$-\infty \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \bullet \cdot \cdot \cdot \cdot \cdot \cdot \cdot +\infty$

Exploring the Boundaries of Quantum Mechanics

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Ontological states with deterministic dynamics ?

$\longrightarrow$ **C**ellular**A**utomaton picture of QM

FQT2016, Frascati, 19-21 December 2016

PROLOGUE for Nikola Buric (1959–2016)

Three questions leading to quantum-classical hybrids (QCH):

- *Nikola:*
  What happens if a Hamiltonian, nonlinear, chaotic, or complex system has strictly classical and quantum mechanical parts?
- *Nikola & Thomas:*
  Can this help with the measurement problem in QM?
- *Thomas:*
  Can a quantum system in contact with a classical one act as a seed spreading quantum features in a classical environment?

$\Longrightarrow$ We need a consistent description of **QCH** !!

PHYSICAL REVIEW A **86**, 034104 (2012)

Statistical ensembles in the Hamiltonian formulation of hybrid quantum-classical systems

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(Received 13 August 2012; published 27 September 2012)

PHYSICAL REVIEW A **85**, 064101 (2012)

Hybrid quantum-classical models as constrained quantum systems

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(Received 6 April 2012; published 4 June 2012)

PHYSICAL REVIEW A **85**, 052109 (2012)

Linear dynamics of quantum-classical hybrids

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(Received 15 December 2011; published 11 May 2012)

On Thu, 11 Jul 2013 18:07:13 +0200 Nikola Buric <buric@ipb.ac.rs> wrote:

Dear Thomas, Yes, I still think that we have proposed a good project.

The people that got grants in Serbia all work experimentally on nano-technologies (or at least sell their work as such).

[...]

It is also very hot in Belgrade but...

All the best,

Nikola.

Linearity ...

- Def.: linearity $\Longleftrightarrow$ dynamics maps states linearly on states.
- Theorem: QM is linear.

  proofs: E.P. Wigner, V. Bargmann
  – assumption: dynamics does not change $|\langle\psi'|\psi\rangle|$

  proof: T.F. Jordan
  – assumption: no influences without interactions
  " ... that the system we are considering can be described as part of a larger system without interaction with the rest of the larger system."
- Experiments testing linearity, test also these assumptions!

... *reasoning about the linearity of QM has led us to CA models*

based on three ingredients ...

- deterministic discrete mechanics – T.D. Lee *et al.*

  $\longrightarrow$ ex. minimal time l and discrete updating rules

- sampling theory for discrete structures – A. Kempf *et al.*

  $\longrightarrow$ ex. map: CA $\leftrightarrow$ continuum QM + corrections

- "oscillator representation" of QM – A. Heslot

  $\longrightarrow$ set $\psi \equiv x + ip$

… the CA models of a particular class show

these quantum features …

- evolution by continuous time Schrödinger equation modified by l-dependent higher-order derivatives w.r.t. time
- l-dependent dispersion relation for stationary states
- l-dependent conservation laws in 1-to-1 corresp. with QM
- multipartite CA obeying Superposition Principle

quantum features ... *(cont.)*

- if space discrete as well: Generalized Uncertainty Principle based on Robertson's inequality, $\Delta A \Delta B \geq |\langle [A,B] \rangle|/2$

  define $X_{mn} := l m \delta_{mn}$, $P_{mn} := -i(\delta_{m,n-1} - \delta_{m,n+1})/2l$

  $\Longrightarrow \quad \Delta X \Delta P \geq \left| 1 + \frac{l^2}{2} \langle P^2 \rangle \right|$, $\Delta P^2 := \langle P^2 \rangle - \langle P \rangle^2$

  $\Longrightarrow$ minimal uncertainty $\Delta X_{min} = l/\sqrt{2}$. [David Gigli]

$\Longrightarrow$ QM results can be obtained in continuum limit, $l \to 0$.

Hamiltonian Cellular Automata (CA) – *"bit machines"*

- classical CA with denumerable degrees of freedom

- state described by integer valued coordinates x_n^α, τ_n and momenta p_n^α, π_n

  $\alpha \in \mathbf{N}_0$: different degrees of freedom
  $n \in \mathbf{Z}$: successive states

- finite differences, $\Delta f_n := f_n - f_{n-1}$

  $\longrightarrow$ no infinitesimals!

The CA Action Principle <•=•= Phys. Rev. A 89, 012111 (2014).

- $\mathcal{A}_n := \Delta\tau_n(H_n + H_{n-1}) + c_n\pi_n$,

  $H_n := \frac{1}{2}S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta}p_n^\alpha x_n^\beta + R_n(x,p)$,

  const. c_n, sym. $\hat{S} \equiv \{S_{\alpha\beta}\}$, antisym. $\hat{A} \equiv \{A_{\alpha\beta}\}$, *remainder* R_n .

- integer valued action:

  $\mathcal{S} := \sum_n[(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha + (\pi_n + \pi_{n-1})\Delta\tau_n - \mathcal{A}_n]$.

- Action Principle: $\delta\mathcal{S} \stackrel{!}{=} 0 \;\Rightarrow$ CA updating rules,

  for $\delta g(f_n) := [g(f_n + \delta f_n) - g(f_n - \delta f_n)]/2$, $\delta f_n \in \mathbf{Z}$, arbitrary

  Remarks ... $\Longrightarrow$ $R_n \equiv 0$, only harmonic CA consistent. ??

CA equations of motion

- $\delta\mathcal{S} \stackrel{!}{=} 0 \;\Rightarrow\;$ finite differences e.o.m.:

$$\dot{x}_n^\alpha \;=\; \dot{\tau}_n(S_{\alpha\beta}p_n^\beta + A_{\alpha\beta}x_n^\beta) \;,$$

$$\dot{p}_n^\alpha \;=\; -\dot{\tau}_n(S_{\alpha\beta}x_n^\beta - A_{\alpha\beta}p_n^\beta) \;,$$

$$\dot{\tau}_n \;=\; c_n \;, \quad \dot{\pi}_n \;=\; \dot{H}_n \;, \quad \text{with } \dot{O}_n := O_{n+1} - O_{n-1} \;.$$

- e.o.m. time reversal invariant, $(n \mp 1, n) \;\rightarrow\; (n \pm 1)$

- $\Longrightarrow \;\; \dot{\psi}_n^\alpha \;=\; -i\dot{\tau}_n H_{\alpha\beta}\psi_n^\beta \;,$ discrete "Schrödinger equation"

with $\hat{H} := \hat{S} + i\hat{A}$, self-adjoint, $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$, CA "time" n

CA conservation laws

- Theorem:

  For any $\hat{G}$ with $[\hat{G}, \hat{H}] = 0$, ex. discrete conservation law:

  $$\psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^{\beta} + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^{\beta} = 0 \quad .$$

  For $\hat{G} = \hat{G}^{\dagger}$ (complex integer) $\rightarrow$ n-indep. two-point fct.:

  $$\Re\left[\psi_n^{*\alpha} G_{\alpha\beta} \psi_{n+1}^{\beta}\right] = \text{const} \in \mathbf{Z} \ .$$

  For $\hat{G} = \hat{1}$: $\Re\left[\psi_n^{*\alpha} \psi_{n+1}^{\alpha}\right] = \text{const}$. [cf. $\psi^{*\alpha}\psi^{\alpha} = 1$] Born ?

- Ex. 1-to-1 correspondence CA $\leftrightarrow$ QM conservation laws .

Consistent anharmonic CA <•=•= JPCS 631, 012069 (2015) .

- action with anharmonic polynomial terms – e.g., $(x_n^\alpha x_n^\alpha)^2$

  $\Longrightarrow$ consistent CA e.o.m., provided

  $\delta g_f(f) := [g(f+\delta f) - g(f-\delta f)]/2\delta f$, $\delta f \in \mathbf{Z}$, arbitrary

  generalized by

  $\delta_f g^{(N)}(f) := \sum_k \gamma_k [g^{(N)}(f + m_k\delta f) - g^{(N)}(f - m_k\delta f)]/2\delta f$,

- such that $\delta_f \;\widehat{=}\; \mathrm{d}/\mathrm{d}f$:

  $\delta_f g^{(N)}(f) = \bar{g}^{(N-1)}(f)$, terms $\propto (\delta f)^j$, $j > 0$ cancel .

  $\Longrightarrow$ discrete nonlinear "Schrödinger equation" . ??

Towards continuum QM ...

- recall $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$, CA "time" n

  introduce minimal time $l \longrightarrow n \cdot l$, physical time?

  $\Longrightarrow$ continuum limit, $l \to 0$, does not work
  – integer valuedness $\Rightarrow$ time derivatives diverge!

- construct invertible map:

  discrete (integer valued) $\longleftrightarrow$ continuous (differentiable)
  – G. 't Hooft

- simultaneously continuous & discrete information
  – C.E. Shannon

The Sampling Theorem

- Consider square integrable bandlimited functions f :

  $$f(t) = (2\pi)^{-1} \int_{-\omega_{max}}^{\omega_{max}} d\omega \; e^{-i\omega t} \tilde{f}(\omega) \; , \; \text{bandwidth } \omega_{max} \; .$$

- Shannon's Theorem:

  Given $\{f(t_n)\}$ for set $\{t_n\}$ of equidistantly spaced times (spacing π/ω_{max}), function f is obtained for all t by:

  $$f(t) = \sum_n f(t_n) \cdot \frac{\sin[\omega_{max}(t-t_n)]}{\omega_{max}(t-t_n)} \quad \text{(reconstruction formula)} \; .$$

- CA "time" n $\sim$ discrete time $t_n := nl$ $\rightarrow$ continuous time t
  bandwidth $\omega_{max} := \pi/l$ (Nyquist rate)

Map: discrete CA $\leftrightarrow$ continuous QM

- by Shannon's *reconstruction formula* ...
  discrete e.o.m., $\dot{\psi}_n^\alpha = -i\hat{H}_{\alpha\beta}\psi_n^\beta$,
  $\longleftrightarrow$ continuous time "Schrödinger equation":
  $$(\hat{D}_l - \hat{D}_{-l})\psi^\alpha(t) = 2\sinh(l\partial_t)\psi^\alpha(t) = -iH_{\alpha\beta}\psi^\beta(t) \ ,$$
  with $\hat{D}_T f(t) := f(t+T)$.
- $\Rightarrow$ l-dependent dispersion relation & conservation laws . *
- $\Rightarrow$ $l \to 0$ reproduces corresponding QM results .
- $\Rightarrow$ different linear reconstructions $\longrightarrow$ same e.o.m. & conservation laws – wave fct. "cut-off" changes .

Note: l-dependence (continuous description)

- l-dependent constants of motion:

  from Theorem on discrete conservation laws,

  for any $\hat{G}$ with $[\hat{G}, \hat{H}] = 0$ and $\hat{G} = \hat{G}^\dagger$ (complex integer),

  $\Rightarrow$ $\Re\big[\psi^{*\alpha}(t) G_{\alpha\beta} \psi^\beta(t+l)\big] = \text{const} \in \mathbf{Z}$.

- l-dependent dispersion relation:

  $\Rightarrow$ $lE_\alpha = \arcsin(\frac{l\epsilon_\alpha}{2}) = \frac{l\epsilon_\alpha}{2}[1 + (\frac{l\epsilon_\alpha}{2})^2/6 + O\big((l\epsilon_\alpha)^4\big)]$,

  where $\hat{H} \to \{l\epsilon_\alpha\}$ and, thus, $|E_\alpha| \leq \pi/2l = \omega_{max}/2$.

What goes wrong with anharmonic CA ...

- by Shannon's *reconstruction formula* ...

  discrete $\psi_n \longleftrightarrow \psi(t)$, bandlimited $\sim \pi/l$

  discrete anharmonic, e.g., $(\psi_n)^2 \longleftrightarrow \psi_{(2)}(t) =$??

- $\psi_n = l^{-1} \int \mathrm{d}t \; s_n(t)\psi(t)$, $\quad s_n(t) := \mathrm{sinc}[\pi(t/l - n)]$

- $\psi_{(2)}(t) = l^{-2} \iint \mathrm{d}t'\mathrm{d}t'' \left[\sum_n s_n(t) s_n(t') s_n(t'') \right] \psi(t')\psi(t'')$,

  $\rightarrow$ correctly bandlimited, but nonlocal in time .

- $\Rightarrow$ anharmonic CA not describable in a local continuous way
  $\Rightarrow$ no l-dep. CA based nonlinear Schrödinger equation .

The formal solution of discrete CA e.o.m.:

- recall: $\partial_t \psi(t) = -i\hat{H}\psi(t) \quad \Rightarrow \quad \psi(t) = \mathrm{e}^{-i\hat{H}t}\psi(0)$.
- here: $\dot{\psi}_n = \psi_{n+1} - \psi_{n-1} = -i\hat{H}\psi_n \quad \Rightarrow$

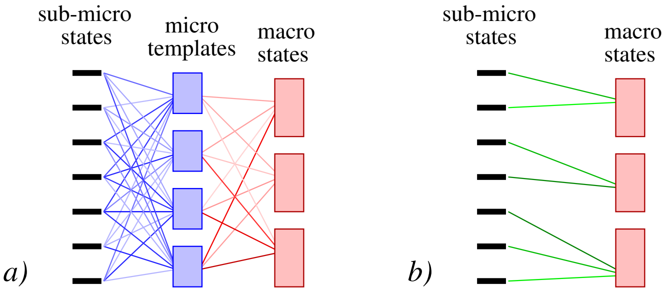
  $$\psi_n = \frac{1}{2\cos\hat{\phi}}\left(\mathrm{e}^{-in\hat{\phi}}[\mathrm{e}^{i\hat{\phi}}\psi_0 + \psi_1] + (-1)^n \mathrm{e}^{in\hat{\phi}}[\mathrm{e}^{-i\hat{\phi}}\psi_0 - \psi_1]\right) ,$$

  where $2\sin\hat{\phi} := \hat{H} \quad \Rightarrow \quad -2 \leq l\epsilon_\alpha \leq 2$. !

  $\longrightarrow$ Ex. (in)finite sets of such symmetric integer $\hat{H}$'s.
- note: solutions exponential for $n \to \infty$, iff $\psi_1 \equiv \psi_0$, ??

  $\Longrightarrow \quad \psi_n = \left(\mathrm{e}^{-in\phi} + \mathrm{O}(1/n)\right)\psi_0$.

  Let $l \equiv t_{Planck}$, even for $n \sim 10^{10}$: $t \equiv nt_{Planck} \sim 10^{-34}\mathrm{s}$.



G 't Hooft *The Cellular Automaton Interpretation of QM* arXiv:1405.1548

Two-state systems:

- note: $1/l = t_{Planck}^{-1}$ sets energy scale for Hamiltonians bounded by $-2/l \leq \epsilon_\alpha \leq 2/l$.
- now: $\psi_n = \left(e^{-in\phi} + O(1/n)\right)\psi_0$, with

  $\phi = \arcsin \frac{\hat{H}}{2} := \arcsin \frac{\vec{m}\cdot\vec{\sigma}}{2}$, $\vec{m} \in \mathbf{Z}_3$, $\vec{\sigma}$: Pauli's $\sigma_{1,2,3}$

  $\Rightarrow$ $\hat{H}^2 = |\vec{m}|^2$, eigenvalues $\pm|\vec{m}|$, bounded by $|\vec{m}| \stackrel{!}{\leq} 2$

  $\Rightarrow$ $\phi = |\vec{m}|^{-1} \arcsin \frac{|\vec{m}|}{2} \cdot \hat{H} =: \hat{H}' = O(1)$.
- with $t \equiv nt_{Planck}$: $\exp(-in\phi) = \exp(-it[\hat{H}'/t_{Planck}])$!?

Magnons (cf. Peierls '56, Feynman '74):

- Consider "ferromagnet" of N two-state components,

  e.g., $\uparrow_1 \uparrow_2 \downarrow_3 \ \ldots \ \downarrow_{N-1} \uparrow_N$, periodic b.c. (1-d, ..., 3-d c.l.)

- $\hat{H} + N := -\sum_{n=1}^{N} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} + N = -2\sum_n (\hat{P}_{n,n+1} - 1)$

  $\longrightarrow$ totally symm. groundstate, $E_0 = \langle c.s.|\hat{H}|c.s.\rangle = -N$.

- lowest exc. state, $\psi = \sum_n a_n \ \uparrow_1 \uparrow_2 \ \ldots \downarrow_n \ \ldots \ \uparrow_N$,

  i.e., "superposition + dephasing", $a_n \equiv e^{i\delta n}$, $a_{N+1} \stackrel{!}{=} a_1$

  $\longrightarrow \ \delta = 2\pi j/N$, $j \in \mathbf{Z}$, $-N/2 \leq j \leq N/2$.

Magnons ... *(cont.)*

- dispersion relation from $Ea_n = -2(a_{n+1} + a_{n-1} - 2a_n)$,

  $\Longrightarrow$ energy band: $E = 4(1 - \cos\delta)$, $\delta = 2\pi j/N$.
- set $\delta \equiv kl$, momentum of plane wave, $k = \frac{2\pi j}{Nl}$

  $\longrightarrow$ long wavelength limit: $E \approx 2k^2 l^2 \equiv \frac{k^2}{2\mu} = 2\left(\frac{2\pi j}{N}\right)^2$
- note: dimensionfull energy is $E_j := E/l = 2\frac{(2\pi j)^2}{lN^2}$.
- $\Longrightarrow$ "coherent superposition + dephasing" (NR model!)

  reduce Planck scale energies $\propto N^{-2}$.

- Common l-dependent aspects of natural CA & QM:

  eqs. of motion, conservation laws; observables, admiss. $\hat{H}$

  multipartite CA <•=•= Int.J.Quant.Info. 14(4), 1640001 (2016)

- Map: CA $\leftrightarrow$ QM based on linear Sampling Theory

  ... fails for nonlinear CA $\longrightarrow$ alternatives ?

- *Desiderata*:

  random / nonunitary aspects **?** better: ontological models **!!**

  relativistic / QFT extension **??**

Discrete Poisson brackets & CA observables:

- recall: only variational derivatives for discrete variables
  $\delta g(f) := [g(f+\delta f) - g(f-\delta f)]/2\ ,\quad f, \delta f \in \mathbf{Z}\ .$
- $\rightarrow$ define: $\{A, B\} := \delta_{x^\alpha} A\, \delta_{p^\alpha} B - \delta_{x^\alpha} B\, \delta_{p^\alpha} A\ .$
- for *constant, linear, or quadratic polynomials* A, B, variational derivatives independent of δ_x, δ_p and bracket corresponds to ordinary Poisson bracket, in all respects.
- $\Rightarrow$ CA observables can be chosen as real quadratic forms in $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$; a closed algebra endowed with $\{\ ,\ \}$.

E.g., $\dot{\psi}^\alpha = \{\psi^\alpha, \mathcal{H}\}$, with $\mathcal{H} := \psi^{*\alpha} H_{\alpha\beta} \psi^\beta/2$.