analytic Results and Numerical Checks
- Summary and Perspectives

Entanglement Entropy



State of a quantum system: ρ

Partial trace: $\rho_A = \mathrm{Tr}_B \rho$

$$\boxed{S_A = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]}$$

Pure State: $\rho = |\psi\rangle\langle\psi| \longrightarrow S_A = S_B$

Good measure of the bipartite entanglement

We focus on CFT_3 and on the vacuum state:

$$\boxed{\textbf{2+1:}\quad S_A = \gamma_1 \frac{A(\partial\Sigma)}{\epsilon} + \gamma_2}$$

$\epsilon = \text{cut-off}$

- **Smooth entangling surfaces**
- **No boundaries**

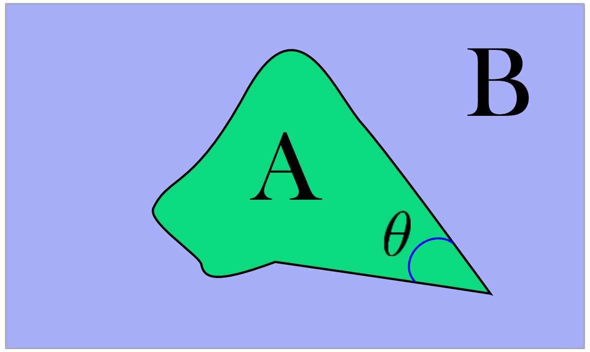
Logarithmic Terms in 2+1 Dimensions

When are they present?

- When there are **corners** in the entangling surface

$$\boxed{S_A = \gamma \frac{P_A}{\epsilon} - f(\theta) \log\left(\frac{P_A}{\epsilon}\right) + \mathcal{O}(1)}$$

[Casini, Huerta (2007)]

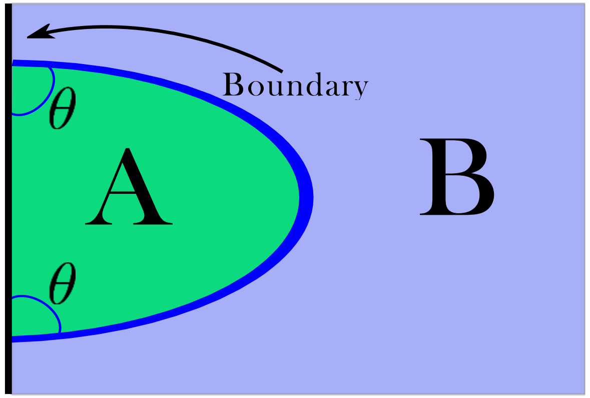


Universal Relation: $\boxed{\frac{f''(\pi)}{C_T} = \frac{\pi^2}{48}}$ $\langle T_{\mu\nu}(x) T_{\alpha\beta}(0)\rangle = \frac{C_T}{|x|^6}\,\mathcal{I}_{\mu\nu,\alpha\beta}(x)$ [Bueno, Myers, Witczak-Krempa (2015)]

Holographic Corner Function: [Drukker, Gross, Ooguri, (1998)]

- In presence of boundaries: **if the entangling surface hits a boundary**

$$\boxed{S_A = \gamma \frac{P_{A,B}}{\epsilon} - 2F(\theta) \log\left(\frac{P_{A,B}}{\epsilon}\right) + \mathcal{O}(1)}$$



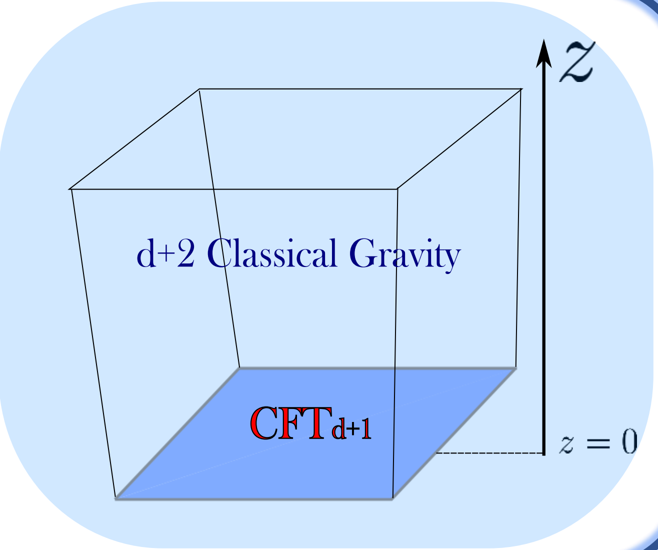
$F(\theta)$ depends on the boundary conditions

CFT results for free theories and infinite line $(\theta = \pi/2)$ [Berthiere, Solodukhin, (2016)]

Holographic Entanglement Entropy

- **CFTs** that admit a **holographic dual**
- **Strong coupling** and **large d.o.f.**

→ **Classical Gravity**



Anti de-Sitter (AdS) metric:

$$ds^2 = \frac{L_{AdS}^2}{z^2}\left(-dt^2 + d\mathbf{x}^2 + dz^2\right)$$

Vacuum state in the CFT

- Holographic dictionary → computation of the correlation functions

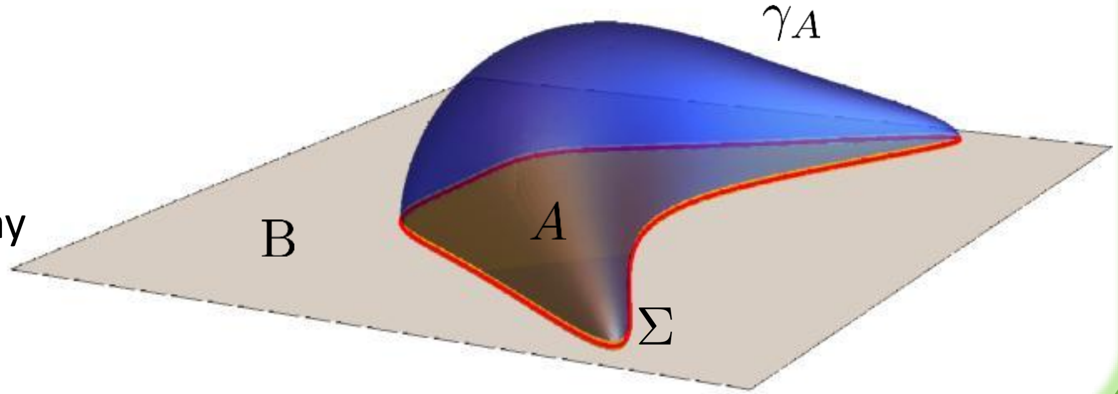
Ryu-Takayanagi Formula

[Ryu, Takayanagi (2006)]

$$S_A = \min_{\gamma_A \sim A}\left[\frac{\mathrm{Area}\,(\gamma_A)}{4G_{d+2}}\right]$$

- The Entanglement Entropy is calculated in a **geometrical** way

$$\mathrm{Area}(\gamma_A) = \frac{P_A}{\epsilon} - F_A + \mathcal{O}(\epsilon)$$



$AdS_4/BCFT_3$ Duality

[Takayanagi (2011)]

How to **extend the boundary** of the CFT into the **bulk**?

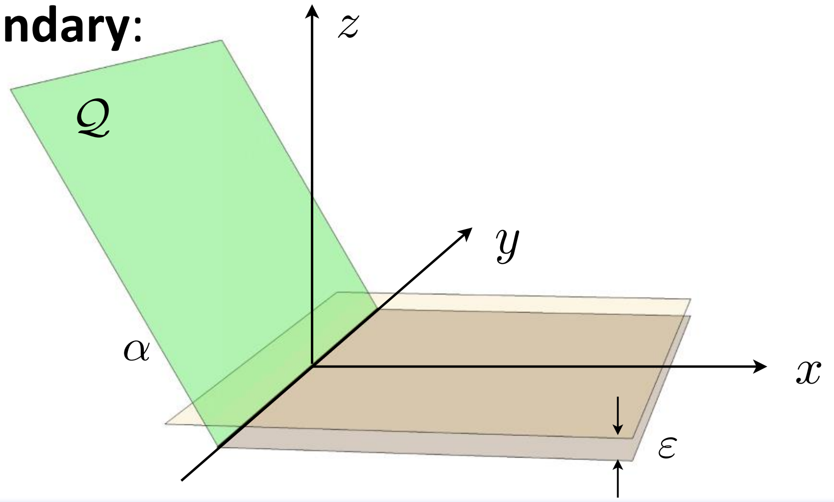
$$I_{gr} = \underbrace{\int_{\mathcal{N}} \sqrt{g}(\mathcal{R} - 2\Lambda) + 2\int_{\mathcal{M}} \sqrt{h_{\mathcal{M}}}K}_{\text{Action without boundaries}} + \underbrace{2\int_{\mathcal{Q}} \sqrt{h_{\mathcal{Q}}}(K - T)}_{\text{brane term}}$$

$K = \text{extrinsic curvature} \qquad T = \text{matter}$

We consider **T constant** and **flat boundary**:

$$K = \frac{d}{d-1}T = const \Rightarrow \quad T = 2\cos\alpha$$

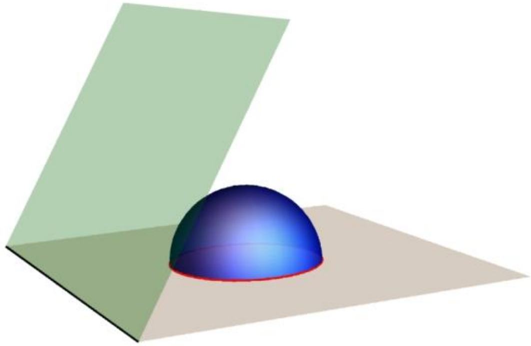
The brane is a hyperplane $\quad \mathcal{Q}: \quad z = -(\tan\alpha)x$



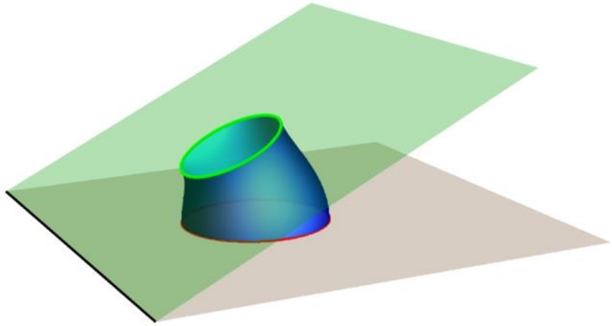
Other discussions: [Astaneh, Berthiere, Fursaev, Solodukhin, (2017)]
[Chu, Miao, Guo, (2017)]

HEE in $AdS_4/BCFT_3$

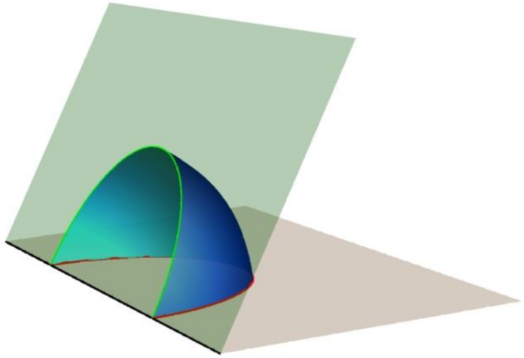
If a **brane** (boundary in BCFT) is present, there are **3 possibilities**:



Like as CFT without boundaries



Boundary effects but no logarithms



Logarithmic divergences

- We **fix** the boundary of the surface **only** on the $BCFT_3$
- We consider **free boundary conditions on the brane** $\Longrightarrow$ $Surface \perp Brane$

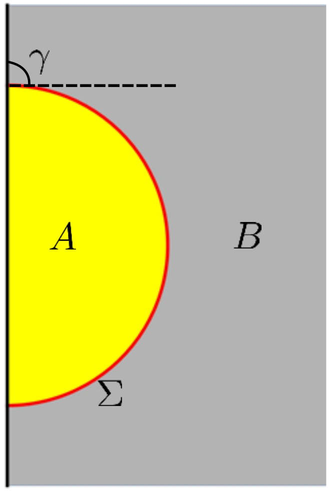
Simplest case: The half-disk $\quad \gamma = \pi/2$

In AdS_4 the minimal surface for a circular region is a **half-sphere**

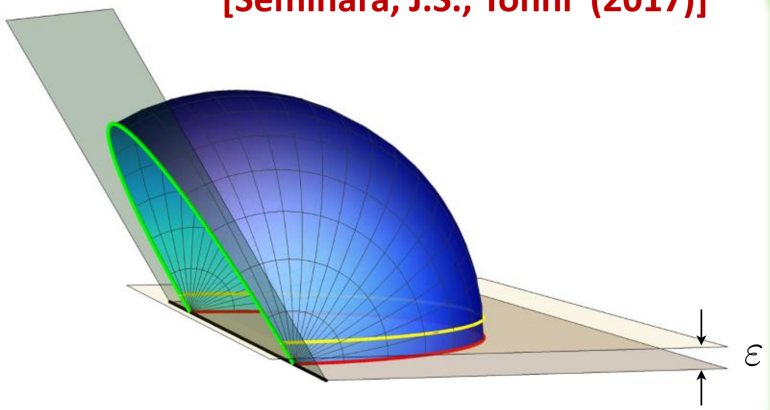
$$\mathcal{A}[\hat{\gamma}_\varepsilon] = L^2_{\text{AdS}}\left(\frac{\pi R}{\varepsilon} + 2\cot(\alpha)\log(R/\varepsilon) + O(1)\right)$$

Analog holographic results found for infinite line $(\gamma = \pi/2)$

[Astaneh, Berthiere, Fursaev, Solodukhin, (2017)]



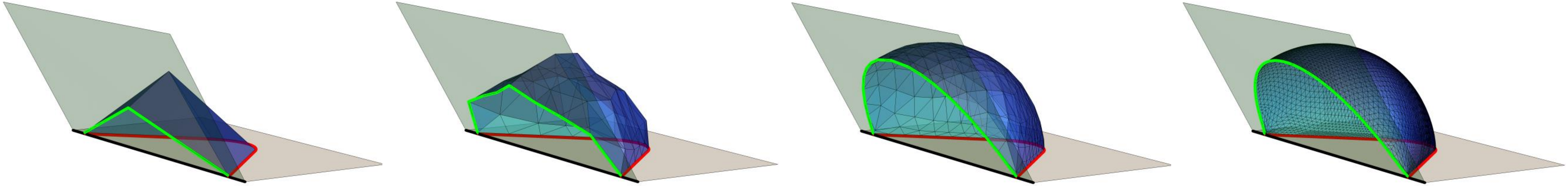
[Seminara, J.S., Tonni (2017)]



Surface Evolver

[K. Brakke (1992)]

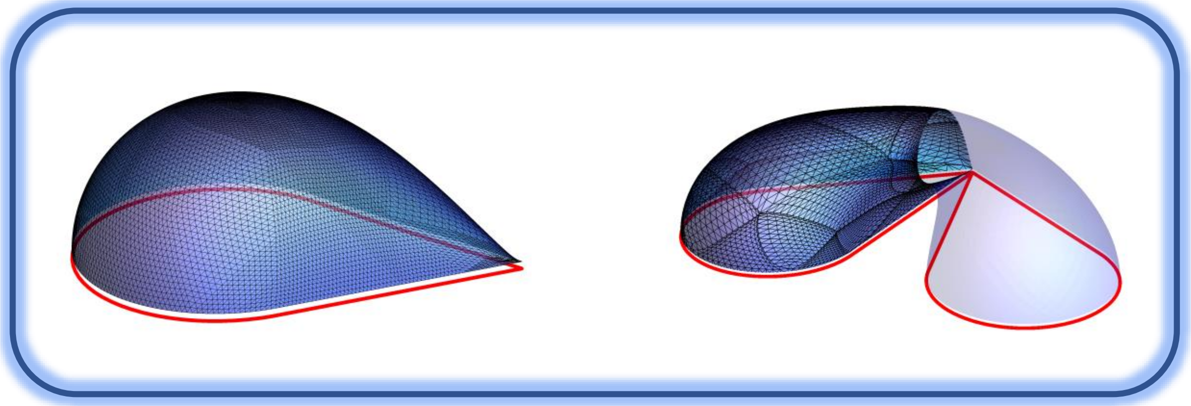
Analytical solutions for arbitrary shapes are very difficult → **Numerical Analysis**

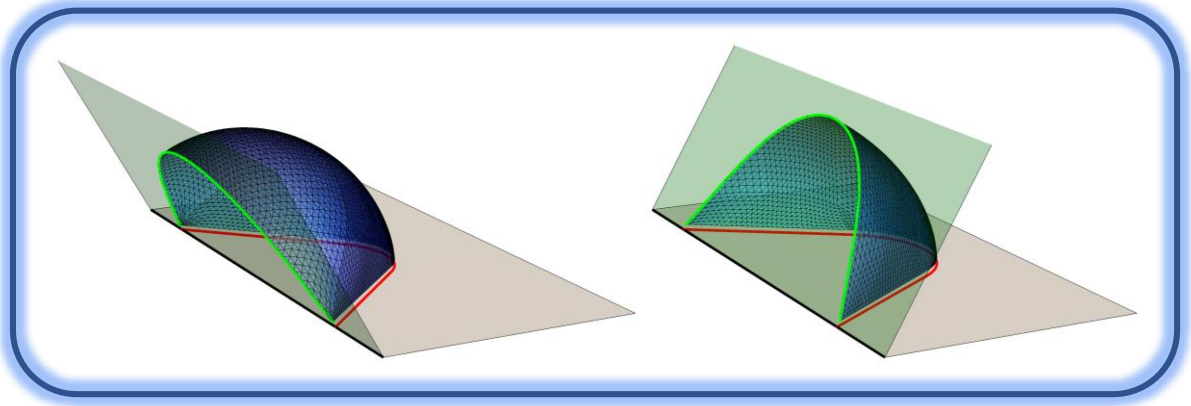


Evolution towards the minimum

Final Result

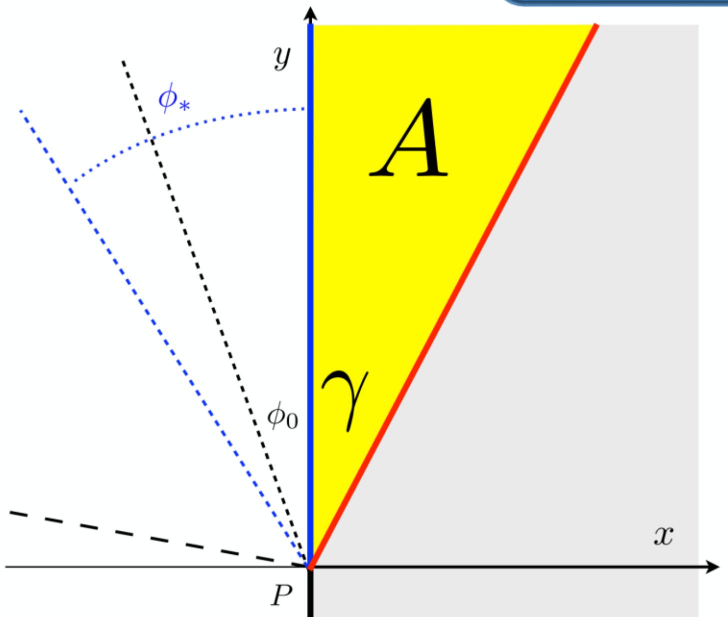
- Due to the **divergence** of the metric as $z \to 0$ we set the boundary at $z = \epsilon$
- The area is found as sum of the areas of all the triangles





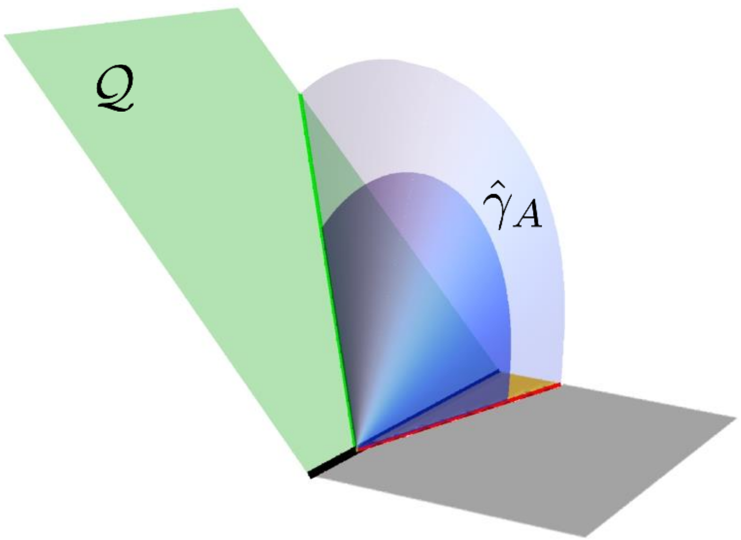
Corner on the boundary

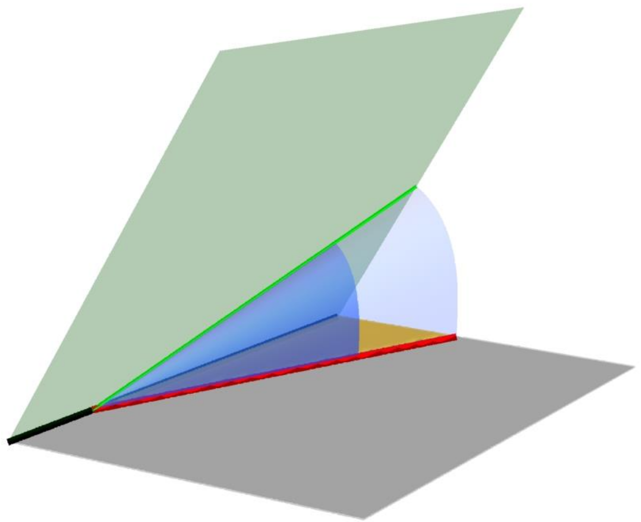
[Seminara, J.S., Tonni (2017)]



- Infinite wedge with an edge on the boundary
- The surface $\hat{\gamma}_A$ intersects $\mathcal{Q}$ orthogonally
- The area can be computed analytically:

$$\boxed{\mathcal{A}[\hat{\gamma}_\varepsilon] = L^2_{\mathrm{AdS}}\left(\frac{L}{\varepsilon} - F_\alpha(\gamma)\,\log(L/\varepsilon) + O(1)\right)}$$





Corner on the boundary

- The corner function $F_A(\gamma)$ can be written in parametric form:

$$\boxed{\begin{cases} F_\alpha = F(q_0) + \eta_\alpha\, \mathcal{G}\big(q_*(\alpha, q_0), q_0\big) \\ \gamma = P_0(q_0) + \eta_\alpha\Big(\arcsin[s_*(\alpha, q_0)] - P\big(q_*(\alpha, q_0), q_0\big)\Big) \end{cases}}$$

$$P(q, q_0) \equiv \frac{1}{q_0(1+q_0^2)}\Big\{(1+2q_0^2)\,\Pi\big(-1/Q_0^2\,,\,\sigma(q,q_0)\,\big|-Q_0^2\big) - q_0^2\,\mathbb{F}\big(\sigma(q,q_0)\,\big|-Q_0^2\big)\Big\}$$

$$\mathcal{G}(q, q_0) \equiv \sqrt{1+q_0^2}\,\left\{\mathbb{F}\big(\sigma(q,q_0)\,\big|-Q_0^2\big) - \mathbb{E}\big(\sigma(q,q_0)\,\big|-Q_0^2\big) + \sqrt{\frac{(q^2+1)(q^2-q_0^2)}{(q_0^2+1)(q^2+q_0^2+1)}}\,\right\}$$

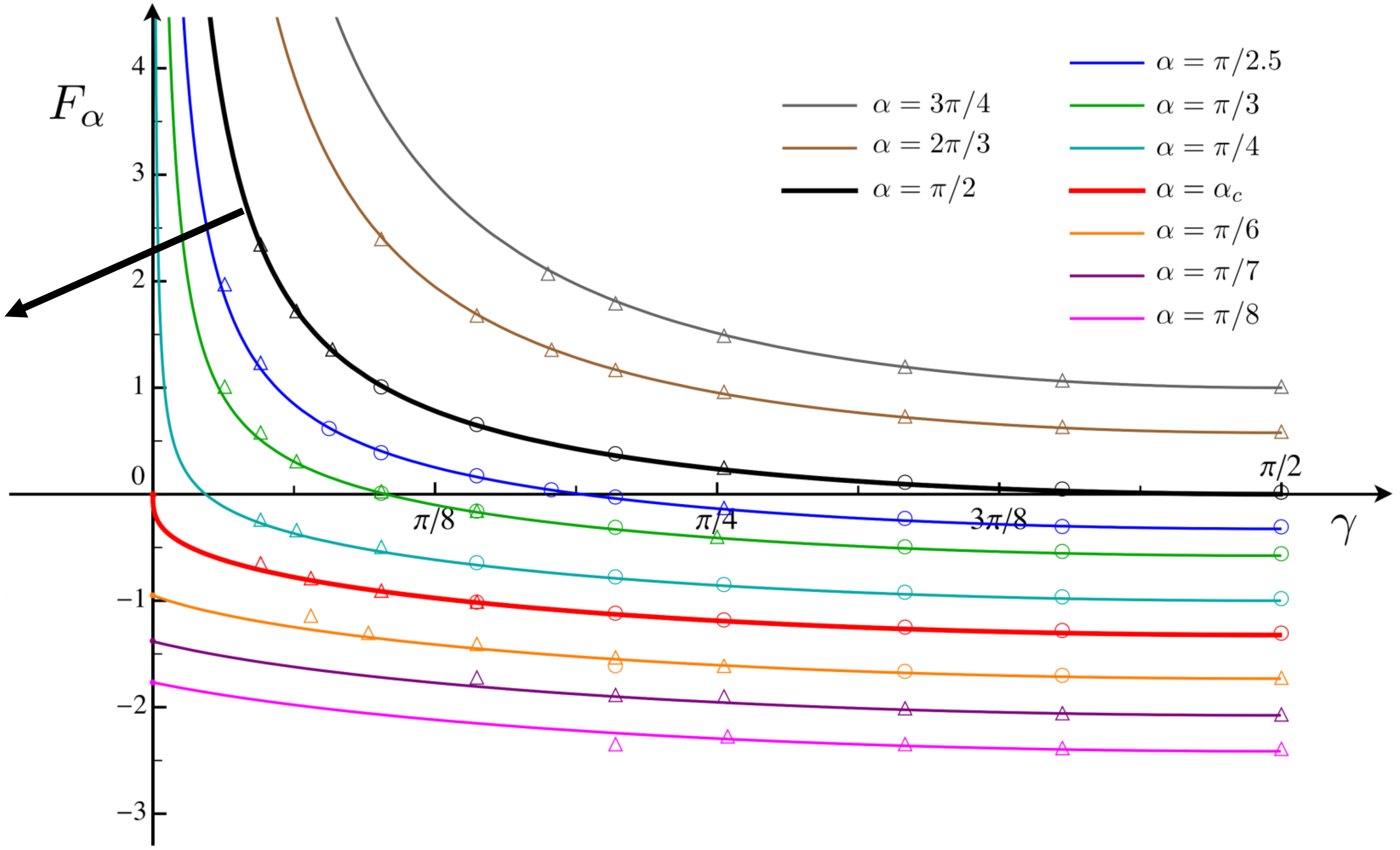
$$s_*(\alpha, q_0) \equiv$$

$$\frac{1}{\sqrt{2}}\left\{\left(1+\frac{q_0^4+q_0^2}{(\cos\alpha)^2}\right)^{-1}\left[1-(\cot\alpha)^2+\sqrt{\Big[1-(\cot\alpha)^2\Big]^2+4\left(1+\frac{q_0^4+q_0^2}{(\cos\alpha)^2}\right)(\cot\alpha)^2}\,\right]\right\}^{\frac{1}{2}}$$

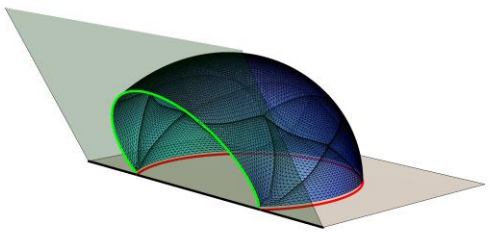
$$q_*(\alpha, q_0) = \frac{|\cot\alpha|}{s_*(\alpha, q_0)} \qquad \sigma(q, q_0) \equiv \arctan\sqrt{\frac{q^2-q_0^2}{1+2q_0^2}} \qquad Q_0^2 \equiv \frac{q_0^2}{1+q_0^2}$$

$$\eta_\alpha \equiv -\,\mathrm{sign}(\cot\alpha)$$

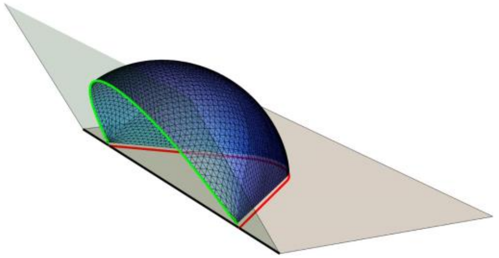
We obtain the Drukker-Gross-Ooguri corner function if $\alpha = \pi/2$



Shapes employed with Surface Evolver:



(empty circles)



(empty triangles)

Relation with the one-point Stress Energy Tensor

- One-point stress energy tensor (**generic BCFT$_3$** and **curved boundary**)

$$\langle T_{ij} \rangle = \frac{A_T}{X^2}\,\kappa_{ij} + \dots \qquad X \to 0^+$$

[Deutsch, Candelas, (1979)]

$$\langle T^{i}_{\ i} \rangle = \frac{1}{4\pi}\big(-\mathfrak{a}\,\mathcal{R} + \mathfrak{q}\,\mathrm{Tr}\kappa^2 \big)\,\delta(\partial\mathcal{B})$$

- Expansion of the boundary corner function:

$$F_\alpha(\gamma) = -\cot\alpha + \frac{(\pi/2-\gamma)^2}{\boxed{2(\pi-\alpha)}} + O\big((\pi/2-\gamma)^4\big)$$

- In the **holographic setup** introduced by Takayanagi

$$\mathfrak{a} = F_\alpha(\pi/2)$$

[Fujita, Takayanagi, Tonni (2011)]

$$\boxed{\frac{F''_\alpha(\pi/2)}{A_T} = -2\pi}$$

[Seminara, J.S., Tonni (2017)]

- The result depends on the boundary condition defining $\mathcal{Q}$
- The relation $\mathfrak{a} = F_\alpha(\pi/2)$ fails for the scalar field due to the minimal coupling with the curvature

[Fursaev, Solodukhin (2016)]

Summary and Conclusions

Summary:

- We study the **logarithmic divergence in $AdS_4/BCFT_3$**
- We found analytically the **boundary corner function**
- A **relation** with the **holographic one-point stress energy tensor** is observed
- Numerical checks with **Surface Evolver**

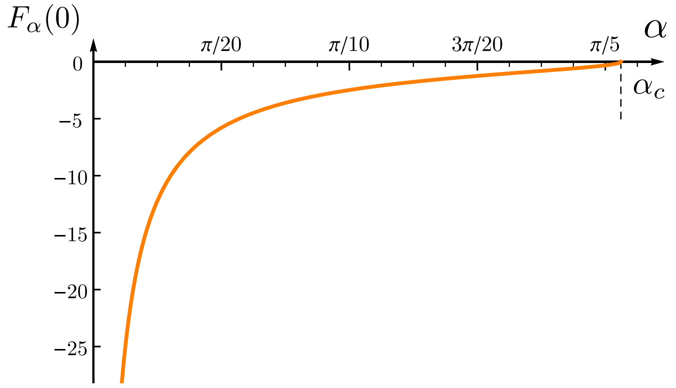
Open Questions:

- Interpretation of α in the **BCFT side**?
- Boundary corner functions in **BCFT**
- Higher dimensional cases

Thank you!

Limits of the Corner Function

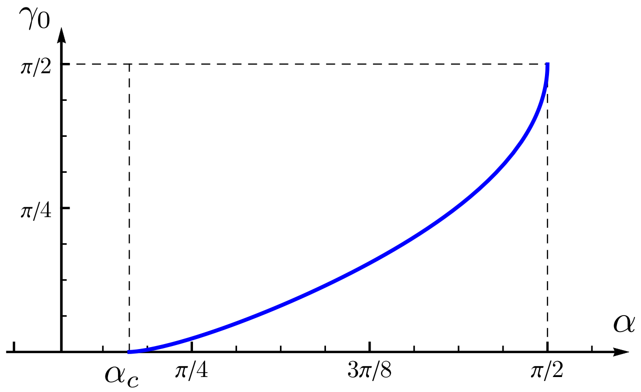
$$\gamma \to 0^+ \qquad \begin{cases} F_\alpha = \dfrac{\mathfrak{g}(\alpha)^2}{\gamma} + O(\gamma) & \alpha \in (\alpha_c, \pi) \\ F_\alpha \to F_\alpha(0) & \alpha \in (0, \alpha_c] \end{cases}$$



$\alpha \in [\alpha_c, \pi/2]$

The logarithmic divergence does not occur for $\gamma = \gamma_0(\alpha)$

$$F_\alpha(\gamma_0) = 0$$



$$\gamma \to \pi/2 \qquad F_\alpha(\gamma) = -\cot\alpha + \frac{(\pi/2-\gamma)^2}{\boxed{2(\pi-\alpha)}} + O\big((\pi/2-\gamma)^4\big)$$

Conformal Field Theory

Replica Trick: $$S_A = \lim_{\alpha\to 1}\left[-\frac{\partial}{\partial\alpha}\log \mathrm{Tr}\rho_A^\alpha\right] \qquad \mathrm{Tr}\rho_A^\alpha = \frac{Z_\alpha}{(Z_1)^\alpha}$$

$$S_A = \lim_{\alpha\to 1}(\alpha\partial_\alpha - 1)W_{CFT}(\mathcal{M}_\alpha)$$

$$W_{CFT}(\mathcal{M}_\alpha) = \frac{a_d}{\epsilon^d} - \frac{a_n}{\epsilon^{d-2n}} - \cdots - a_{d/2}\log\epsilon + \omega(g_{\mu\nu}^{(\alpha)})$$

Manifold with conical singularities: $$a_{d-2n}(\mathcal{M}_\alpha) = \alpha a_{d-2n}^{bulk} + (1-\alpha)a_{d-2n}^{\Sigma} + \mathcal{O}(1-\alpha)^2$$

$$S_A = \frac{s_{d-2}}{\epsilon^{d-2}} - \frac{s_{d-2n}}{\epsilon^{d-2n}} - \cdots - s_0^{\Sigma}\log\epsilon + s \qquad s_{d-2n} = a_n^{\Sigma}$$

1+1: $$S_A = \frac{c}{3}\log\frac{l}{\epsilon} + \gamma_1$$

2+1: $$S_A = \gamma_1\frac{A\,(\partial\Sigma)}{\epsilon} + \gamma_2$$

3+1: $$S_A = \gamma_1\frac{A\,(\partial\Sigma)}{\epsilon^2} + \gamma_2\log\left(\frac{\mu}{\epsilon}\right) + \gamma_3$$

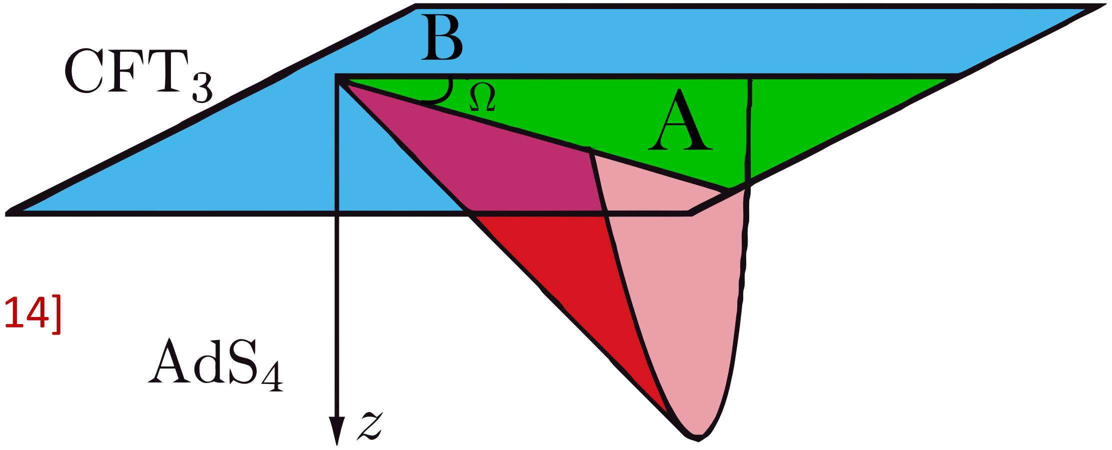
Corner Functions

- Study for **free** field theory

  [H. Casini, M. Huerta; 2006]
- Holographic description

  [T. Hirata, T. Takayanagi; 2006]

  [P. Fonda, L. Giomi, A. Salvio, E. Tonni; 2014]



$$ds^2 = \frac{1}{z^2}(dz^2 + dr^2 + r^2 d\phi^2)$$

Scale invariance $\Longrightarrow$ $z = \dfrac{r}{f(\phi)}$ [N. Drukker , D. J. Gross, H. Ooguri; 1999]

$$\mathcal{A} = \int dr d\phi \sqrt{h_{\rm ind}} = \int dr\, \frac{1}{r} \int d\phi \sqrt{f^4 + f^2 + f'^2} = \int \frac{1}{r} \int d\phi\, \mathcal{L}$$

Conserved quantity

$$\mathcal{H} = \frac{f^2 + f^4}{\sqrt{f^4 + f^2 + f'^2}} = f_0 \sqrt{1 + f_0^2}$$

$$f_0 = f'(\phi_0) = 0$$

$\Longrightarrow$

$$\left[\begin{aligned} \Omega &= 2 f_0 \sqrt{1 + f_0^2} \int_0^\infty \frac{dx}{(x^2 + f_0^2)\sqrt{(z^2 + f_0^2 + 1)(z^2 + 2f_0^2 + 1)}} \\ \mathcal{A} &= 2 \int_{f_0 \epsilon}^{L} \frac{dr}{r} \int_0^{r/\epsilon} dx \sqrt{\frac{x^2 + f_0^2 + 1}{x^2 + 2f_0^2 + 1}} \\ &= \frac{2L}{\epsilon} - 2 f(\Omega) \log\left(\frac{L}{\epsilon}\right) + \mathcal{O}(1) \end{aligned}\right.$$