CP REDUCED DYNAMICAL MAPS INITIAL CORRELATIONS AND COMMUTATIVITY

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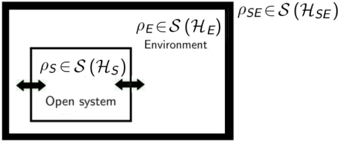
Open quantum systems



$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$$

$$\rho_{SE} \in \mathcal{S}(\mathcal{H}_{SE}), \quad \rho_S = \mathrm{Tr}_E \rho_{SE}$$

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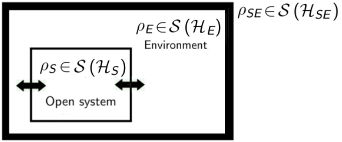

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Composite system ρ_{SE}

$$\frac{d}{dt}\rho_{SE}(t) = -\frac{i}{\hbar}[H_{SE}, \rho_{SE}(t)] \quad \Rightarrow \quad \rho_{SE}(t) = U_{SE}(t)\, \rho_{SE}\, U_{SE}^{\dagger}(t)$$

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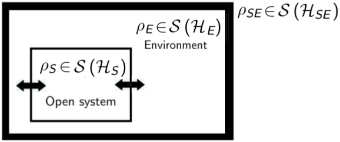

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Open system $\rho_S = \mathrm{Tr}_E\, \rho_{SE}$

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$$\frac{d}{dt}\rho_S(t) = -\frac{i}{\hbar}\mathrm{Tr}_E([H_{SE}, \rho_{SE}(t)]) \quad \Rightarrow \quad \rho_S(t) = \mathrm{Tr}_E \left[U_{SE}(t)\,\rho_{SE}\,U_{SE}^{\dagger} \right]$$

$$\rightarrow \Phi(t)\, \rho_S(0)$$

[H.-P. Breuer, "The theory of open quantum systems"]

Properties of Φ

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2. $\mathrm{Tr}\,\Phi\,[T] = \mathrm{Tr}\ T$
3. Φ linear on $\mathcal{T}\,(\mathcal{H}_S)$

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$$\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \quad \text{CP} \quad \Longleftrightarrow \quad \Phi \otimes I_n : \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^n) \to \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^n) \quad \text{P} \ \forall n$$

$$\rho \otimes \sigma \mapsto \Phi[\rho] \otimes \sigma$$

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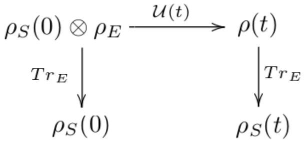
$\Phi, \Psi\ \ CP \quad \Rightarrow \quad \Phi \circ \Psi\ \ CP$

$\Phi : A \to B$ if $A \vee B$ commutative $\quad \Rightarrow \quad \mathrm{CP} \equiv \mathrm{P}$

[Stinespring, Proceedings AMS 1955; Takesaki 2002]

Factorized initial states: CP dynamics

$$\rho_S(t) = \mathrm{Tr}_E\left[U_{SE}(t)\,\rho_{SE}(0)\,U^{\dagger}_{SE}(t)\right] \qquad \longrightarrow$$



$$\rho_S(t) \stackrel{?}{=} \Phi(t)\,\rho_S(0)$$

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$$\rho_S(t) \stackrel{?}{=} \Phi(t)\,\rho_S(0) \qquad \Rightarrow \qquad \rho_S(0) \to \rho_{SE}(0)$$

Assignement map $\quad \mathcal{A}_{\rho_E} : \rho_S(0) \mapsto \rho_S(0)\otimes\rho_E, \;\; \mathrm{Tr}_E \circ \mathcal{A}_{\rho_E} = I_{\mathcal{T}(\mathcal{H}_S)},$ CP

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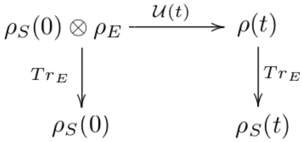

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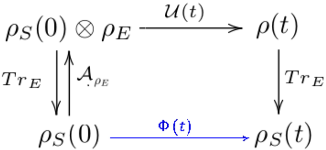
Dynamical map

$$\Phi(t) = \mathrm{Tr}_E \circ \mathcal{U}_{SE}(t) \circ \mathcal{A}_{\rho_E}$$
$$\Phi(0) = I_{\mathcal{T}(\mathcal{H}_S)}$$
$$\mathrm{Tr}_E,\; \mathcal{U}_{SE}(t) \;\; \mathrm{CP} \Rightarrow \Phi(t) \;\; \mathrm{CP}$$



Factorized initial states: Kraus form

$\sum_\gamma |\gamma\rangle\langle\gamma| = I_{\mathcal{H}_E}$, $\rho_E = \sum_\alpha \lambda_\alpha |\alpha\rangle\langle\alpha|$

$$\begin{aligned}\rho_S(t) = \Phi(t)\,\rho_S(0) &= \mathrm{Tr}_E\left[U_{SE}(t)\,\rho_S(0)\otimes\rho_E(0)\,U_{SE}^\dagger(t)\right]\\ &= \sum_\gamma \langle\gamma|\,U_{SE}(t)\,\rho_S(0)\otimes\left(\sum_\alpha \lambda_\alpha|\alpha\rangle\langle\alpha|\right)U_{SE}^\dagger(t)\,|\gamma\rangle\\ &= \sum_{\gamma,\alpha}\underbrace{\sqrt{\lambda_\alpha}\,\langle\gamma|\,U_{SE}(t)\,|\alpha\rangle}_{M_{\gamma,\alpha}(t)}\,\rho_S(0)\,\underbrace{\sqrt{\lambda_\alpha}\,\langle\alpha|\,U_{SE}^\dagger(t)\,|\gamma\rangle}_{M_{\gamma,\alpha}^\dagger(t)}\end{aligned}$$

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Kraus theorem (linear maps)

$$\Lambda[T] = \sum_l M_l T M_l^\dagger,\ \sum_l M_l^\dagger M_l = I_{\mathcal{T}(\mathcal{H})} \quad \Longleftrightarrow \quad \Lambda\ \mathrm{CPT}$$

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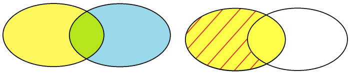
We now introduce initial correlations

Factorized states $\rightarrow$ Classically correlated states $\rightarrow$ ...

Quantum Discord

Classical setting: mutual information

$$I(p_{AB}) = H(p_A) + H(p_B) - H(p_{AB})$$
$$J(p_{AB}) = H(p_A) - H\left(p_{A|B}\right)$$



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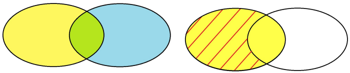

Quantum setting $\quad p_{AB} \to \rho_{AB},\ p_A \to \mathrm{Tr}_B\, \rho_{AB},\ H \to S,$

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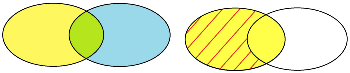

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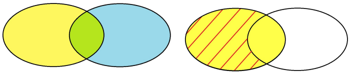

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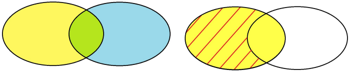

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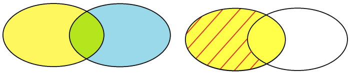

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- Classically correlated $\subset$ Separable states $\subset \mathcal{S}(\mathcal{H}_{AB})$

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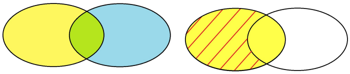

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- Classically correlated $\subset$ Separable states $\subset \mathcal{S}\left(\mathcal{H}_{AB}\right)$
- CP reduced dynamical maps.. not unique

[H. Ollivier, W. H. Zurek, PRL **88**, 017901 (2001)]

Zero quantum discord initial states: Kraus form

$\rho_{SE}(0) = \sum_i p_i \Pi_S^i \otimes \rho_E^i$, fixed $\left\{\Pi_S^i\right\}_i$, fixed $\left\{\rho_E^i = \sum_\alpha \lambda_\alpha^i \left|\alpha^i\right\rangle \left\langle\alpha^i\right|\right\}_i$

[C.A. Rodrìguez-Rosario *et al.*, J. Phys. A: Math. Theor. **41**, 205301 (2008)]

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$\mathcal{A}^{1,2}_{\{\rho_E^i\}} : \sum_i p_i \Pi_S^i \mapsto \sum_i p_i \Pi_S^i \otimes \rho_E^i$ CP

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$\mathcal{A}^{1,2}_{\{\rho_E^i\}} : \sum_i p_i \Pi_S^i \mapsto \sum_i p_i \Pi_S^i \otimes \rho_E^i$ CP $\Rightarrow$ $\Phi_{1,2} = \mathrm{Tr}_E \circ \mathcal{U}_{SE} \circ \mathcal{A}^{1,2}_{\{\rho_E^i\}}$ CP

$$\rho_S(t) = \Phi(t)\rho_S(0) = \mathrm{Tr}_E\left[U_{SE}(t) \sum_i \overbrace{p_i \Pi_S^i}^{\to \rho_S} \otimes \rho_E^i U_{SE}^\dagger(t)\right]$$

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$$\mathcal{A}^{1,2}_{\{\rho_E^i\}} : \sum_i p_i \Pi_S^i \mapsto \sum_i p_i \Pi_S^i \otimes \rho_E^i \quad \text{CP} \Rightarrow \quad \Phi_{1,2} = \mathrm{Tr}_E \circ \mathcal{U}_{SE} \circ \mathcal{A}^{1,2}_{\{\rho_E^i\}} \quad \text{CP}$$

$$\rho_S(t) = \Phi(t)\rho_S(0) = \mathrm{Tr}_E\left[U_{SE}(t)\sum_i \overbrace{p_i \Pi_S^i}^{\to \rho_S} \otimes \rho_E^i U_{SE}^\dagger(t)\right] \qquad \Pi_S^i \Pi_S^j = \delta_{ij}\Pi_S^j$$

$$\Phi_1 \rho_S = \sum_{\gamma,\alpha,j} M^j_{\gamma,\alpha} \rho_S {M^j_{\gamma,\alpha}}^\dagger \qquad \Phi_2 \rho_S = \sum_{\gamma,\alpha} M_{\gamma,\alpha} \rho_S M^\dagger_{\gamma,\alpha}$$

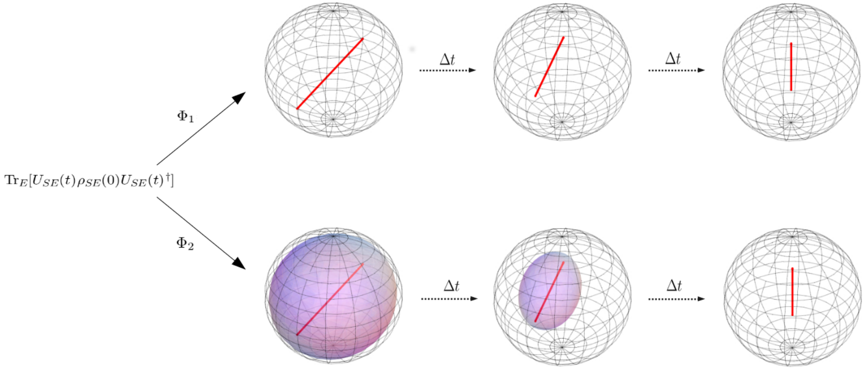
$$M^j_{\gamma,\alpha} = \sqrt{\lambda^j_\alpha}\langle\gamma| U_{SE}(t)|\alpha^j\rangle \Pi_S^j \qquad M_{\gamma,\alpha} = \sum_j \sqrt{\lambda^j_\alpha}\langle\gamma| U_{SE}(t)|\alpha^j\rangle \Pi_S^j$$

$$\forall \rho_S \in \mathcal{C}(\mathcal{H}_S) \qquad \Phi_1(t)\rho_S = \Phi_2(t)\rho_S$$

Zero quantum discord initial states: map action extension

Once we have a Kraus form, we can extend its action $\mathcal{C}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$

$\Rightarrow \Phi_1 \neq \Phi_2$



$$\Phi_{1,2}(t \to 0) = I_{\mathcal{C}(\mathcal{H}_S)} \quad \Leftarrow \quad \mathrm{Tr}_E \circ \mathcal{A}^{1,2}_{\{\rho^i_E\}} = I_{\mathcal{C}(\mathcal{H}_S)}$$

Commutativity of the compatibility domain

Commutative *compatibility domain*

$$\mathcal{C}\left(\mathcal{H}_S\right) = \left\{\rho_S = \sum_i p_i \sigma_S^i \mid [\sigma_S^i, \sigma_S^j] = 0\right\} \qquad \left\{\sigma_S^i\right\}_i = \left\{\Pi_S^1, \ldots, \Pi_S^{n-1}, W\right\}$$

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Crucial QM $\quad W = \sum_{i=n}^{\dim \mathcal{H}_S} w_i \Pi_S^i = \overbrace{\sum_{k=1}^{r} \mu_k \left|\psi_k\right\rangle \left\langle\psi_k\right|}^{\text{MANY!}} \quad r > \dim \mathcal{H}_S - (n-1)$

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Crucial QM

$$W = \sum_{i=n}^{\dim\mathcal{H}_S} w_i \Pi_S^i = \overbrace{\sum_{k=1}^{r} \mu_k \left|\psi_k\right\rangle \left\langle\psi_k\right|}^{\text{MANY!}} \qquad r > \dim\mathcal{H}_S - (n-1)$$

$$\mathcal{C}^I\left(\mathcal{H}_{SE}\right) = \left\{\rho_{SE} = \sum_{i=1}^{n-1} p_i \Pi_S^i \otimes \rho_E^i + p_n \sum_{i=n}^{\dim\mathcal{H}_S} w_i \Pi_S^i \otimes \rho_E^i\right\}$$

$$\Downarrow$$

Zero quantum discord states

$$\mathcal{C}^{II}\left(\mathcal{H}_{SE}\right) = \left\{\rho_{SE} = \sum_{i=1}^{n-1} p_i \Pi_S^i \otimes \rho_E^i + p_n \sum_{k=1}^{r} \mu_k \left|\psi_k\right\rangle \left\langle\psi_k\right| \otimes \varrho_E^k\right\}$$

Commutativity of the compatibility domain

Commutative *compatibility domain*

$$\mathcal{C}(\mathcal{H}_S) = \{\rho_S = \sum_i p_i \sigma_S^i \mid [\sigma_S^i, \sigma_S^j] = 0\} \qquad \{\sigma_S^i\}_i = \{\Pi_S^1, \ldots, \Pi_S^{n-1}, W\}$$

Crucial QM

$$W = \sum_{i=n}^{\dim\mathcal{H}_S} w_i \Pi_S^i = \overbrace{\sum_{k=1}^{r} \mu_k |\psi_k\rangle\langle\psi_k|}^{\text{MANY!}} \qquad r > \dim\mathcal{H}_S - (n-1)$$

$$\mathcal{C}^I(\mathcal{H}_{SE}) = \{\rho_{SE} = \sum_{i=1}^{n-1} p_i \Pi_S^i \otimes \rho_E^i + p_n \sum_{i=n}^{\dim\mathcal{H}_S} w_i \Pi_S^i \otimes \rho_E^i\}$$

$$\Downarrow$$

Zero quantum discord states

$$\mathcal{C}^{II}(\mathcal{H}_{SE}) = \{\rho_{SE} = \sum_{i=1}^{n-1} p_i \Pi_S^i \otimes \rho_E^i + p_n \sum_{k=1}^{r} \mu_k |\psi_k\rangle\langle\psi_k| \otimes \varrho_E^k\}$$

$$\Downarrow$$

States with quantum correlations

CP maps on commutative compatibility domain

- $\rho_S(0) \in \mathcal{C}(\mathcal{H}_S) \rightarrow \rho_{SE}(0) \in \mathcal{C}''(\mathcal{H}_{SE})$ P
- $\mathcal{C}(\mathcal{H}_S)$ commutative

CP maps on commutative compatibility domain

- $\rho_S(0) \in \mathcal{C}(\mathcal{H}_S) \rightarrow \rho_{SE}(0) \in \mathcal{C}^{II}(\mathcal{H}_{SE})$ P
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$\Rightarrow$ P $\equiv$ CP

CP maps on commutative compatibility domain

- $\rho_S(0) \in \mathcal{C}(\mathcal{H}_S) \to \rho_{SE}(0) \in \mathcal{C}^{II}(\mathcal{H}_{SE})$ P
- $\mathcal{C}(\mathcal{H}_S)$ commutative

$\Rightarrow$ P $\equiv$ CP

$$\rho_S(0) \to \rho_S(t) = \mathrm{Tr}_E\left[U_{SE}\ \underbrace{\rho_{SE}(0)}_{\in \mathcal{C}^{II}(\mathcal{H}_{SE})}\ U_{SE}^{\dagger}\right] \quad \mathrm{CP}$$

CP maps on commutative compatibility domain

- $\rho_S(0) \in \mathcal{C}(\mathcal{H}_S) \to \rho_{SE}(0) \in \mathcal{C}^{II}(\mathcal{H}_{SE})$ P
- $\mathcal{C}(\mathcal{H}_S)$ commutative

$\Rightarrow$ P $\equiv$ CP

$$\rho_S(0) \to \rho_S(t) = \mathrm{Tr}_E\left[U_{SE}\underbrace{\rho_{SE}(0)}_{\in\mathcal{C}^{II}(\mathcal{H}_{SE})}U_{SE}^{\dagger}\right] \quad \text{CP}$$

Thanks to the GHJW theorem

[Gisin, Helv. Phys. Acta 1989; Hughston, Jozsa and Wootters PLA 1993]

$$\rho_S(0) = \sum_{i=1}^{n-1}\Pi_S^i\rho_S(0)\Pi_S^i + \sum_{k=1}^{r}\sum_{j=n}^{\dim\mathcal{H}_S}K_{jk}\rho_S(0)K_{jk}^{\dagger}$$

And can display a Kraus form

$$\begin{aligned}\rho_S(t) &= \Phi^{II}(t)\,\rho_S(0)\\ &= \sum_{i=1}^{n-1}\sum_{\gamma,\alpha}M^i_{\gamma\alpha}(t)\rho_S(0)M^i_{\gamma\alpha}(t)^{\dagger} + \sum_{k=1}^{r}\sum_{j=n}^{\dim\mathcal{H}_S}\sum_{\gamma,\beta}M^{jk}_{\gamma\beta}(t)\rho_S(0)M^{jk}_{\gamma\beta}(t)^{\dagger}\end{aligned}$$

Summary

Summing up

- Simple approach to obtain CP maps from class of correlated states
- Include both zero and non zero quantum discord states
- Non uniqueness of definition of the map
- Dynamical meaning only on commutativity subset

Many (many!) thanks to

B. Vacchini (Unimi)

Reference

B. Vacchini, G. Amato, *Scientific Reports* **6**, 37328 (2016)