Cylindrical coordinates

Equations of motion in cylindrical coordinates and Hamiltonian

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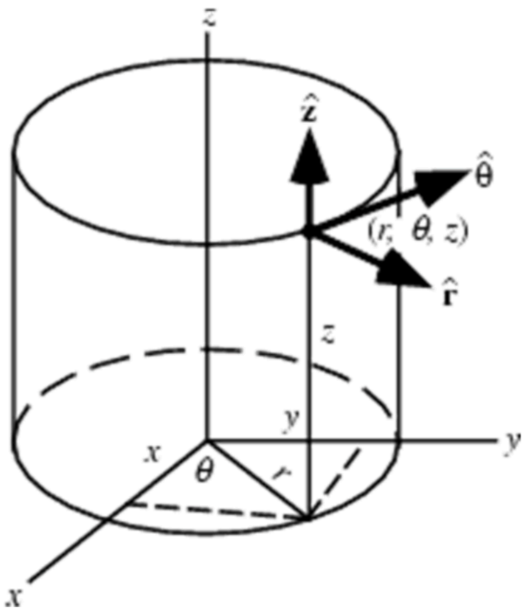
Equations of motion in cylindrical coordinates

The relativistic Lagrangian is

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{v} \cdot \mathbf{A} - q\phi$$

Let us use the cylindrical coordinates



$$\begin{aligned}
\mathbf{v} &= v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_z \hat{\mathbf{z}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{z}\hat{\mathbf{z}} \ , \\
\mathbf{A}(r, \theta, z) &= A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_z \hat{\mathbf{z}} \ , \\
\phi &= \phi(r, \theta, z) \ .
\end{aligned}$$

$$L = -m_0 c^2 \sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2}{c^2}} + q(\dot{r}A_r + r\dot{\theta}A_\theta + \dot{z}A_z) - q\phi$$

Equations of motion in cylindrical coordinates

$$L = -m_0 c^2 \sqrt{\underbrace{1 - \frac{\dot r^2 + r^2\dot\theta^2 + \dot z^2}{c^2}}} + q(\dot r A_r + r\dot\theta A_\theta + \dot z A_z) - q\phi$$

$$= 1 - \frac{v^2}{c^2} = 1 - \beta^2 = \frac{1}{\gamma^2}$$

$$= \frac{\partial}{\partial \dot r} = -m_0 c^2 \frac{\partial \quad /\partial \dot r}{2/\gamma} = -m_0 c^2 \gamma \frac{-2\,\dot r/c^2}{2}$$

The generalized momentum has then components:

$$P_r = \frac{\partial L}{\partial \dot r} = m_0\gamma\dot r + qA_r \ ,$$

$$P_\theta = \frac{\partial L}{\partial \dot\theta} = m_0\gamma r^2\dot\theta + qrA_\theta \ ,$$

$$P_z = \frac{\partial L}{\partial \dot z} = m_0\gamma\dot z + qA_z \ ,$$

Equations of motion in cylindrical coordinates

$$L = -m_0 c^2 \sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2}{c^2}} + q(\dot{r}A_r + r\dot{\theta}A_\theta + \dot{z}A_z) - q\phi$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m_0\gamma\dot{r} + qA_r \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial r} = m_0\gamma r\dot{\theta}^2 + q\dot{\theta}A_\theta + q\dot{r}\frac{\partial A_r}{\partial r} + qr\dot{\theta}\frac{\partial A_\theta}{\partial r} + q\dot{z}\frac{\partial A_z}{\partial r} - q\frac{\partial \phi}{\partial r}$$

Equations of motion in cylindrical coordinates

$$P_r = \frac{\partial L}{\partial \dot{r}} = m_0 \gamma \dot{r} + q A_r$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial r} = m_0 \gamma r \dot{\theta}^2 + q \dot{\theta} A_\theta + q \dot{r} \frac{\partial A_r}{\partial r} + q r \dot{\theta} \frac{\partial A_\theta}{\partial r} + q \dot{z} \frac{\partial A_z}{\partial r} - q \frac{\partial \phi}{\partial r}$$

$$q \frac{dA_r}{dt} = q \frac{\partial A_r}{\partial t} + q \frac{\partial A_r}{\partial \theta} \dot{\theta} + q \frac{\partial A_r}{\partial r} \dot{r} + q \frac{\partial A_r}{\partial z} \dot{z}$$

$$\frac{d}{dt}\left(m_0 \gamma \dot{r}\right) - m_0 \gamma r \dot{\theta}^2 = \boxed{q r \dot{\theta}\left(\frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r}\right) - q \dot{\theta} \frac{\partial A_r}{\partial \theta}} + \boxed{q \dot{z} \frac{\partial A_z}{\partial r} - q \dot{z} \frac{\partial A_r}{\partial z}} - q\left(\frac{\partial \phi}{\partial r} + \frac{\partial A_r}{\partial t}\right) + q \dot{r} \frac{\partial A_r}{\partial r} - q \dot{r} \frac{\partial A_r}{\partial r}$$

Equations of motion in cylindrical coordinates

We first need to express the electric and magnetic fields in terms of the vector and scalar potentials in cylindrical coordinates

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$$

$$B_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \qquad B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$B_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right) = \frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

$$E_r = -\left(\frac{\partial \phi}{\partial r} + \frac{\partial A_r}{\partial t}\right) \qquad E_\theta = -\left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial A_\theta}{\partial t}\right) \qquad E_z = -\left(\frac{\partial \phi}{\partial z} + \frac{\partial A_z}{\partial t}\right)$$

Equations of motion in cylindrical coordinates

$$\frac{d}{dt}\left(m_0\gamma\dot{r}\right) - m_0\gamma r\dot{\theta}^2 = q r\dot{\theta}\left(\frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r}\right) + q\dot{z}\frac{\partial A_z}{\partial r} - q\left(\frac{\partial \phi}{\partial r} + \frac{\partial A_r}{\partial t}\right) + q\dot{r}\frac{\partial A_r}{\partial r}$$

$$- q\dot{\theta}\frac{\partial A_r}{\partial \theta} - q\dot{z}\frac{\partial A_r}{\partial z} - q\dot{r}\frac{\partial A_r}{\partial r}$$

$$B_z = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r A_\theta\right) - \frac{\partial A_r}{\partial \theta}\right) = \frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

$$B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$E_r = -\left(\frac{\partial \phi}{\partial r} + \frac{\partial A_r}{\partial t}\right)$$

Equations of motion in cylindrical coordinates

So the first equation (along *r*) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\dot{r}) - m_0\gamma r\dot{\theta}^2 = q(E_r + r\dot{\theta}B_z - \dot{z}B_\theta)$$

Proceeding in a similar way for the others:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma r^2\dot{\theta}) = q(E_\theta + \dot{z}B_r - \dot{r}B_z)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\dot{z}) = q(E_z + \dot{r}B_\theta - r\dot{\theta}B_r)$$

Hamiltonian in cylindrical coordinates

$$\mathbf{P} = m\mathbf{v} + q\mathbf{A} \qquad\qquad H = \sqrt{m_0^2c^4 + c^2|\mathbf{P} - q\mathbf{A}|^2} + q\phi$$

$$H = \sqrt{m_0^2c^4 + c^2m_0^2\gamma^2v^2} + q\phi \qquad\qquad \mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{z}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{z}\hat{\mathbf{z}}$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m_0\gamma\dot{r} + qA_r \ ,$$
$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_0\gamma r^2\dot{\theta} + qrA_\theta \ ,$$
$$P_z = \frac{\partial L}{\partial \dot{z}} = m_0\gamma\dot{z} + qA_z \ ,$$

$$\dot{r}^2 = \left(\frac{P_r - qA_r}{m_0\gamma}\right)^2$$

$$r\dot{\theta}^2 = \left(\frac{P_\theta - qrA_\theta}{m_0\gamma r}\right)^2$$

$$\dot{z}^2 = \left(\frac{P_z - qA_z}{m_0\gamma}\right)^2$$

Hamiltonian in cylindrical coordinates

$$H = \sqrt{m_0^2 c^4 + c^2 \cancel{m_0^2 \gamma^2} \left[\frac{(P_r - qA_r)^2 + \left(\frac{P_\theta - qrA_\theta}{r}\right)^2 + (P_z - qA_z)^2}{\cancel{m_0^2 \gamma^2}} \right]} + q\phi$$

$$H = c\sqrt{m^2c^2 + (P_r - qA_r)^2 + \left(\frac{P_\theta - qrA_\theta}{r}\right)^2 + (P_z - qA_z)^2} + q\phi$$