# The optomechanical experiment of Marshall *et al.* in a quantum-classical hybrid theory

Aniello Lampo (ICFO, Barcelona)

Training School of Frascati - 20/12/2016

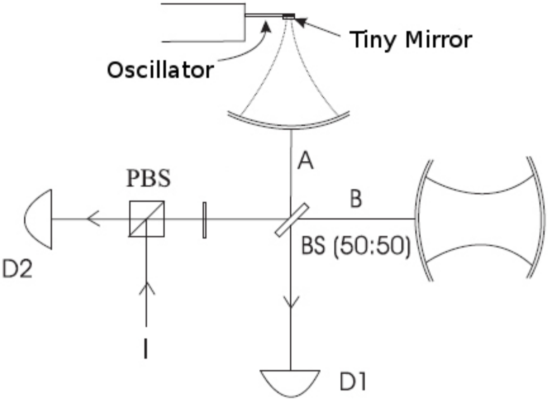
Collaborators: H.-T. Elze and L. Fratino

ICFO Institut de Ciències Fotòniques

erc

# The Marshall *et al.* Optomechanical System

The **Marshall et al. optomechanical setup** has the following structure:




[W. Marshall, C. Simon, R. Penrose and D. Bouwmeester, *Towards quantum superpositions of a mirror*, Phys. Rev. Lett. 91, 130401 (2003)]

# Previous Results - State of the System

The Marshall *et al.* system is described by the **Hamiltonian**:

$$\hat{H} = \hbar\omega_c\left(\hat{c}_A^\dagger\hat{c}_A + \hat{c}_B^\dagger\hat{c}_B\right) + \hbar\Omega\hat{b}^\dagger\hat{b} - \hbar g\hat{c}_A^\dagger\hat{c}_A\left(\hat{b} + \hat{b}^\dagger\right)$$

The initial state of the whole system is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle + |B\rangle\right)|\beta\rangle$$

at a generic instant:

$$\begin{aligned}|\psi(t)\rangle = & \frac{1}{\sqrt{2}}e^{ik^2\theta(t)}e^{-i\mathrm{Im}\left[\alpha^*(t)\beta e^{-i\Omega t}\right]}|A\rangle|\beta e^{-i\Omega t} + \alpha(t)\rangle \\ & + \frac{1}{\sqrt{2}}|B\rangle|\beta e^{-i\Omega t}\rangle\end{aligned}$$

with $k = \frac{g}{\Omega}$, $\theta(t) \equiv \Omega t - \sin(\Omega t)$ and $\alpha(t) \equiv -k\left(e^{-i\Omega t} - 1\right)$.

Generally, is a **Schrödinger cat state**; however, **at $\mathbf{t = \frac{2\pi n}{\Omega}}$, no entanglement between photon and mirror.**

# Previous Results - Visibility of Interference

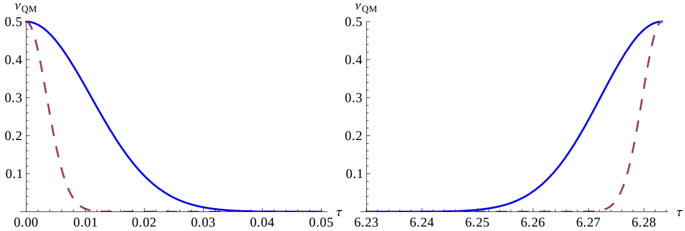
Assume the mirror is in a **thermal mixture** of coherent states:

$$\rho_{AB} = \int_{-\infty}^{+\infty} \frac{d^2\beta}{2\pi\bar{n}} \mathsf{Tr}_n \left[\langle A|\Psi\rangle\langle\Psi|B\rangle\right] e^{-\frac{|\beta|^2}{\bar{n}}}$$

The **visibility of interference** is:

$$\nu_{QM} = |\rho_{AB}| = \frac{1}{2} e^{-(2\bar{n}+1)k^2[1-\cos(\tau)]}$$

where $\tau = \Omega t$ and $\bar{n} = 1/\left(e^{\frac{\hbar\Omega}{k_B T}} - 1\right)$ .




Visibility of interference for $k = 1$ and $\Omega = 2\pi \cdot 500$Hz. Blue and purple line refer respectively to $T = 10^{-4}$ and $T = 10^{-3}$ K.

# Hybrid Theory - Essentials

A **hybrid theory** aims to study systems where a classical and quantum sector coexist and interact.

Every generic **quantum state** $|\psi\rangle \in \mathcal{H}$ can be expanded in the "oscillator representation":

$$|\psi\rangle = \sum_j \frac{(X_j + iP_j)}{\sqrt{2\hbar}} |j\rangle$$

How do we treat **observables** in this framework?

$$\langle\psi|\hat{O}|\psi\rangle = \frac{1}{2\hbar} \sum_{i,j} (X_i - iP_i)(X_j + iP_j) \langle i|\hat{O}|j\rangle$$

We can also define a **Poisson bracket**:

$$\{f, g\}_{QM} = \sum_i \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial X_i} \frac{\partial f}{\partial P_i} \right)$$

# Hybrid Hamiltonian

We use the hybrid approach to study the Marshall *et al.* system **considering the mirror as a classical rather than quantum object**.

The quantum and classical sector are described by:

$$\hat{H}_{QM} = \hbar\omega_c \hat{c}_A^\dagger \hat{c}_A + \hbar\omega_c \hat{c}_B^\dagger \hat{c}_B \quad , \quad H_{CL} = \frac{p^2}{2M} + \frac{M\omega_m^2}{2} x^2$$

and the hybrid coupling by:

$$\hat{I} = -\hbar\tilde{g} x \hat{c}_A^\dagger \hat{c}_A$$

Introducing "oscillator representation", the **full hybrid Hamiltonian**:

$$\begin{aligned} H = & \frac{p^2}{2M} + \frac{M\omega_m^2}{2} x^2 + \frac{\omega_c}{2}\left(X_A^2 + P_A^2\right) + \\ & + \frac{\omega_c}{2}\left(X_B^2 + P_B^2\right) + \frac{\tilde{g}}{2} x \left(X_A^2 + P_A^2\right) \end{aligned}$$

# Hybrid Density Matrix

Solving the equation of motion following by the Hamiltonian we can obtain the off-diagonal matrix element of the **photon reduced density operator**:

$$
\begin{aligned}
\rho_{AB}(t) =& \frac{1}{2\hbar}\left(X_A + iP_A\right)\left(X_B - iP_B\right) \\
=& \frac{1}{2}\exp\{i\omega_c t\}\exp\left[-i\Omega t - \frac{iA_m\tilde{g}}{\Omega}\left[\cos\left(\Omega t + \phi_m\right) - \cos\left(\phi_m\right)\right]\right]
\end{aligned}
$$

Unlike in purely quantum case, **off-diagonal matrix element is just a phase**.

$\Longrightarrow$**if mirror initial conditions consist in one fixed point of the classical phase space, there is no decoherence** (interference effects of the photon are preserved by interaction with the classical mirror).

# Thermal Average

Next, we assume more realistically that only a **thermal distribution** of mirror initial conditions in classical phase space is known:

$$f\left(x_0, p_0, T\right) = \frac{\beta\Omega}{2\pi} \exp\left[-\beta\left(\frac{p_0^2}{2M} + \frac{M\omega_m^2 x_0^2}{2}\right)\right]$$

where $\beta \equiv \frac{1}{k_B T}$.

Adopting this distribution, the averaged off-diagonal element becomes:

$$< \rho_{AB} >_f = \frac{1}{2} e^{-ik^2\theta(t)} \exp\{-z^2\left[1 - \cos\left(\Omega t\right)\right]\}$$

with $z = \sqrt{\frac{\tilde{g}^2}{M\omega_m^4\beta}}$, and $\theta\left(t\right) \equiv \Omega t - \sin\left(\Omega t\right)$.

$\Longrightarrow$**SAME RESULT AS FOR A QUANTUM MIRROR!**

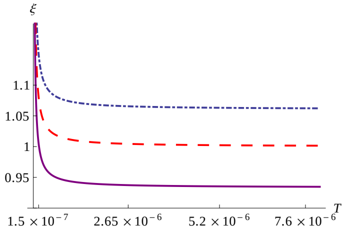
# Final Comparison

Visibilities of interference in the two cases:

- $\nu_{CL} = \frac{1}{2} e^{-z_{CL}^2 [1 - \cos(\Omega t)]}, \quad z_{CL}^2 = 2 \frac{k_B T}{\hbar \Omega} k^2$
- $\nu_{QM} = \frac{1}{2} e^{-z_{QM}^2 [1 - \cos(\Omega t)]}, \quad z_{QM}^2 = (2\bar{n} + 1) \, k^2$

The following relation subsists:

$$z_{QM} = \xi(\chi) z_{CL} \quad , \quad \xi(\chi) = \frac{\chi}{e^{\chi} + 1} + \frac{\chi}{2}, \quad \chi(\Omega, T) = \frac{\hbar \Omega}{k_B T}.$$



# Conclusions and Perspectives

Our study of the **visibility of interference** both in a purely quantum and hybrid descriptions yields the following results:

- in hybrid theory, **if the mirror initial conditions consist in one fixed point of classical phase space, there is no decoherence** and the photon remains a pure state.
- in hybrid theory, mirror induced decoherence appears if mirror initial conditions are statistically distributed. **Adopting a thermal distribution, we obtain the same result as in purely QM description**.

PHYSICAL REVIEW A **90**, 042120 (2014)

**Mirror-induced decoherence in hybrid quantum-classical theory**

Aniello Lampo,[1,*] Lorenzo Fratino,[2,†] and Hans-Thomas Elze[3,‡]

# Hamiltonian: qualitative derivation

The interaction term represents **the mechanical work of the radiation pressure on the mirror displacement**.

Accordingly, apart some constants, it can be expressed as:

$$H_{int} = PS\left(\Delta x\right) \quad ,$$

but:

- $\Delta x \sim \left(\hat{b} + \hat{b}^{\dagger}\right)$;
- $P \sim I \sim \hat{c}_A^{\dagger}\hat{c}_A$;

Finally:

$$\hat{H}_{int} = \hbar g \hat{c}_A^{\dagger}\hat{c}_A \left(\hat{b} + \hat{b}^{\dagger}\right)$$

# Hamiltonian: semi-quantitative derivation

The interaction between mirror and photon consists in a displacement of the former, and therefore in a **variation of the photon frequency**:

$$\hat{H}_{int} = \hbar\,(\delta\omega_c)\,\hat{c}_A^\dagger \hat{c}_A$$

The variation of photon frequency is:

$$\delta\omega_c = \frac{\partial\omega_c}{\partial L}\delta L = -\frac{\omega_c}{L}\delta L$$

where we have used $\omega_c = \frac{2\pi c}{L}$.

Finally:

$$\hat{H}_{int} = \hbar g \hat{c}_A^\dagger \hat{c}_A \left(\hat{b} + \hat{b}^\dagger\right)$$

[S.Mancini, V.I. Man'ko and P.Tombesi, *Ponderomotive control of quantum macroscopic coherence* (1997)]

# Coherent state

We define coherent state the **eigenstates of the destruction operator**. A coherent state has the following form:

$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

We list some of the most important properties of a coherent state:

- the ground state of a driven harmonic oscillator is a coherent state;
- in a coherent state we have $\Delta x \Delta p = \frac{\hbar}{2}$;
- the temporal evolution of coherent state is a coherent state;
- in configuration space a coherent state is represented by a gaussian.

According to the last three properties **coherent states are considered as the "most classical" quantum states**.

# Temporal Evolution

The initial state of the system is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B\Big)|\beta\rangle \ ,$$

Employing the following time evolution operator:

$$\begin{aligned} e^{-\frac{i}{\hbar}\hat{H}t} =& e^{-ik^2\left(\hat{c}_A^\dagger \hat{c}_A\right)^2\left(e^{-i\Omega t}+i\Omega t-1\right)} e^{-k\hat{c}_A^\dagger \hat{c}_A\left(e^{-i\Omega t}-1\right)\hat{b}^\dagger} \cdot \\ & e^{-k\hat{c}_A^\dagger \hat{c}_A\left(e^{-i\Omega t}-1\right)\hat{b}} e^{-i\Omega \hat{b}^\dagger \hat{b} t} e^{-i\omega_c \hat{c}_A^\dagger \hat{c}_A t} e^{-i\omega_c \hat{c}_B^\dagger \hat{c}_B t} \ , \end{aligned}$$

We obtain the state at a generic instant:

$$\begin{aligned} |\psi(t)\rangle =& \frac{1}{\sqrt{2}} e^{ik^2\theta(t)} e^{-i\mathrm{Im}\left(\alpha^*\beta e^{-i\Omega t}\right)}|1\rangle_A|0\rangle_B|\beta e^{-i\Omega t} + \alpha(t)\rangle \\ & + \frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B|\beta e^{-i\Omega t}\rangle \ , \end{aligned}$$

# Where do Poisson brackets come from?

We write the inner product of $\mathcal{H}$ in oscillator representation:

$$\langle\psi|\phi\rangle = \frac{1}{2\hbar}\sum_{i,j=1}^{N}\left(X_{\psi,i} - iP_{\psi,i}\right)\left(X_{\phi,j} + iP_{\phi,j}\right)\langle i|j\rangle =$$

$$\equiv \frac{1}{2\hbar}\left[\bar{\psi}^T \hat{G}_{\mathcal{M}}\bar{\phi} + i\bar{\psi}^T\hat{\Omega}_{\mathcal{M}}\bar{\phi}\right] \quad ,$$

where:

$$\hat{G}_{\mathcal{M}} \equiv \begin{pmatrix} \hat{\mathbb{I}}_N & \hat{0} \\ \hat{0} & \hat{\mathbb{I}}_N \end{pmatrix} \quad , \quad \hat{\Omega}_{\mathcal{M}} \equiv \begin{pmatrix} \hat{0} & \hat{\mathbb{I}}_N \\ -\hat{\mathbb{I}}_N & \hat{0} \end{pmatrix} \quad .$$

Accordingly, we introduce Poisson bracket as:

$$\{f,g\}_{QM} = \sum_{i,j}\Omega_{ij}\frac{\partial f}{\partial \bar{X}_i}\frac{\partial f}{\partial \bar{X}_j} \quad ,$$

in which $\bar{X}_i = (X_1, ..., X_N, P_1, ..., P_N)$.

# Hybrid ensemble

We describe a **quantum-classical hybrid ensemble** by a normalized, real-valued, positive semi-definite, possibly time-dependent regular function on $X$:

$$\rho\left(x_j, p_j, X_i, P_i\right) = \frac{1}{2\hbar} \sum_{n,m=1}^{N} \rho_{mn}\left(X_m - iP_m\right)\left(X_n + iP_n\right) \quad ,$$

where $\rho_{mn}\left(x_j, p_j\right) = \langle m|\hat{\rho}\left(x_j, p_j\right)|n\rangle$.

The temporal evolution of this function is governed by the equation:

$$\frac{\partial \rho}{\partial t} = \left\{\rho, H\left(x_j, p_j, X_i, P_i\right)\right\}_X \quad ,$$

which retains the form of the Liouville equation.

# Equation of Motion

Full hybrid Hamiltonian allows to obtain the **equations of motion** which encode the dynamical information of the system:

$$\frac{\partial x}{\partial t} = \{x, H\}_X = \frac{p}{M} \quad , \tag{1}$$

$$\frac{\partial p}{\partial t} = \{p, H\}_X = -M\omega_m^2 x + \frac{\tilde{g}}{2}\left(X_A^2 + P_A^2\right) \quad , \tag{2}$$

$$\frac{\partial X_A}{\partial t} = \{X_A, H\}_X = \omega_c P_A - \tilde{g} x P_A \quad , \tag{3}$$

$$\frac{\partial P_A}{\partial t} = \{P_A, H\}_X = -\omega_c X_A + \tilde{g} x X_A \quad , \tag{4}$$

$$\frac{\partial X_B}{\partial t} = \{X_B, H\}_X = \omega_c P_B \quad , \tag{5}$$

$$\frac{\partial P_B}{\partial t} = \{P_B, H\}_X = -\omega_c X_B \quad . \tag{6}$$

# Solution of equations

The equations of motion can be solved **analytically**:

$$x\left(t\right) = A_m \sin\left[\Omega t + \phi_m\right] - \frac{\hbar\tilde{g}}{2M\omega_m^2} \quad , \tag{7}$$

$$p\left(t\right) = \Omega M A_m \cos\left[\Omega t + \phi_m\right] \quad , \tag{8}$$

$$X_A\left(t\right) = \sqrt{\hbar}\cos\{\Omega t + \frac{A_m\tilde{g}}{\Omega}\left[\cos\left(\Omega t + \phi_m\right) - \cos\left[\phi_m\right]\right]\} \quad , \tag{9}$$

$$P_A\left(t\right) = -\sqrt{\hbar}\sin\{\Omega t + \frac{A_m\tilde{g}}{\Omega}\left[\cos\left(\Omega t + \phi_m\right) - \cos\left[\phi_m\right]\right]\} \quad , \tag{10}$$

$$X_B\left(t\right) = \sqrt{\hbar}\cos\left(\omega_c t\right) \quad , \tag{11}$$

$$P_B\left(t\right) = -\sqrt{\hbar}\sin\left(\omega_c t\right) \quad , \tag{12}$$

# Boundary Conditions

The constants $A_m$ and $\phi_m$ are related to the initial conditions of the mirror.

If we define:

$$x\left(t=0\right)=A_m\sin\left(\phi_m\right)-\frac{\hbar\tilde{g}}{2M\omega_m^2}\equiv x_0 \quad ,$$

and:

$$p\left(t=0\right)=A_m\Omega M\cos\left(\phi_m\right)\equiv p_0 \quad .$$

we have:

$$A_m\left(x_0,p_0\right)=\frac{p_0}{\Omega M}\frac{1}{\cos\left(\arctan\left[\Omega M\frac{x_0}{p_0}+\frac{\hbar\tilde{g}}{2p_0\Omega}\right]\right)} \quad ,$$

and:

$$\phi_m\left(x_0,p_0\right)=\arctan\left[\frac{1}{p_0}\left(\Omega M x_0+\frac{\hbar\tilde{g}}{2\Omega}\right)\right] \quad .$$

# Decoherence timescales

The decrease of the interference is quantified by the **decoherence timescale**:

$$t_{HD} = \sqrt{\frac{2\hbar\beta}{\Omega k^2}} \quad \text{with} \quad k = \frac{\hbar\omega_c^2}{2ML^2\omega_m^3} \quad .$$

In the following table we list estimates of $t_{HD}$ for different values of the temperature:

| Temperature [K] | $t_{HD}$ [sec] | $t_{QM}$ [sec] |
|---|---|---|
| $10^{-3}$ | $2,15 \cdot 10^{-6}$ | $3,01 \cdot 10^{-6}$ |
| $10^{-4}$ | $0,68 \cdot 10^{-5}$ | $0,98 \cdot 10^{-5}$ |
| $10^{-5}$ | $2,15 \cdot 10^{-5}$ | $3,13 \cdot 10^{-5}$ |
| $10^{-6}$ | $0,68 \cdot 10^{-4}$ | $0,96 \cdot 10^{-4}$ |

(13)

where $t_{QM} = \frac{1}{k\Omega\sqrt{\bar{n}+1}}$.