Longitudinal beam dynamics

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Longitudinal beam dynamics: terminology

- **The beam will be described with reference to a synchronous particle that follows a particular space-time trajectory. The 'space' trajectory is the central orbit of the transverse motion and the 'time' trajectory is defined by initial conditions.**
- **When crossing an RF cavity, the synchronous particle receives a kick in momentum ($\Delta_s p$). Non-synchronous particles receive slightly different kicks ($\Delta_s p + \Delta p$).**
- **The motion of the non-synchronous particles is then expressed in terms of how much they lead or lag (Δs) the synchronous particle in their flight through the lattice and by how much they deviate from the synchronous particle in momentum ($\Delta p/p$).**
- **Δs-$\Delta p/p$ defines the longitudinal phase space.**
- **A large number of particles concentrated around a synchronous particle are referred to as bunch.**
- **Without longitudinal focusing, a bunch will progressively spread out.**

Longitudinal beam dynamics: terminology

- **A focusing region in longitudinal phase-space around the synchronous particle is known as an RF bucket.**
- **A stationary RF bucket is one that does not alter the momentum of the synchronous particle ($\Delta_s p=0$), but does modify the momenta of the non-synchronous particles ($\Delta p \neq 0$).**
- **An accelerating bucket applies a positive momentum kick to the synchronous particle ($\Delta_s p>0$).**
- **RF cavities are usually configured to bring non-synchronous particles closer to the synchronous one.**
- **In Linacs, this is called longitudinal focusing.**
- **In a ring, it is called phase stability.**

Path length and velocity

- **The variable Δs has two components:**

$$s = vt \Rightarrow \Delta s = v\Delta t + t\Delta v = \Delta s_{path} + \Delta s_{velocity}$$

- **The first term is the geometric difference in path length, given by the velocity of the reference particle multiplied by the extra time taken by the given particle to traverse the element. The second term is the distance due to the difference in velocity between the given particle and the synchronous one applied for the time needed for the reference particle to traverse the element.**
- **If the change in path length compensates the effect of the velocity difference (i.e. $\Delta s_{path}=-\Delta s_{velocity}$), so that $\Delta s = 0$, the transit time is the same for particles of all momenta and the lattice is known as an isochronous lattice.**

Path length and velocity

- **Drift spaces, quadrupoles, multipoles and solenoids are considered to have the same geometric path length to first order for all momenta, so Δs_{path} is zero in these cases.**
- **$\Delta s_{velocity}$ is derived from the basic relativistic expression, $p_0=m_0\gamma\beta c$ by differentiation:**

$$\frac{\Delta p}{p_0}=\gamma^2\frac{\Delta\beta}{\beta}\Rightarrow\Delta v=\frac{v}{\gamma^2}\frac{\Delta p}{p_0}\Rightarrow\Delta s=\boxed{\Delta s_{velocity}=t\Delta v=\frac{l}{\gamma^2}\frac{\Delta p}{p_0}}$$

NB:

$$p_0=m_0c\frac{\beta}{\sqrt{1-\beta^2}}\Rightarrow dp_0=m_0c\frac{d\beta}{\left(1-\beta^2\right)^{3/2}}=m_0c\gamma^3d\beta$$

$$\frac{dp_0}{p_0}=m_0c\gamma^3d\beta/m_0c\beta\gamma$$

term used in 6x6 transfer matrices

6D transfer matrix for non bending elements

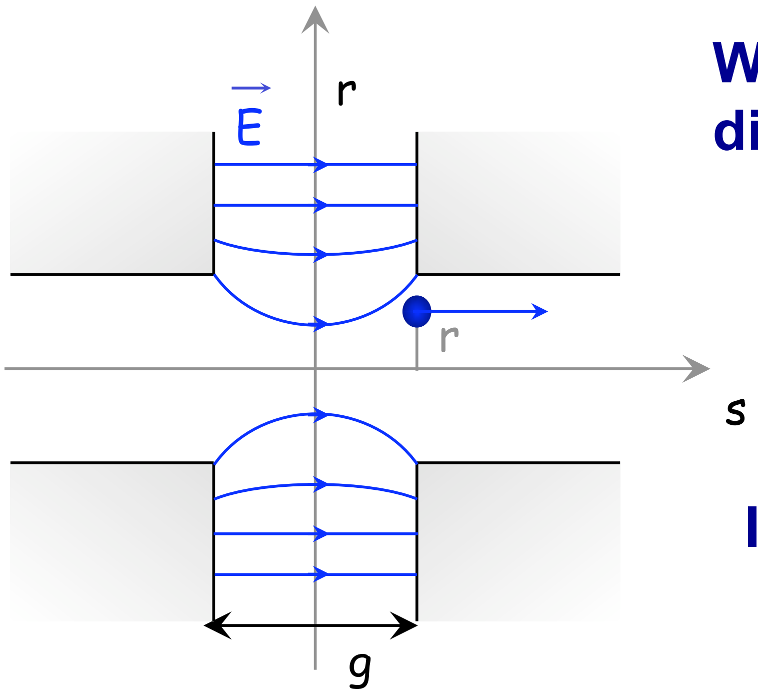
Transverse coupling off-axis
sub-matrices

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \ell/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \\ \Delta s \\ \Delta p / p_0 \end{bmatrix}$$

Uncoupled transverse
sub-matrices

Longitudinal sub-matrix

EM fields in RF devices



When a particle crosses the gap g at a distance r, its energy gain is

$$\Delta E = q\int_{-g/2}^{g/2} \vec{E}(s,r,t)\,d\vec{s}$$

In the cavity gap the electric field is

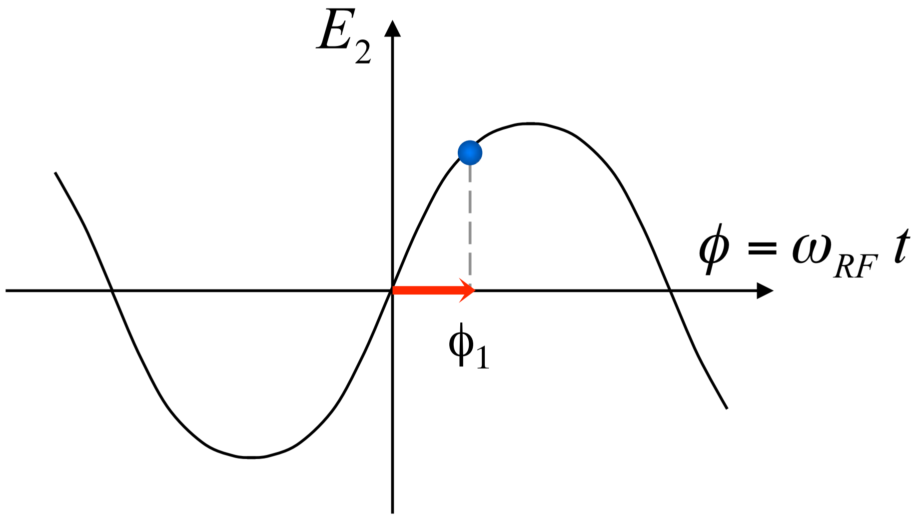
$$E(s,r,t) = E(s,r)\,g(t)$$

In general g(t) is a sinusoidal function

$$g(t) = \sin\Phi(t) \qquad \Phi(t) = \int_{t_0}^{t} \omega_{rf}\,dt + \Phi_0$$

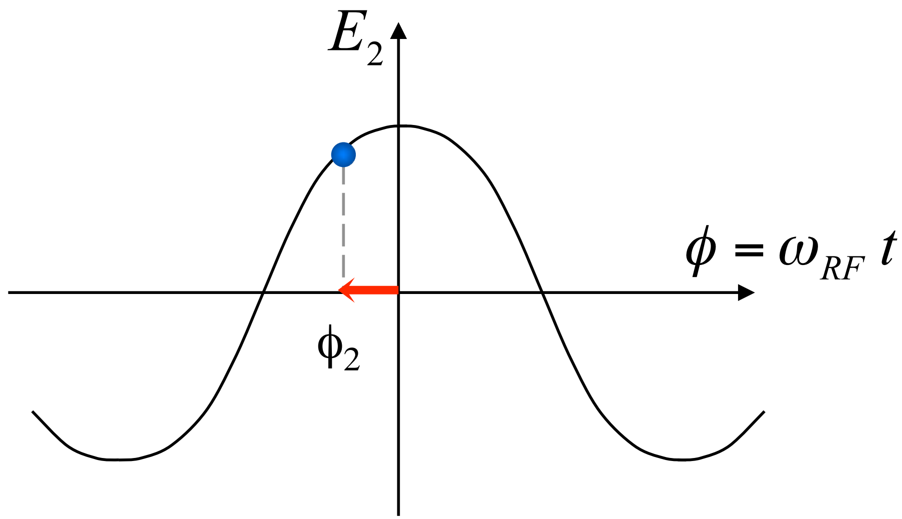
EM fields in RF devices: convention

For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope



$$g(t) = \sin\Phi(t)$$

For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

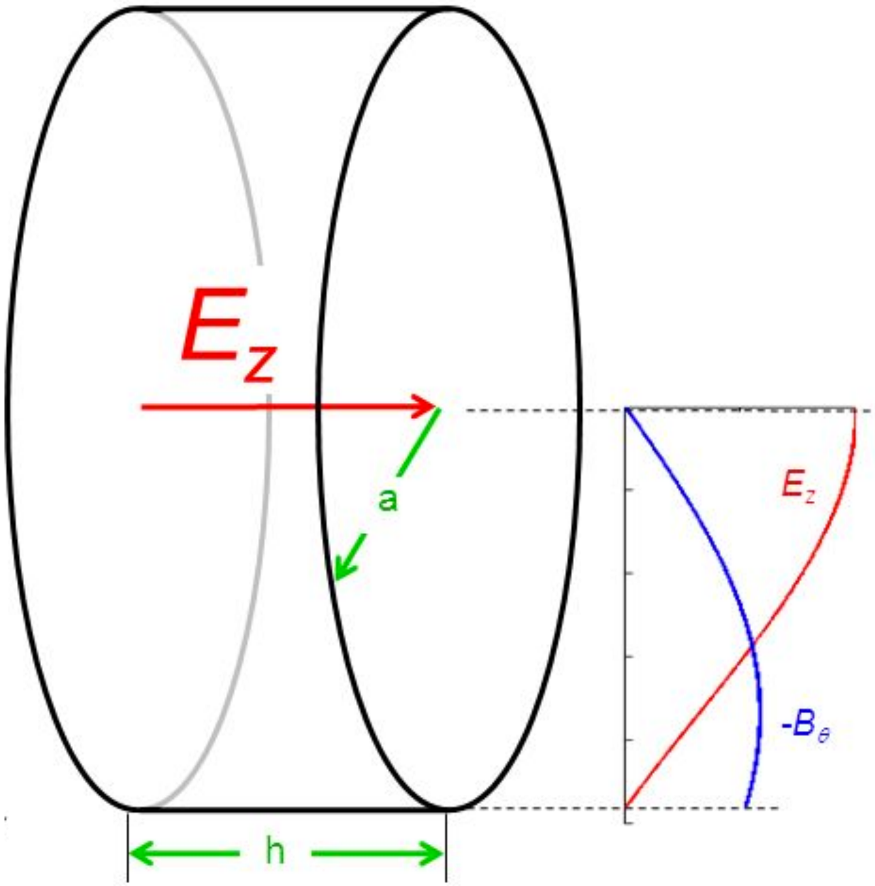


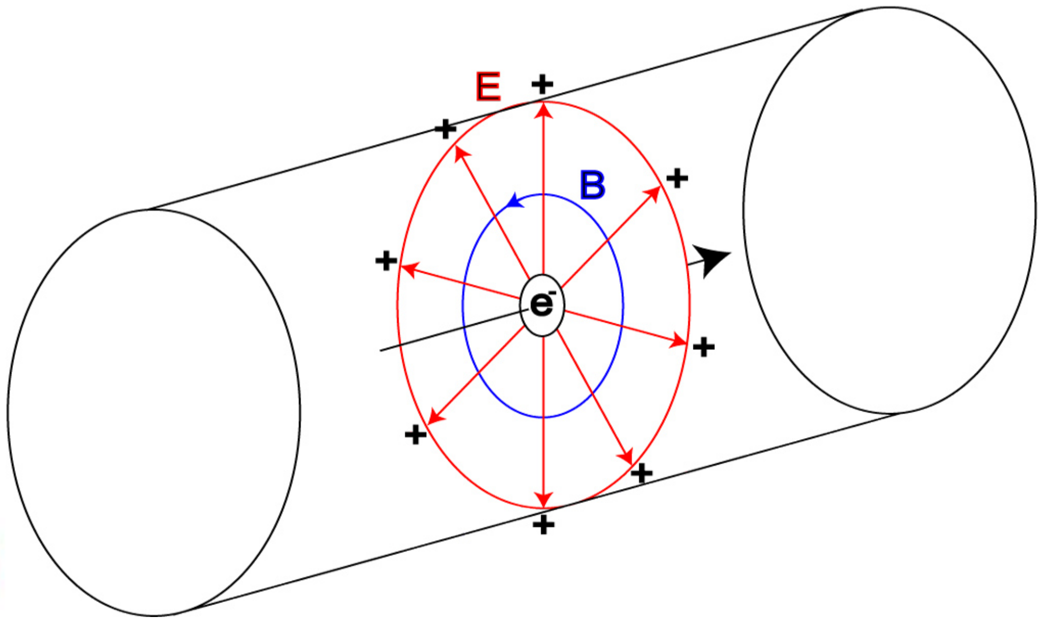
$$g(t) = \cos\Phi(t)$$

EM fields in RF devices: TM010 modes

- The most common accelerating structures in standing wave structures have rotational symmetry excited by a TM010 mode.
- The term TM (transverse magnetic) indicates that magnetic fields are normal to the longitudinal direction.
- The other class of modes, TE, have longitudinal components of B, and E_z=0.
- The first number in the subscript is the azimuthal mode number: it is zero for azimuthally symmetric modes.
- The second number is the radial mode number. The radial mode number minus one is the number of nodes in the radial variation of E_z.
- The third number is the longitudinal mode number. It is zero if E_z is constant in the z direction: in the TM010 mode, only the E_r, E_z and B_θ components are non-zero.

EM fields in RF devices: TM010 modes



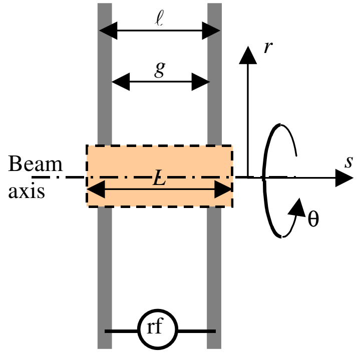


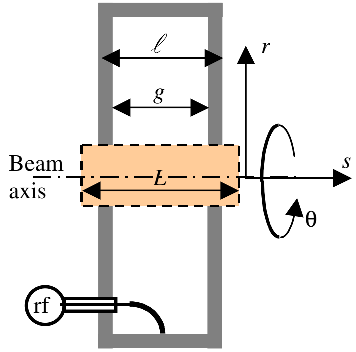
$$\vec{B} = B_\theta \hat{\theta}$$
$$\vec{E} = E_z \hat{z} + E_r \hat{r}$$

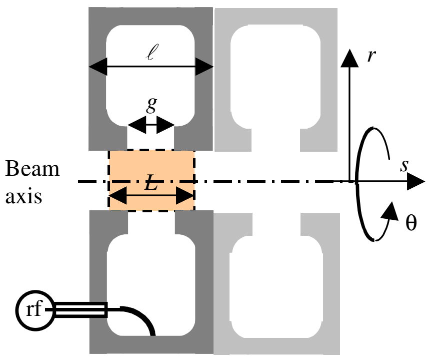
EM fields in RF devices: TM010 modes

- The wavenumber and frequency of TM0N0 modes depends only on R_0, not g. This is not generally true for other types of modes.
- TM0N0 modes are optimal for particle acceleration. The longitudinal electric field is uniform along the propagation direction of the beam and its magnitude is maximum on axis.
- The transverse magnetic field is zero on axis; this is important for electron acceleration where transverse magnetic fields could deflect the beam.
- Whether the standing-wave structure is called a gap, a cavity, or a tank with drift tubes depends on the external geometry.
- The basic modules can be used individually or in a periodic array operating in the so-called π-mode in which the fields of adjacent cells are π out of phase, or the 2π-mode for drift tubes in a tank.

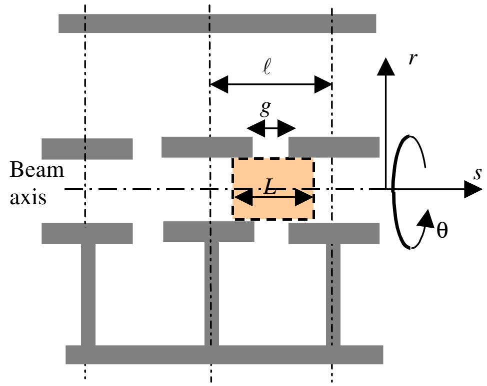
RF standing waves structures





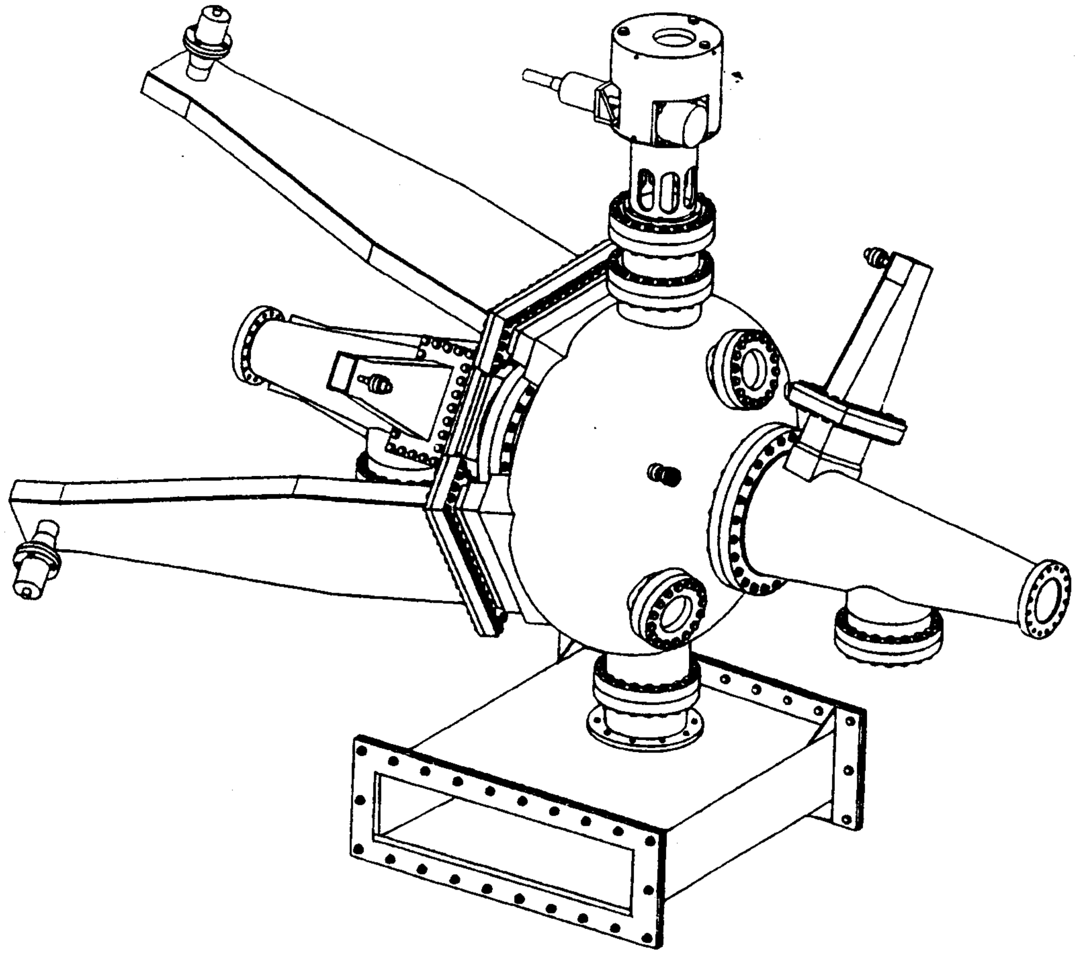


Cavities with noses



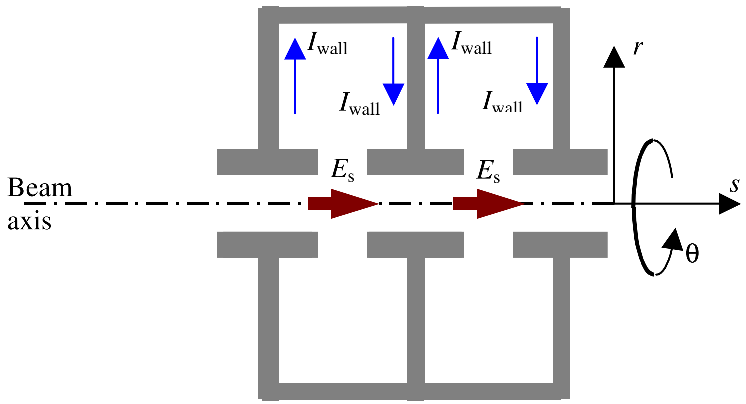
Drift tubes in a tank

DAΦNE MAIN RING CAVITY



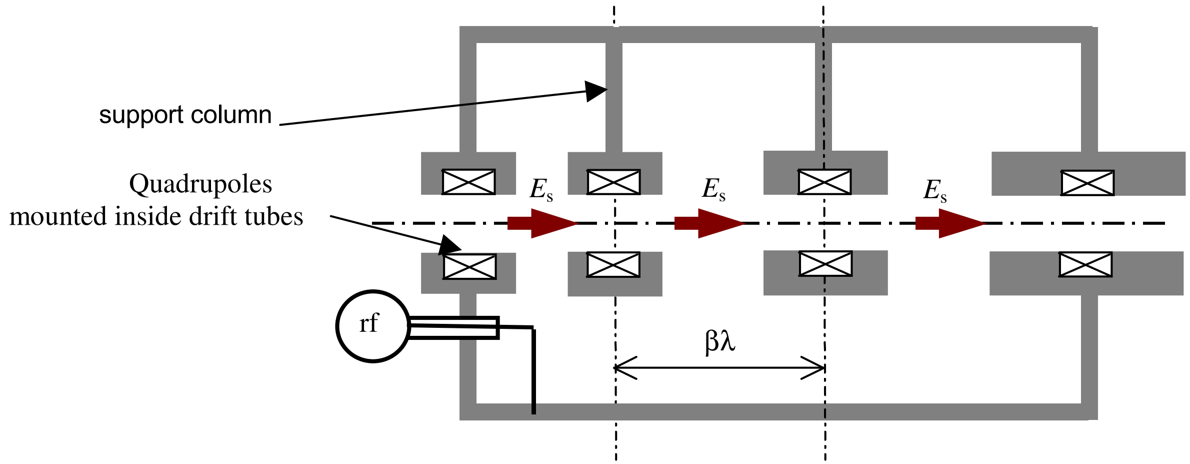
Alvarez structure (non-relativistic)

- **Start with a series of cavities with 'noses' or drift tubes and excite all cavities in phase (2π mode). Note that wall currents cancel.**



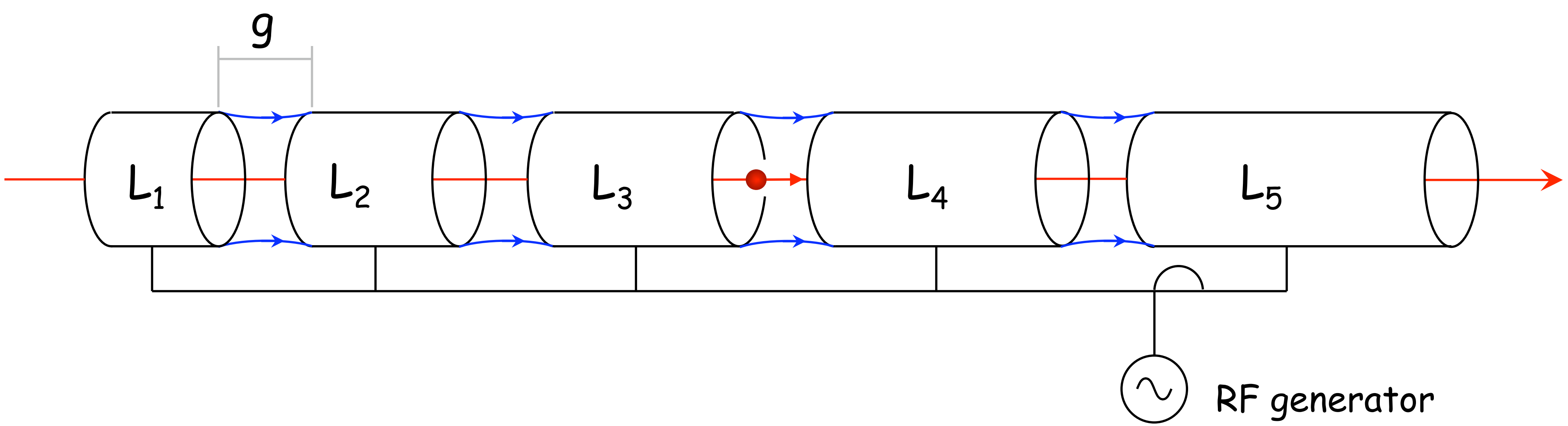
- **As wall currents cancel, remove walls except for a support column for the drift tubes, then adjust the drift tube lengths for the velocity.**

- **Note that there are quadrupoles lodged inside the drift tubes for additional focusing.**



Alvarez structure: proton and ion LINACS

- **Used for protons and ions (50 – 200 MeV, f ~ 200 MHz)**



- **Synchronism condition (g<<L) is such that the time the particle takes to cross a drift tube has to be the RF period:**

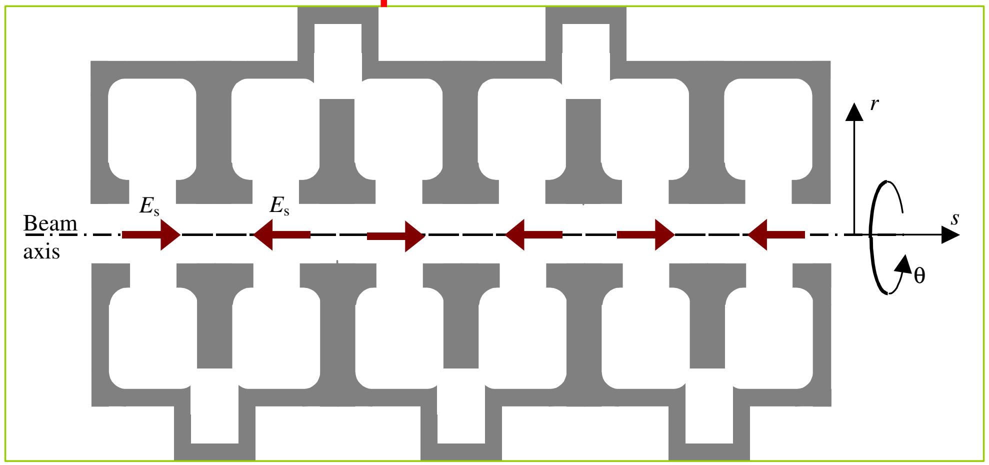
$$\boxed{L = v_s T_{RF} = \beta_s \lambda_{RF}}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

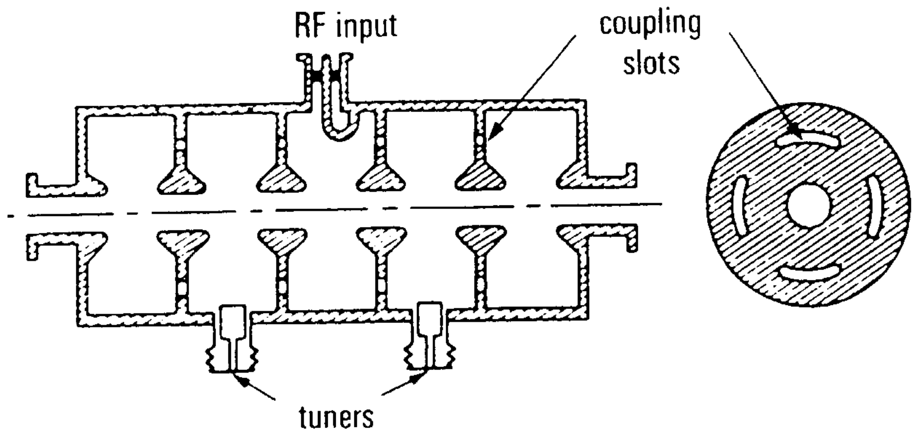
Standing wave structures

- **The beam velocity is virtually that of light, so the cavities are identical. These LINACS are used for electrons.**

side coupled structure



nose-cone structure

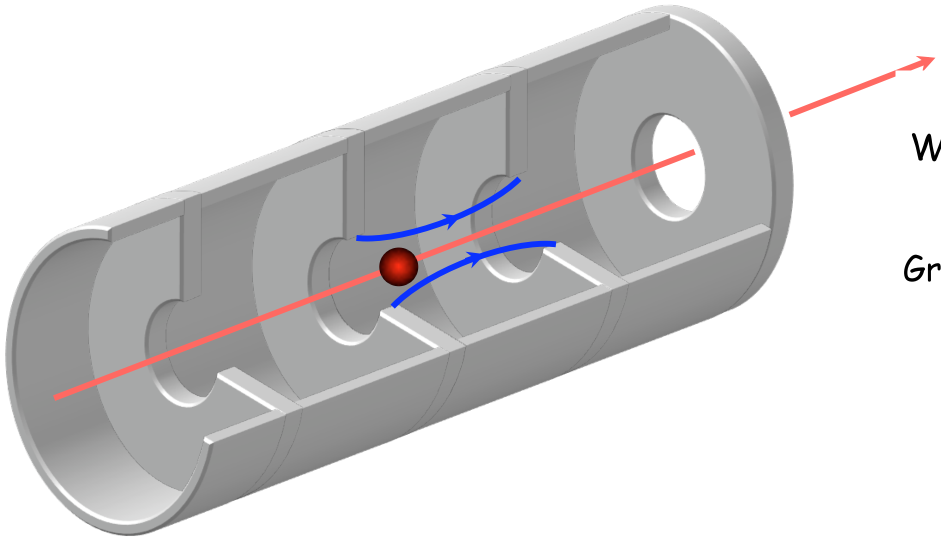


- **The cavities are coupled to be excited in the π-mode. This saves having an RF source for each cavity and synchronising them.**

Travelling wave LINAC

- **In a resonant structure the standing wave pattern can be expanded into two travelling waves, a forward one synchronous with the particle and a backward one which has no mean effect on the particle energy.**
- **However TM modes (with an electric field in the direction of propagation) in rectangular or cylindrical guides have phase velocities bigger than c. Then it is necessary to bring the phase velocity of the forward wave at the level of the particle velocity ($v_{ph} \sim c$) and to do so the simplest method consists of loading the structure with disks: the size of the holes determines the degree of coupling and so determines the relative phase shift from one cavity to the next. When the dimensions have been tailored correctly the phase changes from cavity to cavity along the accelerator to give an overall phase velocity corresponding to the particle velocity.**

Travelling wave LINAC



$$E(z,t) = E_0 \, e^{i(\omega t - kz)}$$ Electric field

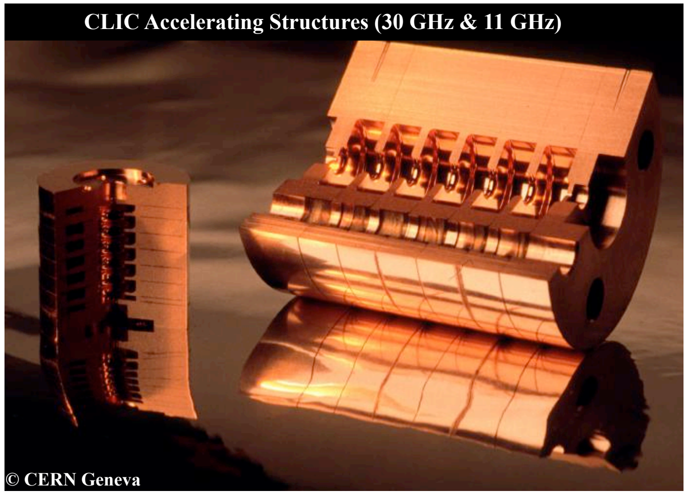
Wave number $k = \dfrac{2\pi}{\lambda_{RF}}$

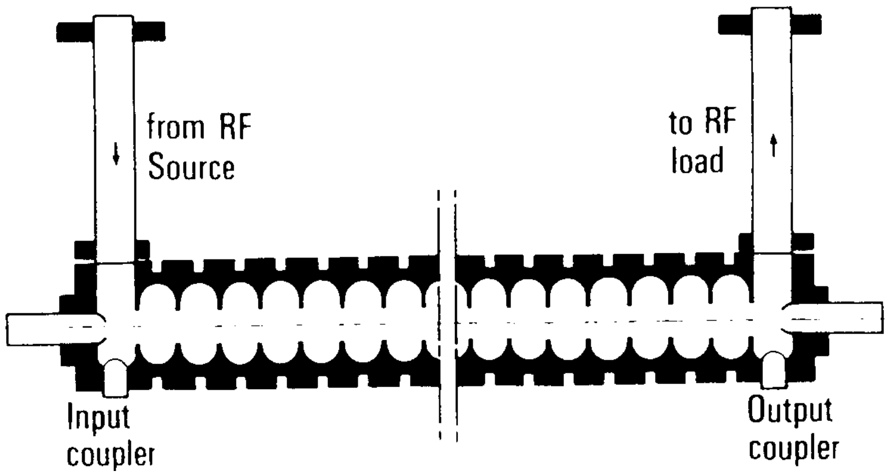
Phase velocity $v_{ph} = \dfrac{\omega}{k}$

Group velocity $v_g = \dfrac{d\omega}{dk}$

synchronism:

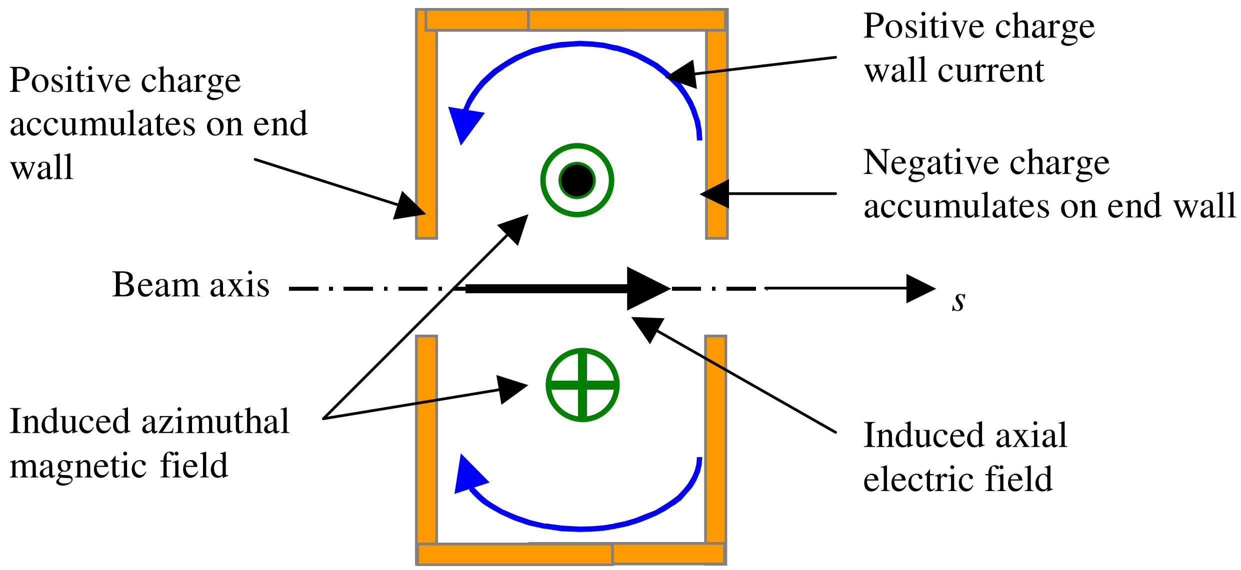
$$\boxed{v_{el} \approx c = \frac{\omega}{k} = v_{ph}}$$





Physical description of em fields in a cavity

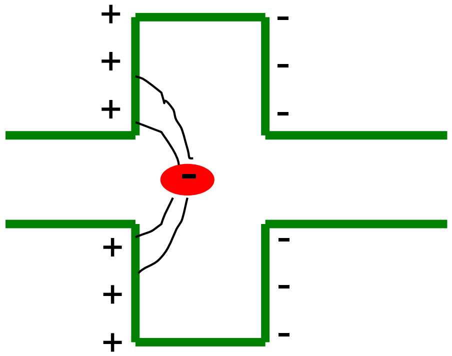
- **When a charge crosses a resonant structure, it excites the fundamental mode and high order modes (HOMs). Each mode can be treated as an electric RLC circuit loaded by an impulsive current.**

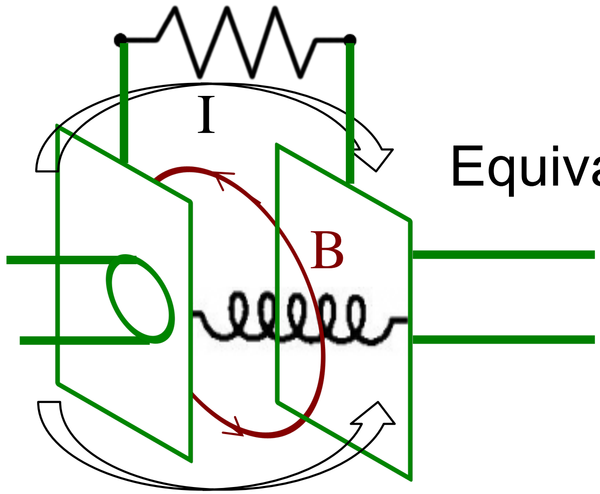


- **Wall currents flow back and forth between the two end plates of the cavity**
- **The current flow supports an azimuthal magnetic field.**
- **The charge accumulation on the end plates drives an electric field acting on the beam.**
- **To relate the azimuthal magnetic field to the induced axial electric field use Faraday's law.**

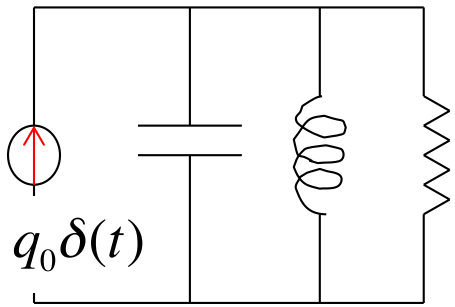
$$\nabla \times E = -\partial B/\partial t$$

Physical description of em fields in a cavity

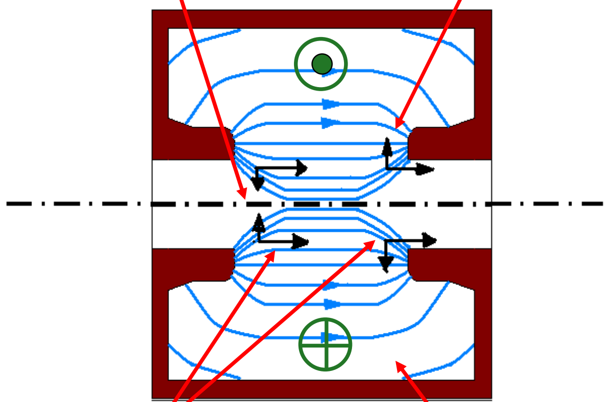




Equivalent circuit



$$I_b(t) = q_0 \delta(t)$$



Nose-cone cavity

- The ‘noses’ shield the incoming particle from the axial field.
- As the particle enters the gap, the axial field rises with a cosine-like form.
- The radial focusing at the entry slightly exceeds the defocusing at the exit because the ion has a higher energy and is stiffer.

Transit time factor

$$E(s,r,t) = E(s,r)g(t)$$ **simplified model**

$$\boxed{\begin{aligned} E(s,r) &= \frac{V_{rf}}{g} = const \\ g(t) &= \sin\left(\omega_{rf} t + \Phi_0\right) \end{aligned}}$$

At t=0 and s=0, with v≠0 parallel to the axis (s=vt), the energy gain is

$$\Delta E = q\int_{-g/2}^{g/2} \vec{E}(s,r,t)\,d\vec{s} = \frac{qV_{rf}}{g}\int_{-g/2}^{g/2} \sin\left(\omega_{rf}\frac{s}{v} + \Phi_0\right)ds = \frac{qV_{rf}}{g}\sin\Phi_0 \int_{-g/2}^{g/2}\cos\left(\omega_{rf}\frac{s}{v}\right)ds$$

$$= \frac{2qV_{rf}}{\omega_{rf} g / v}\sin\left(\frac{\omega_{rf} g}{2v}\right)\sin\Phi_0 = qV_{rf}T_t\sin\Phi_0$$

$$T_t = \frac{\sin\left(\frac{\omega_{rf} g}{2v}\right)}{\frac{\omega_{rf} g}{2v}}$$

transit time factor

T_t<1 and $T_t \to 1$ if $g \to 0$

Transit time factor

In the general case, the transit time factor is given by

$$T_t = \frac{\int_{-\infty}^{\infty} E(s,r)\cos\left(\omega_{rf}\frac{s}{v}\right)ds}{\int_{-\infty}^{\infty} E(s,r)\,ds}$$

It is defined as the ratio of the peak energy gained by a particle with velocity v to the peak energy gained by a particle with infinite velocity.

The energy gain is therefore $\Delta E = qV_{rf}T_t \sin\Phi_0$

NB: the field distribution with distance in the gap is not constant but close to a cosine so it is possible to improve the approximation.

RF parameters

- The transit time factor is mostly important for LINACS.
- In circular machines it possible to take a very simplified model of the RF cavities.
- The RF period T_{rf} is related to the revolution period T_0 by the harmonic number h because at every turn the particle must see the same voltage $T_0 = hT_{rf}$ $f_{rf} = hf_0$
- In most cases, the time to cross the gap in a ring is very small compared to the RF period, so that the transit time factor is close to unity.
- In this case, the energy gained by the particle is

$$\Delta E = qV_{rf} \sin \Phi_s$$

- where Φ_s is called synchronous phase

Momentum compaction

- **In a circular accelerator a nominal closed orbit is defined for a particle with a nominal momentum p_0.**
- **For a particle with momentum $p_0 + \Delta p$ the trajectory length can be different from the length L_0 of the nominal trajectory due to the different bending radius in the dipoles.**
- **We call ΔL this extra length, and define a new quantity, the momentum compaction, as**

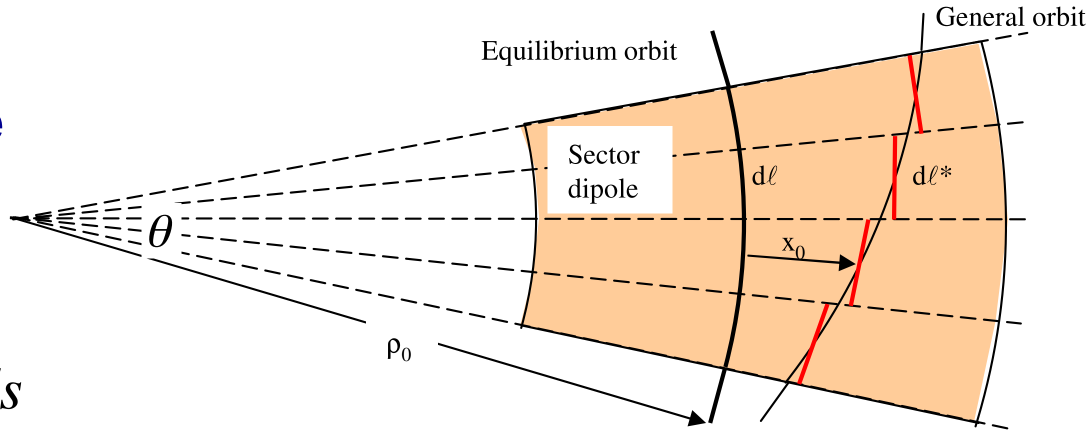
$$\alpha_c = \frac{\Delta L / L_0}{\Delta p / p_0}$$

- **so that**

$$\frac{\Delta L}{L_0} = \alpha_c \frac{\Delta p}{p_0}$$

Momentum compaction

- **Example for a dipole**



$$L_0 = \rho_0 \theta$$

$$dL = (\rho_0 + x_0) d\theta = \frac{(\rho_0 + x_0)}{\rho_0} ds$$

$$L = \int_0^{L_0} \frac{(\rho_0 + x_0)}{\rho_0} ds = L_0 + \int_0^{L_0} \frac{x_0}{\rho_0} ds$$

$$\Delta L = \int_0^{L_0} \frac{x_0}{\rho_0} ds = \frac{\Delta p}{p_0} \int_0^{L_0} \frac{D}{\rho_0} ds$$

by definition of dispersion

$$\frac{\Delta L}{L_0} = \alpha_c \frac{\Delta p}{p_0}$$

if ρ=ρ(s)

$$\alpha_c = \frac{1}{L_0} \int_0^{L_0} \frac{D}{\rho} ds$$

Momentum compaction

- **For a circular machine we can also write**

$$\alpha_c = \frac{\Delta L / L_0}{\Delta p / p_0} = \frac{\Delta R / R_0}{\Delta p / p_0}$$

- **In most circular machines α_c is positive: higher momentum means larger circumference.**
- **This does not necessarily means larger revolution time: higher momentum means also higher velocity and lower revolution time (if β<1)**

$$\omega_{rev} = \frac{2\pi v}{L_0} \qquad d\omega_{rev} = \frac{2\pi}{L_0} dv - \frac{2\pi v}{L_0^2} dL \Rightarrow \frac{d\omega_{rev}}{\omega_{rev}} = \frac{dv}{v} - \frac{dL}{L_0}$$

Slippage factor

$$\frac{d\omega_{rev}}{\omega_{rev}} = \frac{dv}{v} - \frac{dL}{L_0}$$ remember that $$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0}$$

so that, including also the definition of momentum compaction

$$\frac{d\omega_{rev}}{\omega_{rev}} = \frac{df}{f_0} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p_0} = \eta\frac{dp}{p_0}$$

η is called slippage factor and it depends on the beam energy.

For a given machine (with momentum compaction α_c) there is an energy at which η=0: transition energy

Transition energy

$$\eta = \left(\frac{1}{\gamma^2} - \alpha_c \right) = 0 \Rightarrow \frac{1}{\gamma_{tr}^2} = \alpha_c \Rightarrow \gamma_{tr} = \sqrt{\frac{1}{\alpha_c}}$$

- **Below transition, $\gamma<\gamma_{tr}$ ⇒ ($\eta>0$): higher momentum gives higher revolution frequency (velocity is important: proton machines).**
- **Above transition, $\gamma>\gamma_{tr}$ ⇒ ($\eta<0$): higher momentum gives lower revolution frequency (dispersion is important: electron machines).**
- **For LINACS $\alpha_c=0$ ⇒ $\eta>0$ (either protons and electrons).**

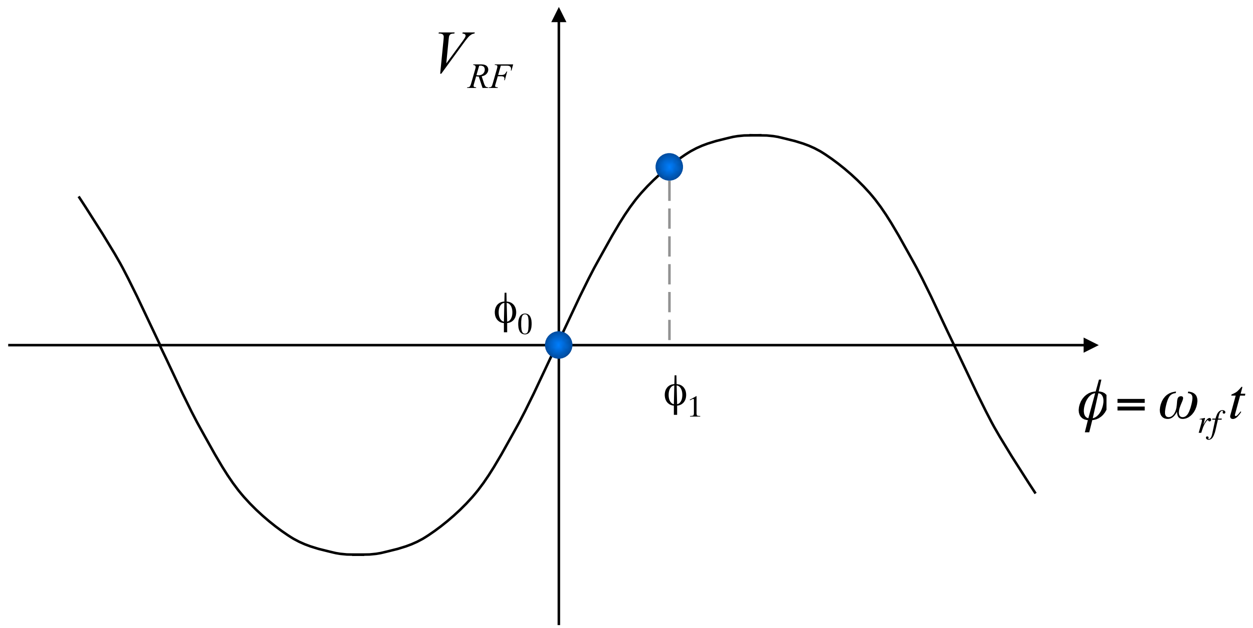
the slippage factor can be also written as $\eta = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2} \right)$

Synchronous particle in storage rings

- **Let's consider a simplified assumption of no acceleration of particles and time constant magnetic fields (circular accelerators, not synchrotrons but storage rings).**
- **A synchronous particle is a particle that, at each turn, sees always the same phase in the RF cavity.**

$$\omega_{rev} = \frac{\omega_{rf}}{h}$$

$$\Delta E = qV_{rf} \sin \Phi_0$$



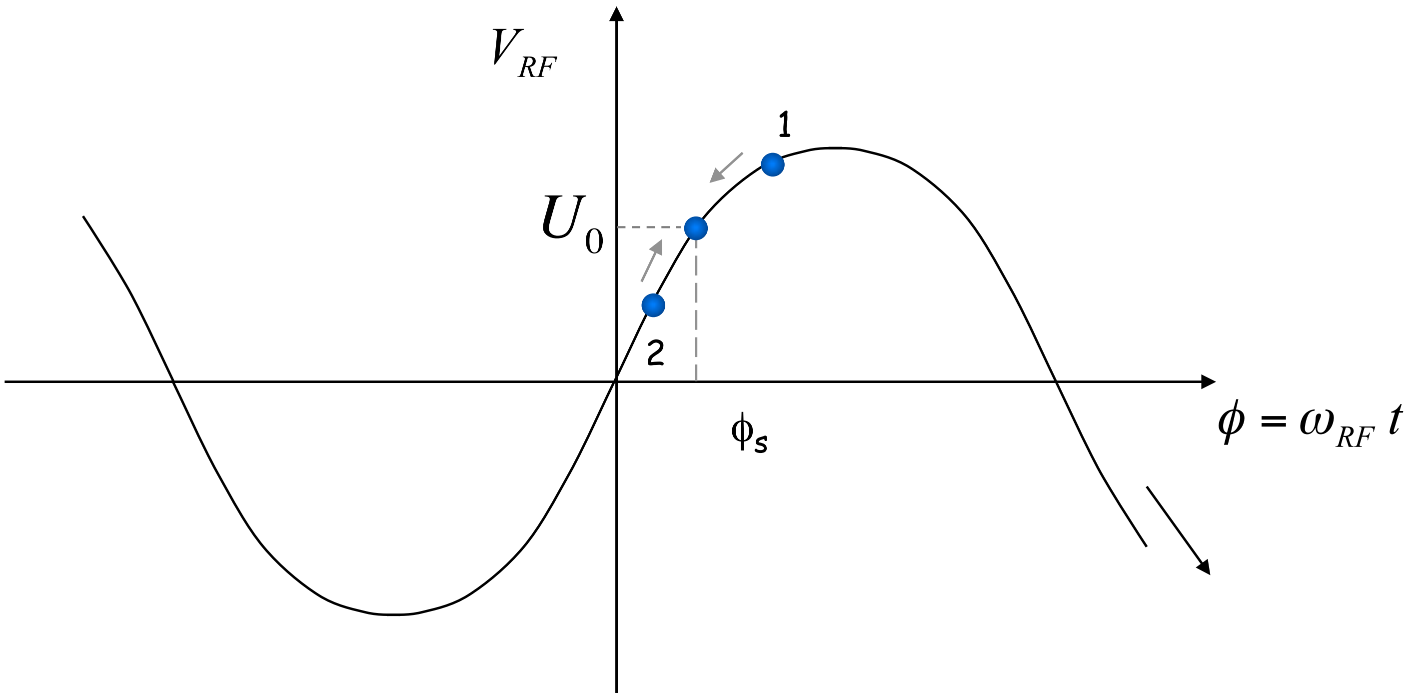
Synchronous particle in storage rings

- **In order to keep the synchronous condition, the particle must keep a constant energy**
- **The phase of the synchronous particle must therefore be $\phi_0 = 0$ if there are no losses in the machine (good approximation for protons).**
- **If there are losses U_0, due, for example, to synchrotron radiation, then**

$$\sin\phi_0 = \frac{U_0}{qV_{rf}}$$

- **There are two values of ϕ_0 satisfying this relation. Let's see that for stability, below transition energy, we must choose the value below $\pi/2$.**

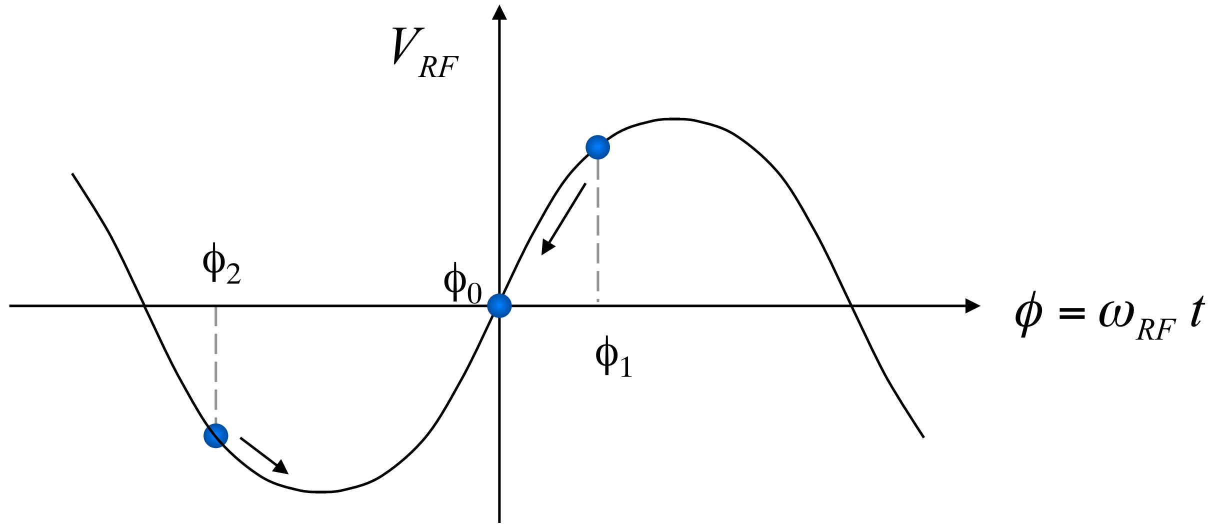
Synchrotron oscillations



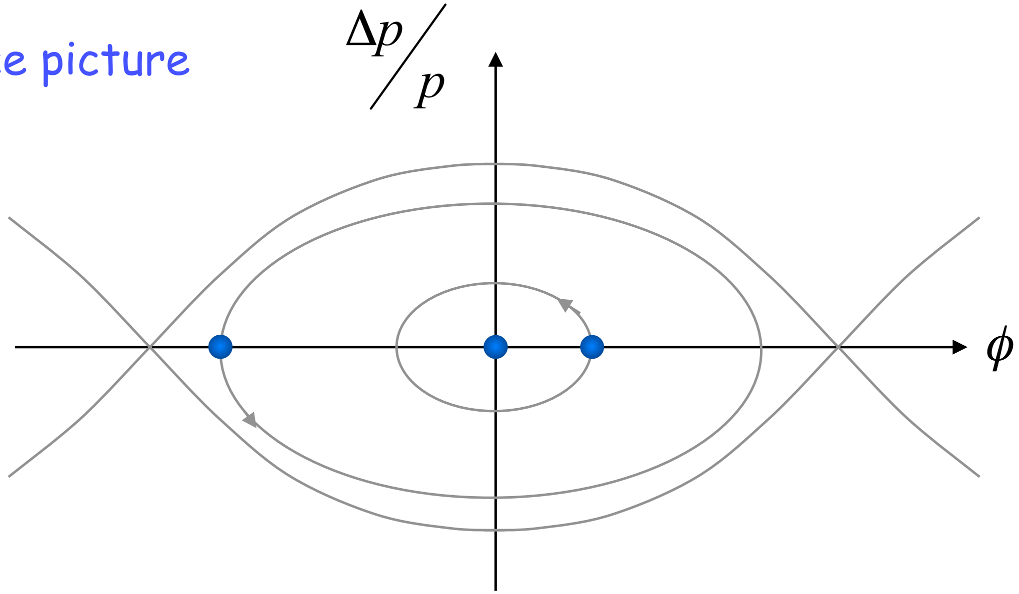
- **A particle that enters in 1 gains energy and it is accelerated.**
- **Below transition its revolution frequency increases.**
- **The particle arrives in the cavity earlier, its phase tends to ϕ_s.**

Synchrotron oscillations

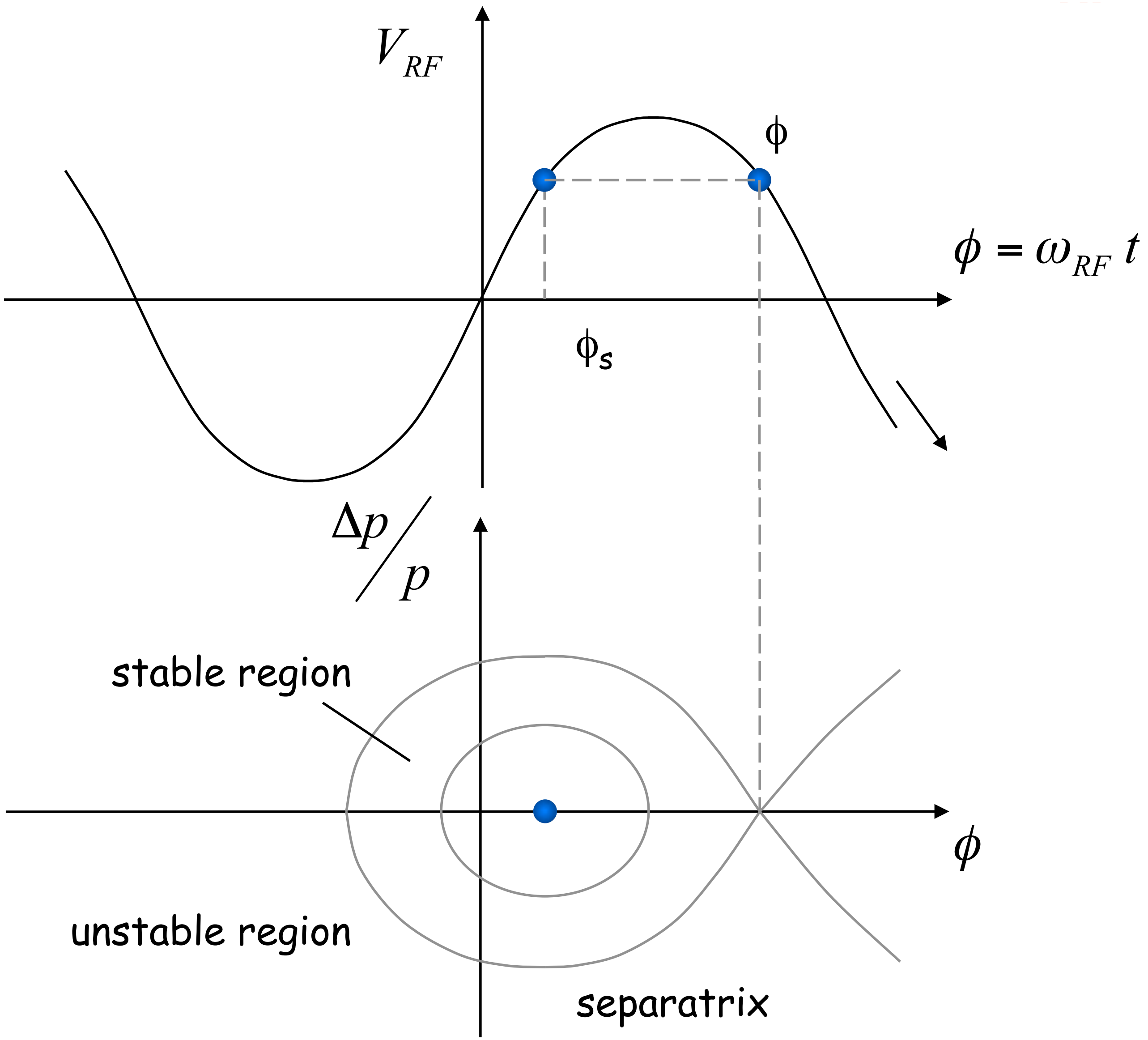
$$U_0 = 0$$



Phase space picture



Synchrotron oscillations: unstable phase



Synchronous particle: acceleration

$$qB\rho = mv = p$$

- **In order to maintain ρ constant, if the particle accelerates, the magnetic field must increase**

$$\frac{dp}{dt} = q\rho\frac{dB}{dt} = q\rho\dot{B} \Longrightarrow (\Delta p)_{turn} = q\rho\dot{B}T_{rev} = q\rho\dot{B}\frac{\overbrace{2\pi R}^{L_0}}{\beta c}$$

- **Δp is related to energy ΔE by the relativistic expression**

$$p = m_0 c\beta\gamma = m_0 c\sqrt{\gamma^2 - 1} \Rightarrow dp = \frac{m_0 c}{\beta}d\gamma = \frac{dE}{\beta c}$$

$$\boxed{(\Delta E)_{turn} = 2\pi R q\rho\dot{B}}$$

Synchronous particle: acceleration

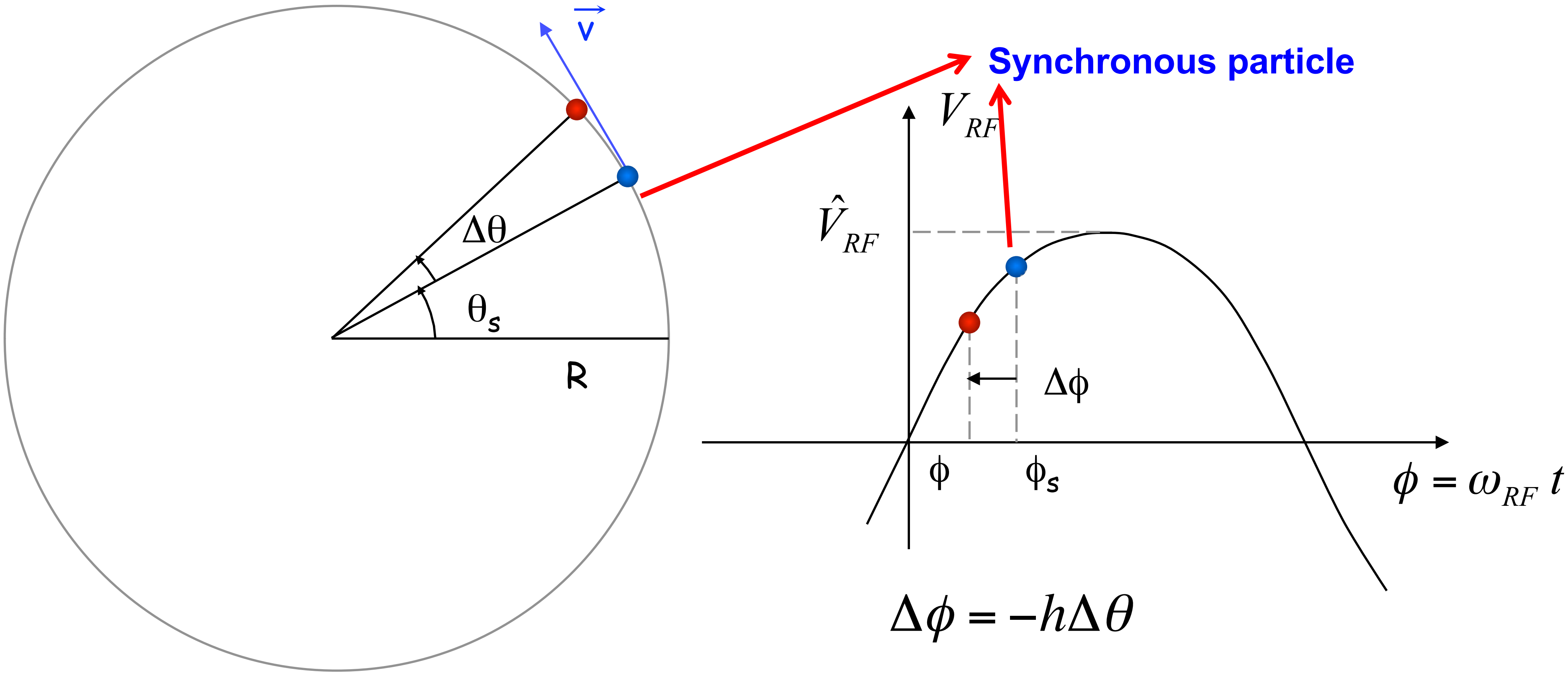
- **ΔE is related to the synchronous phase** $\Delta E = qV_{rf} \sin\phi_S$

$$2\pi R \rho \dot{B} = V_{rf} \sin\phi_s \Longrightarrow \sin\phi_s = \frac{2\pi R \rho \dot{B}}{V_{rf}}$$

- **Also the RF frequency must change with the magnetic field**

$$\left.\begin{aligned} f_{rf} &= hf_0 = \frac{hv}{2\pi R} = \frac{hc\beta}{2\pi R} = \frac{hc}{2\pi R}\sqrt{1-\frac{1}{\gamma^2}} \\ B &= \frac{mv}{q\rho} = \frac{m_0 c\gamma\beta}{q\rho} = \frac{m_0 c}{q\rho}\sqrt{\gamma^2 - 1} \end{aligned}\right\} f_{rf} = \frac{hc}{2\pi R}\frac{q\rho}{m_0 c}B\frac{1}{\sqrt{1+\left(\frac{q\rho}{m_0 c}B\right)^2}}$$

Synchrotron oscillations



$$\Delta\phi = -h\Delta\theta$$

The azimuthal angle θ is related to the azimuthal position by ds = Rdθ. In one revolution this angle varies by 2π while the RF phase varies by 2πh.

The - sign comes from the fact that a particle behind the synchronous particle (Δθ < 0) arrives later in the gap

Synchrotron oscillations

$$\theta = \int_{t_0}^{t} \omega(\tau)\, d\tau$$

constant

$$\Delta\omega = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d}{dt}(\phi - \phi_s) = -\frac{1}{h}\frac{d\phi}{dt}$$

remember $\frac{\Delta\omega}{\omega_0} = \eta \frac{\Delta p}{p_0}$ $\Longrightarrow$ $\Delta p = -\frac{p_0}{h\eta\omega_0}\frac{d\phi}{dt}$

synchronous particle

in energy
($\Delta E = v\Delta p = \omega_0 R_0 \Delta p$) $\Delta E = -\frac{R_0 p_0}{h\eta}\frac{d\phi}{dt}$

NB: remember that ΔE is the energy difference from that of the synchronous particle

$$\boxed{\frac{d\phi}{dt} = -\frac{h\eta}{R_0 p_0}\Delta E}$$

time phase dependence

Synchrotron oscillations

The energy gain for any particle in one turn is $$E_n - E_{n-1} = qV_{rf}\sin\phi$$

For the synchronous particle we have, as well $$E_{n,s} - E_{n-1,s} = qV_{rf}\sin\phi_s$$

$$\Rightarrow \quad \underbrace{\left(E_n - E_{n,s}\right)} - \left(E_{n-1} - E_{n-1,s}\right) = qV_{rf}\left(\sin\phi - \sin\phi_s\right)$$

$$\frac{2\pi}{\omega_0}\underbrace{\frac{1}{T_0}\left(\Delta E_n - \Delta E_{n-1}\right)} = qV_{rf}\left(\sin\phi - \sin\phi_s\right)$$

$$\frac{2\pi}{\omega_0}\frac{d\left(\Delta E\right)}{dt} = qV_{rf}\left(\sin\phi - \sin\phi_s\right)$$

introducing $W = 2\pi\dfrac{\Delta E}{\omega_0}$ $\Rightarrow$ $$\boxed{\frac{dW}{dt} = qV_{rf}\left(\sin\phi - \sin\phi_s\right)}$$

Synchrotron oscillations

The two equations of motion of the non-synchronous particle are then

$$\boxed{\frac{dW}{dt} = qV_{rf}\left(\sin\phi - \sin\phi_s\right)}$$

$$\boxed{\frac{d\phi}{dt} = -\frac{h\eta\omega_0}{2\pi R_0 p_0}W}$$

NB: W and ϕ are canonical variables since the equations of motion can be derived from the Hamiltonian

$$H = eV_{rf}\left[\cos\phi - \cos\phi_s + \left(\phi - \phi_s\right)\sin\phi_s\right] - \frac{h\eta\omega_0}{4\pi R_0 p_0}W^2 \quad \left\{\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial H}{\partial W} \\ \frac{dW}{dt} &= -\frac{\partial H}{\partial \phi}\end{aligned}\right.$$

This Hamiltonian, although legitimate, is inconsistent with the Hamiltonian for transverse betatron oscillations, where *s* is the independent coordinate. To simplify our discussion, we will disregard the inconsistency and study only the synchrotron motion. A fully consistent treatment is needed in the study of synchro-betatron coupling resonances.

Synchrotron oscillations

The second order equation for the phase can be obtained

$$\frac{d}{dt}\left(\frac{R_0 p_0}{h\eta\omega_0}\frac{d\phi}{dt}\right)+\frac{qV_{rf}}{2\pi}\left(\sin\phi-\sin\phi_s\right)=0$$

Let's consider R_0, p_0, ω_0, η and V_{rf} constant or slowly changing with time compared to $\Delta\phi=\phi-\phi_s$

$$\frac{d^2\phi}{dt^2}+\frac{\omega_s^2}{\cos\phi_s}\left(\sin\phi-\sin\phi_s\right)=0$$

with

$$\omega_s^2=\frac{qV_{rf}h\eta\omega_0\cos\phi_s}{2\pi R_0 p_0}=\frac{qV_{rf}h\eta c^2\cos\phi_s}{2\pi R_0^2 E_0}$$

Synchrotron oscillations: small amplitude

$$\sin\phi = \sin(\phi_s + \Delta\phi) = \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi$$

small Δφ $\Rightarrow \sin\phi \cong \sin\phi_s + \Delta\phi\cos\phi_s$

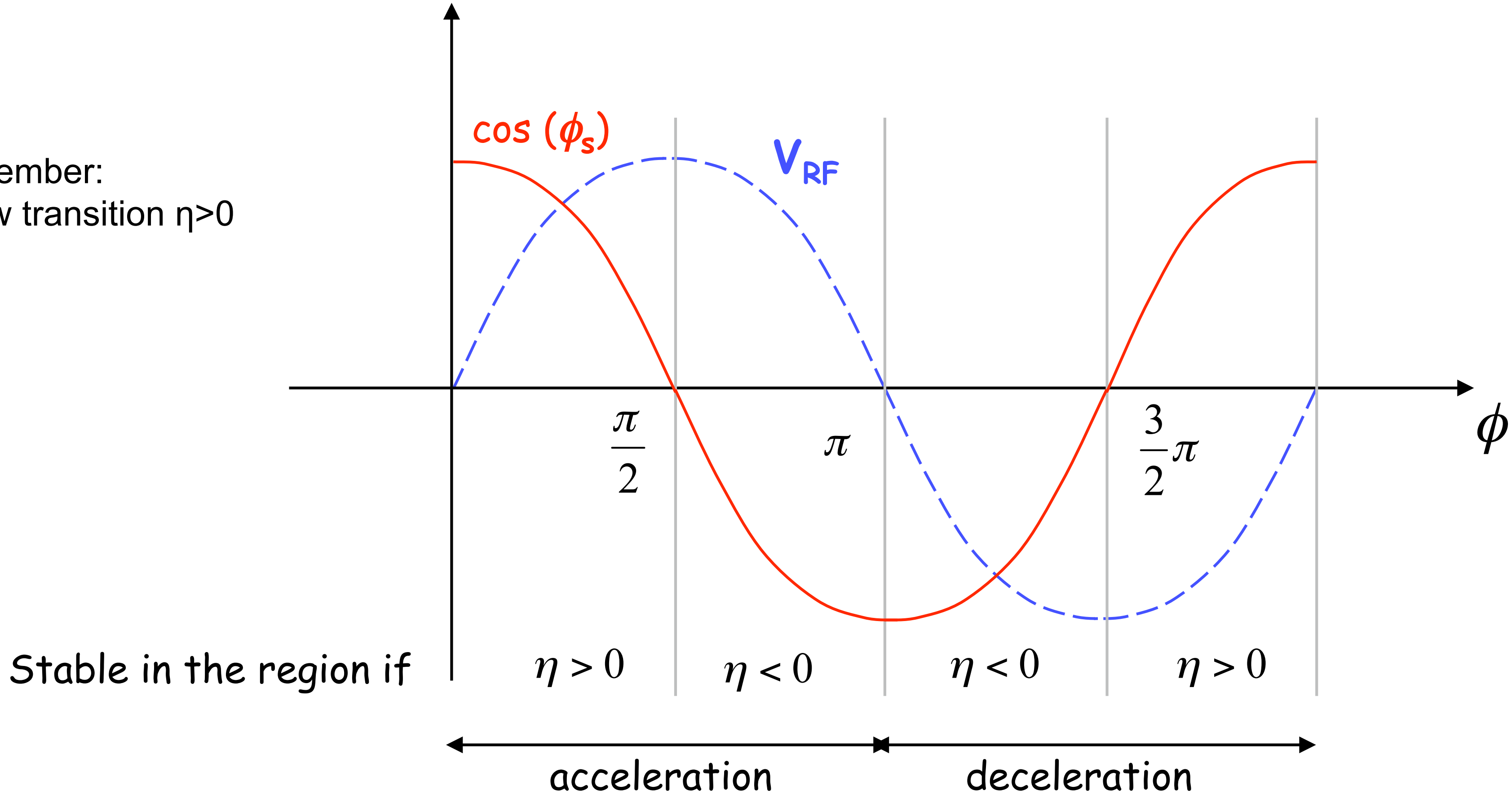
$$\frac{d^2\phi}{dt^2} + \omega_s^2\Delta\phi = 0$$ **or, equivalently** $$\frac{d^2\Delta\phi}{dt^2} + \omega_s^2\Delta\phi = 0$$

which represents the differential equation of an harmonic oscillator with ω_s called synchrotron frequency. It must be real for the stability condition

$$\omega_s^2 = \frac{qV_{rf}h\eta c^2\cos\phi_s}{2\pi R_0^2 E_0} \quad \Rightarrow \quad \eta\cos\phi_s > 0$$

Synchrotron oscillations: stability condition

Remember:
below transition η>0



NB: at transition energy η vanishes, ω_s goes to zero and there is no more phase stability. During acceleration through transition energy, in a proton synchrotron, the RF phase must be switched rapidly from ϕ_s to $\pi - \phi_s$ in order to maintain stability above transition.

Synchrotron oscillations: small amplitude

The solution for small amplitude oscillations is

$$\Delta\phi = \Delta\phi_{\max}\cos\left(\omega_s t + \theta_0\right)$$

and

$$W = -\frac{2\pi R_0 p_0}{h\eta\omega_0}\frac{d\phi}{dt} = \frac{2\pi R_0 p_0}{h\eta\omega_0}\omega_s \Delta\phi_{\max}\sin\left(\omega_s t + \theta_0\right)$$

$$\Delta p = \frac{p_0}{h\eta\omega_0}\omega_s \Delta\phi_{\max}\sin\left(\omega_s t + \theta_0\right) = \Delta p_{\max}\sin\left(\omega_s t + \theta_0\right)$$

the motion is an ellipse (circumference) in the phase space

Synchrotron oscillations: lepton machines

$\beta \cong 1$, γ large ➔ $\eta \cong -\alpha_c$, $\omega_0 \cong c/R_0$

$$\omega_s^2 = -\frac{eV_{rf} h \alpha_c c^2 \cos\phi_s}{2\pi R_0^2 E_0}$$

the synchrotron tune = number of synchrotron oscillations per turn is

$$Q_s = \frac{\omega_s}{\omega_0} = \sqrt{-\frac{eV_{rf} h \alpha_c \cos\phi_s}{2\pi E_0}}$$

The rf frequency does not change

Synchrotron oscillations: large amplitude

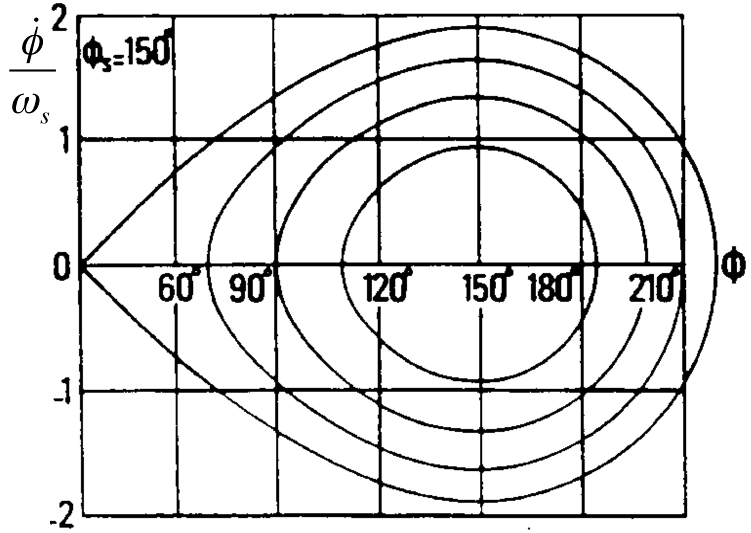
$$\ddot{\phi} + \frac{\omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

multiplying by dφ/dt and integrating

depends on the initial conditions

$$\frac{\dot{\phi}^2}{2} - \frac{\omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = const$$

The small amplitude motions are pure harmonic oscillations which correspond to circles in the frame $\left(\dot{\phi}/\omega_s, \phi\right)$. For larger amplitudes the circles are distorted by the non-linearity of the motion but the curves will still close on themselves.



Remember that Φ_s (150°) is stable, and π-Φ_s unstable

Synchrotron oscillations: large amplitude

Equation of the separatrix: $d\phi/dt=0$ and $\phi=\pi-\phi_s$ (unstable point)

$$\frac{\dot{\phi}^2}{2}-\underbrace{\frac{\omega_s^2}{\cos\phi_s}\left(\cos\phi+\phi\sin\phi_s\right)}_{\text{potential energy U from which the equation of motion can be derived}}=-\frac{\omega_s^2}{\cos\phi_s}\left[\cos\left(\pi-\phi_s\right)+\left(\pi-\phi_s\right)\sin\phi_s\right]$$

potential energy U from which the equation of motion can be derived

$$U=\frac{\omega_s^2}{\cos\phi_s}\left(\cos\phi+\phi\sin\phi_s\right)$$

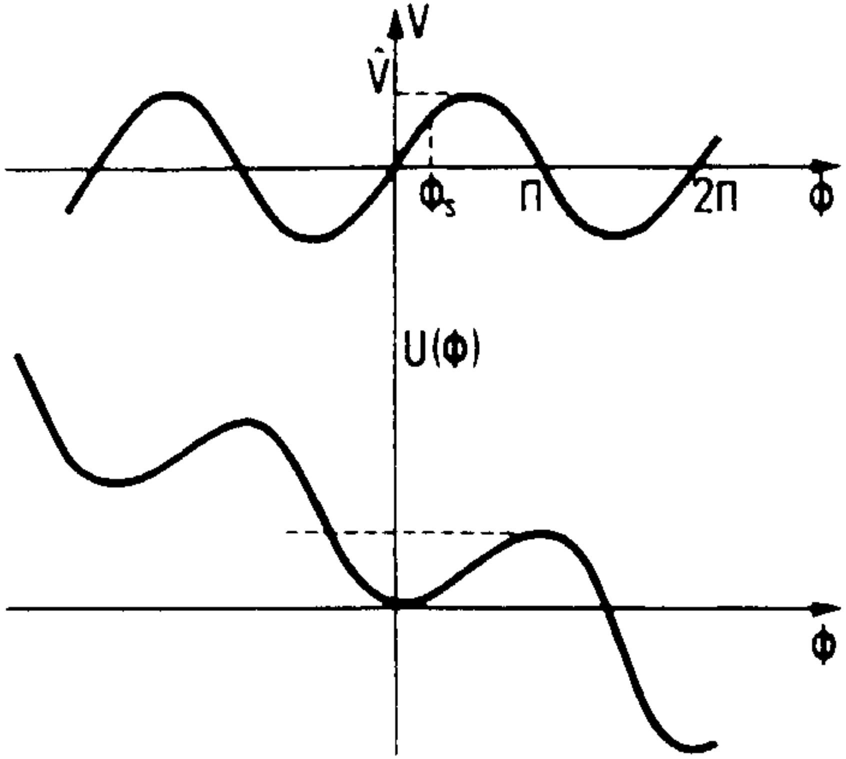
$$F\left(\phi\right)=-\frac{\partial U}{\partial\phi} \qquad \frac{\partial^2\phi}{\partial t^2}=F\left(\phi\right)$$

The sum of the potential energy and the kinetic energy is constant

Synchrotron oscillations: large amplitude

RF voltage and corresponding potential energy function

The shape of the potential energy corresponds to the sum of a linear function and a sinusoidal one. An oscillation can only take place if the particle is trapped in the potential well which means that the total energy cannot exceed a certain value (dotted line) otherwise the particle will slide along the curve. Hence the maxima of the curve correspond to unstable equilibrium points for the synchrotron motion.



Synchrotron oscillations: RF acceptance

The maximum value of $d\phi/dt$ is reached when $d^2\phi/dt^2=0$, that is when $\phi=\phi_s$. Introducing this value in the equation of the separatrix

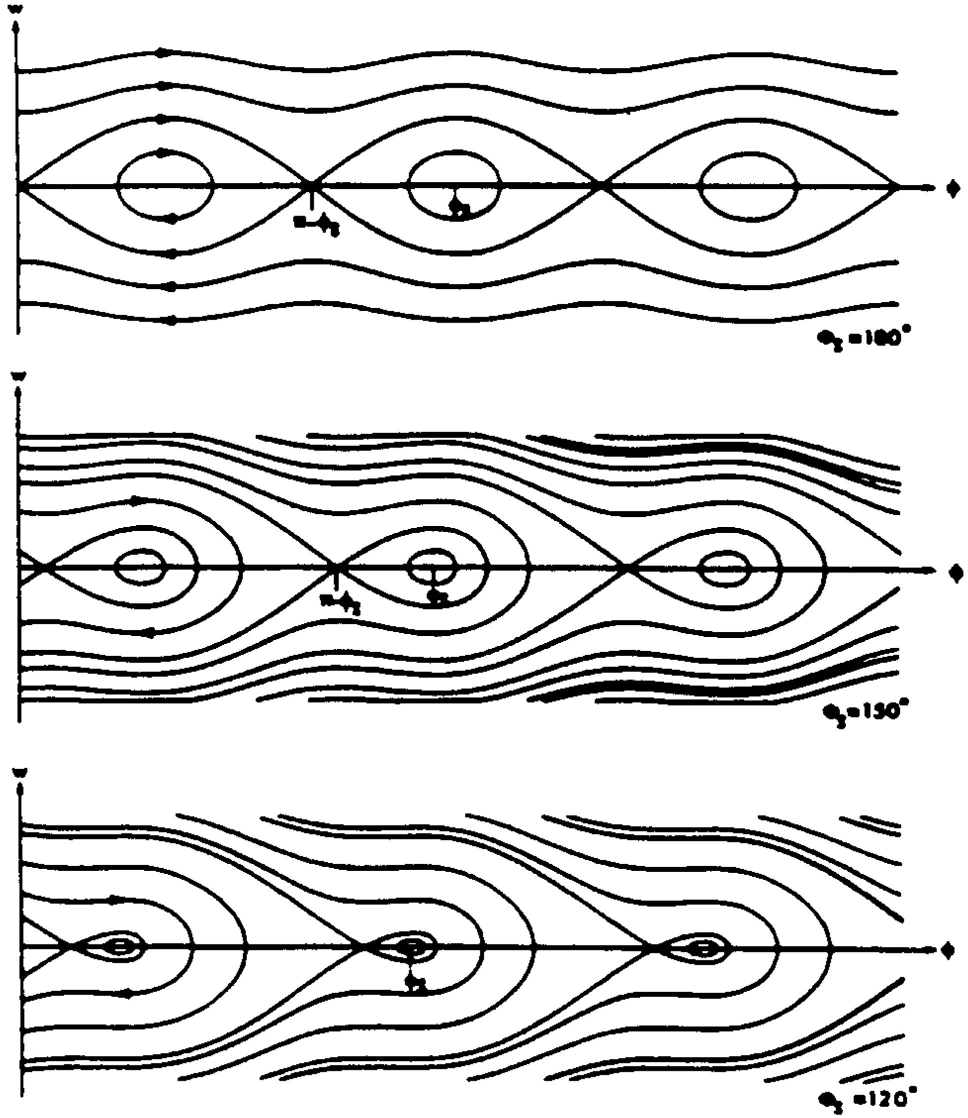
$$\dot{\phi}_{\max}^2 = \frac{2\omega_s^2}{\cos\phi_s}\left[2\cos\phi_s - (\pi - 2\phi_s)\sin\phi_s\right] = \frac{2\omega_s^2}{\cos\phi_s} G(\phi_s)$$

From the previous equations it is possible to obtain that $\left(\dot{\phi} \propto W \propto \Delta E\right)$

RF acceptance: $$\left(\frac{\Delta E}{E_0}\right)_{\max} = \pm\beta\left(\frac{qV_{rf}}{\pi h \eta E_0} G(\phi_s)\right)^{1/2}$$

The RF acceptance plays an important role when designing a machine, since it determines the capture efficiency at injection and the lifetime of stored beams

Synchrotron oscillations: RF acceptance



Adiabatic damping of synchrotron oscillations

- **The expressions that we have seen are valid when the parameters R_0, p_0, ω_0, η and V_{rf} are constant or slowly changing with time compared to $\Delta\phi=\phi-\phi_s$ (slow variation in a synchrotron period).**
- **However in a synchrotron these parameters vary over a large range, even slowly, during an acceleration cycle.**
- **Let's then study the long term evolution of the motion under adiabatic changes of these parameters.**
- **This is possible with the help of the Boltzman-Ehrenfest adiabatic theorem which states that, if p and q are canonically conjugate variables of an oscillatory system with slowly changing parameters, then the action integral over one period of oscillation is constant:**

$$I = \oint p\,dq = \text{const}$$

Adiabatic damping of synchrotron oscillations

- **The variables W and ϕ are canonical variables, so for them the theorem is valid:**

$$I = \oint W \, d\phi = \text{const}$$

- **Let's write again the Hamiltonian**

$$H = eV_{rf}\left[\cos\phi - \cos\phi_s + \left(\phi - \phi_s\right)\sin\phi_s\right] - \frac{h\eta\omega_0}{4\pi R_0 p_0} W^2$$

- **For small oscillation amplitudes it becomes**

$$H = -\frac{eV_{rf}}{2}\cos\phi_s \Delta\phi^2 - \frac{h\eta\omega_0}{4\pi R_0 p_0} W^2$$

- **The harmonic solutions are**

$$\Delta\phi = \Delta\phi_{\max} \sin\left(\omega_s t + \theta_0\right)$$

$$W = \Delta W_{\max} \cos\left(\omega_s t + \theta_0\right)$$

Adiabatic damping of synchrotron oscillations

- **Since**

$$\frac{d\phi}{dt}=\frac{\partial H}{\partial W}=-\frac{h\eta\omega_0}{2\pi R_0 p_0}W$$

- **the action integral is**

$$I=\oint W\frac{d\phi}{dt}dt=-\frac{h\eta\omega_0}{2\pi R_0 p_0}\oint W^2\,dt=-\frac{h\eta\omega_0}{2R_0 p_0\omega_s}\Delta W_{\max}^2=\text{const}$$

- **ΔW_{max} is related to $\Delta\phi_{max}$ so that**

$$\left(\Delta W_{\max}\right)^4=\text{const}^2\,\frac{2qV_{rf}R_0^2E_0\cos\phi_s}{\pi h\eta c^2}$$

$$\left(\Delta\phi_{\max}\right)^4=\text{const}^2\,\frac{h\eta c^2}{2\pi^3 qV_{rf}R_0^2E_0\cos\phi_s}$$

Radiation damping of synchrotron oscillations

- The product ΔW_{max} *$\Delta\phi_{max}$ is constant, which means that the phase space area is invariant and Liouville's theorem still holds in adiabatic conditions. The phase space area is not damped, only the shape of the ellipse is modified.

- However, in particular for electrons, if we take into account also the energy lost by synchrotron radiation, we have another term in the harmonic oscillator equation, which produces a damping of synchrotron oscillations.