Examples of application of relativistic Hamiltonian

Charge in a constant electric and magnetic field

Dipartimento di Scienze
di Base e Applicate
per l'Ingegneria

Sapienza
Università di Roma

M. Migliorati

Charge in a constant electric field

We have introduced the generalized momentum defined as $\mathbf{P} = m\mathbf{v} + q\mathbf{A}$

The relativistic Hamiltonian of a charged particle is $H = \sqrt{m_0^2 c^4 + c^2 |\mathbf{P} - q\mathbf{A}|^2} + q\phi$

From the Hamiltonian it is possible to derive the equations of motion

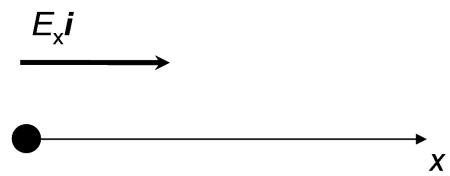
$$\begin{cases} \dot{q}_j = \dfrac{\partial H}{\partial P_j} \\ \dot{P}_j = -\dfrac{\partial H}{\partial q_j} \end{cases}$$

where q_j are the generalized coordinates

(remember that the generalized coordinates are independent from the constraints)

Charge in a constant electric field

Let us suppose a charged particle q in a constant electric field along a fixed direction (example x): $\vec{E} = E_x \hat{\imath}$



At $t = 0$ the charge is at rest in the origin of the coordinate system ($x_0 = 0 \quad v_0 = 0$)

Since the magnetic field is zero, also the vector potential is zero, so that

$$\begin{cases} q \longleftrightarrow x \\ P = p \longleftrightarrow p_x \end{cases}$$

$$H = \sqrt{m_0^2 c^4 + c^2 p_x^2} + q\phi$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{2c^2 p_x}{2\sqrt{m_0^2 c^4 + c^2 p_x^2}}$$

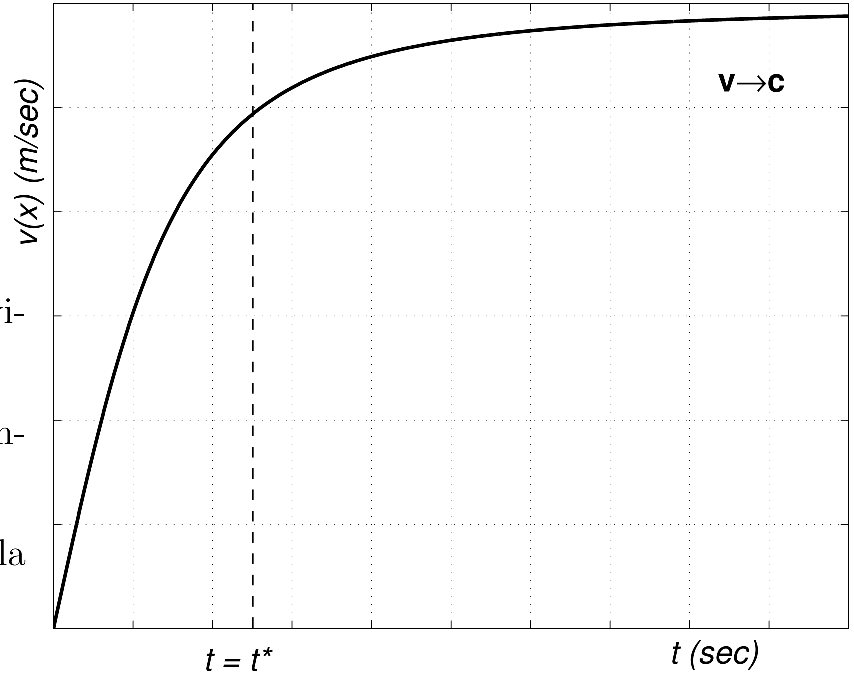
$$\dot{p}_x = -\frac{\partial H}{\partial x} = -q\frac{\partial \phi}{\partial x} = qE_x \quad \Rightarrow \quad p_x(t) = qE_x t$$

Charge in a constant electric field

$$v(t) = \dot{x}(t) = \frac{c^2 q E_x t}{\sqrt{m_0^2 c^4 + (cqE_x t)^2}} = \frac{qE_x t}{m_0 \sqrt{1 + (\frac{cqE_x t}{m_0 c^2})^2}}$$

$$t = t^* = \frac{m_0 c^2}{cqE_x}$$

1. *classico* ($t \ll t^*$): è l'andamento lineare di una carica q "non relativistica" sotto l'azione di un campo elettrico E_x, ossia $v(t) = \frac{qE_x}{m} t$;
2. *relativistico* ($t = t^*$): la carica intorno al tempo t^* si allontana dall'andamento lineare;
3. *ultra relativistico* ($t \gg t^*$): la velocità della carica tende a quella della luce c.

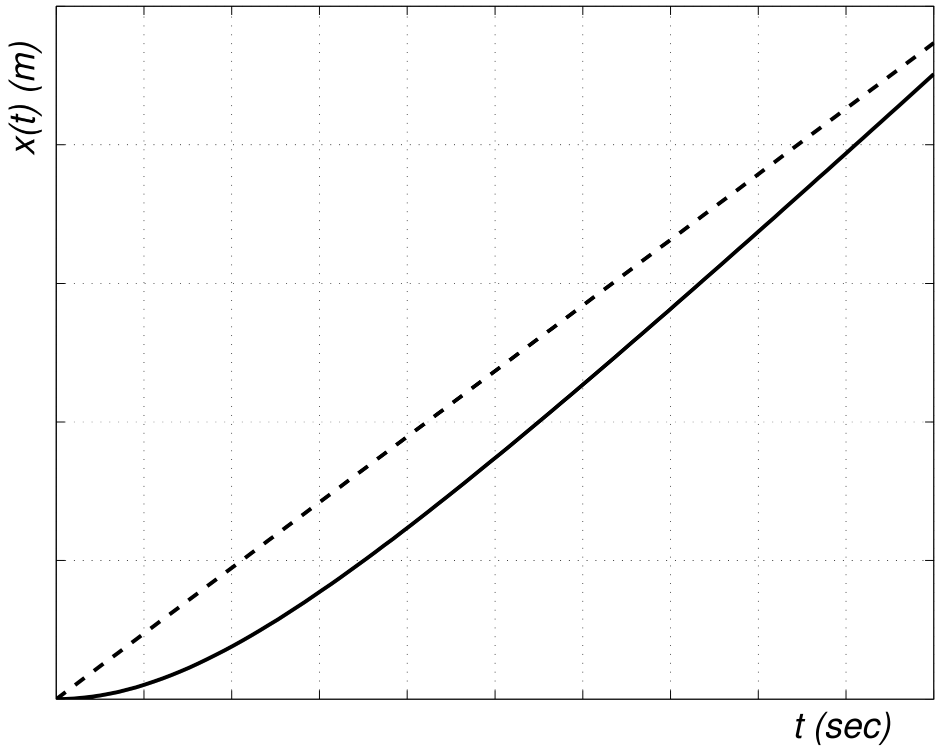


Charge in a constant electric field

$$x(t) = \frac{1}{qE_x}\sqrt{m_0^2c^4 + (cqE_xt)^2} + cost = \frac{m_0c^2}{qE_x}\sqrt{1 + \left(\frac{cqE_xt}{m_0c^2}\right)^2} + cost \Rightarrow x(t) = \frac{m_0c^2}{qE_x}\left[\sqrt{1 + \left(\frac{cqE_xt}{m_0c^2}\right)^2} - 1\right]$$

$$se \quad t \ll t^* \quad \Longrightarrow \quad x(t) = \frac{1}{2}\left(\frac{qE_x}{m_0}\right)t^2$$

$$se \quad t \gg t^* \quad \Longrightarrow \quad x(t) = ct$$

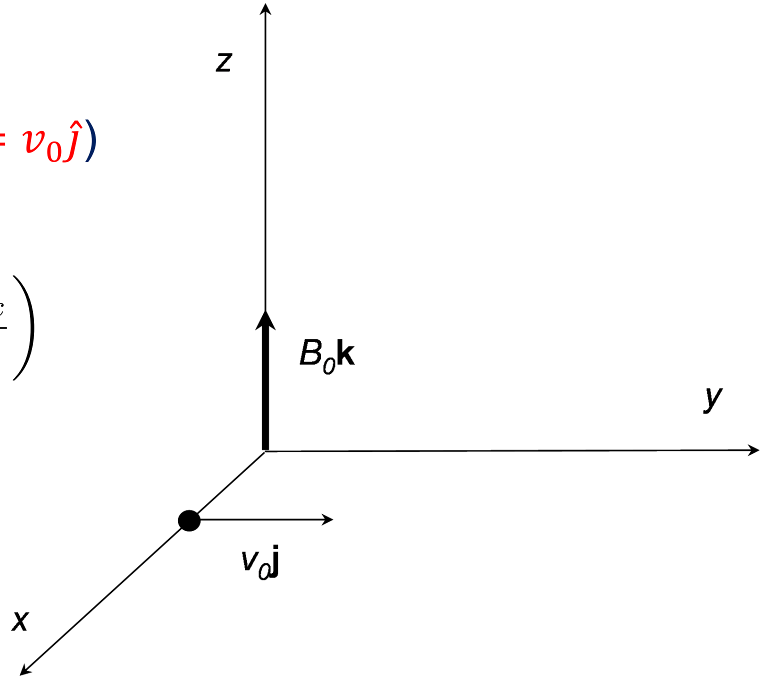


Charge in a constant magnetic field

Let us suppose a charged particle q in a constant magnetic field along a fixed direction (example z): $\vec{B} = B_0 \hat{k}$

At $t = 0$ the charge has an initial velocity along y $\vec{v}(0) = v_0 \hat{j}$)

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{\kappa}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \implies B_0 = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$



The scalar potential is zero and

$$\begin{cases} q_1 \longleftrightarrow x \\ q_2 \longleftrightarrow y \\ P_1 \longleftrightarrow P_x \\ P_2 \longleftrightarrow P_y \end{cases}$$

$$H = \sqrt{m_0^2 c^4 + c^2 \left[(P_x - qA_x)^2 + \left(P_y - qA_y\right)^2 \right]}$$

Charge in a constant magnetic field

$$H=\sqrt{m_0^2c^4+c^2\left[(P_x-qA_x)^2+\left(P_y-qA_y\right)^2\right]}=\sqrt{m_0^2c^4+m_0^2c^2\gamma^2v^2}=\quad (v=\beta c)$$

$$=\sqrt{m_0^2c^4+m_0^2c^4\gamma^2\beta^2}=$$

$$=m_0c^2\sqrt{1+\gamma^2\beta^2}=\gamma m_0c^2$$

$$(\beta^2\gamma^2=\gamma^2-1)$$

$$\begin{cases} P_x-qA_x=p_x=m_0\gamma v_x \\ P_y-qA_y=p_y=m_0\gamma v_y \end{cases}$$

$$\dot{P}_x=-\frac{\partial H}{\partial x}=-\frac{\left[(P_x-qA_x)(-q\frac{\partial A_x}{\partial x})+(P_y-qA_y)(-q\frac{\partial A_y}{\partial x})\right]}{m_0\gamma}$$

$$\dot{P}_y=-\frac{\partial H}{\partial y}=-\frac{\left[(P_x-qA_x)(-q\frac{\partial A_x}{\partial y})+(P_y-qA_y)(-q\frac{\partial A_y}{\partial y})\right]}{m_0\gamma}$$

$$\Rightarrow$$

$$\dot{P}_x=q\left[v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}\right]$$

$$\dot{P}_y=q\left[v_x\frac{\partial A_x}{\partial y}+v_y\frac{\partial A_y}{\partial y}\right]$$

Charge in a constant magnetic field

$$\dot{P}_x = q\left[v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x}\right]$$

$$\dot{P}_y = q\left[v_x\frac{\partial A_x}{\partial y} + v_y\frac{\partial A_y}{\partial y}\right]$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} = v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y}$$

=0 because the magnetic field is constant with time

remember that $\mathbf{P} = m\mathbf{v} + q\mathbf{A}$ ➡ $\dot{P}_x - q\frac{dA_x}{dt} = \frac{dp_x}{dt} = qv_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = qv_yB_0$

$$B_0 = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Similarly, using $\frac{dA_y}{dt}$ $\quad \dot{P}_y - q\frac{dA_y}{dt} = \frac{dp_y}{dt} = -qv_xB_0$

Charge in a constant magnetic field

The final equations are then

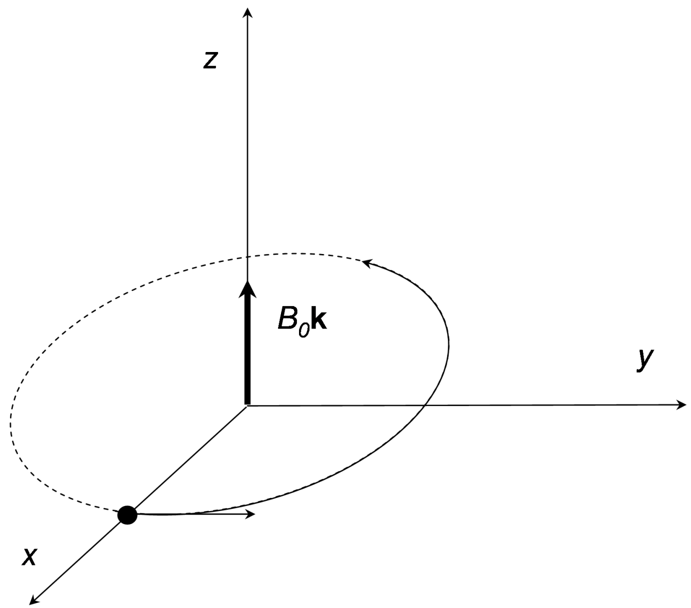
$$\begin{cases} \dot{p}_x = q v_y B_0 \\ \dot{p}_y = -q v_x B_0 \end{cases} \Longrightarrow \begin{cases} \ddot{v}_x = -\left(\frac{qB_0}{m}\right)^2 v_x \\ \ddot{v}_y = -\left(\frac{qB_0}{m}\right)^2 v_y \end{cases}$$

The solutions are (harmonic oscillator)

$$\begin{cases} v_x(t) = -v_0 \sin(\Omega t) \\ v_y(t) = v_0 \cos(\Omega t) \ , \end{cases} \qquad \Omega = \frac{qB_0}{m}$$

and by integrating we finally get

$$\begin{cases} x(t) = \frac{v_0}{\Omega} \cos(\Omega t) \\ y(t) = \frac{v_0}{\Omega} \sin(\Omega t) \end{cases} \qquad r = \frac{v_0}{\Omega} = \frac{m v_0}{q B_0}$$



NB: $(m = \gamma m_0)$