

PhD School
Tutorial on
 ν oscillation probabilities

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- Most of our current knowledge of neutrino properties is based on the observed phenomenon of ν flavor oscillations: $\nu_\alpha \rightarrow \nu_\beta$.
- This tutorial contains a set of worked-out exercises of ν oscillation probabilities in vacuum and matter, with increasing difficulty.
- Basically all the results of the exercises are of interest for the current phenomenology, e.g., as described in the review on Neutrino masses & mixings of the Particle Data Group (PDG).
- Some exercises will bring the student at the frontier of the current discussion of precision oscillation experiments; e.g., we shall prove eq. (2.1) of arXiv: 1507.05613 (interesting for future reactor experiments such as JUNO or RENO-50) and eq. (3.5) of arXiv: 1512.06148 (interesting for current and future accelerator experiments such as DUNE).
- Apologies for hand-writing and possible typos!

INDEX:

- Conventions about mixing matrix U
- Conventions about mixing: U versus U^*
- Conventions about mass states
- Conventions about flavor indices
- Exercise: 3 ν oscillations in vacuum - general case for $P_{\alpha\beta}$
- Exercise: changing units in vacuum
- Exercise: CP(T) properties of $P_{\alpha\beta}$ in vacuum
- Exercise: conditions to observe ~~CP~~ in vacuum
- Exercise: $P_{\alpha\beta}$ in vacuum for $(\Delta m^2 x / 4E) \sim \Theta(1)$ and $(\delta m^2 x / 4E) \ll 1$
- Exercise: $P_{\alpha\beta}$ in vacuum for $(\Delta m^2 x / 4E) \gg 1$ and $(\delta m^2 x / 4E) \sim \Theta(1)$
- Exercise: $P_{\alpha\beta}$ in vacuum, general case (3 ν)
- Exercise: $P_{\alpha\beta}$ in vacuum, general case in alternative formulation \leftarrow Useful for JUNO, RENO-50
- Sketchy proof of $V = \sqrt{2} G_F N_e$ in matter
- Exercise: changing units in matter
- Exercise: 2 ν oscillations in matter with constant density
- Exercise: 2 ν oscillations in matter with slowly varying density
- 3 ν oscillations in matter: general reduction tools
- Calculation of $P_{\alpha\mu}^{3\nu}$ in matter at 2nd order in θ_{13} , δm^2 \leftarrow Useful for precision LBL accelerators

Conventions about mixing matrix U

indices:
 $\alpha = \text{flavor}$
 $i = \text{mass}$

- $$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad [\nu_\alpha = U_{\alpha i} \nu_i]$$
- If these are the only ν states in nature, U is unitary : $UU^+ = I$
- For antineutrinos: $U \rightarrow U^*$ (see next page)
- Particle Data Group convention for U :

$$U = O_{23} \Gamma_\delta O_{13} \Gamma_\delta^+ O_{12} \quad \leftarrow \text{with } \Gamma_\delta = \text{diag}(1, 1, e^{i\delta})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -S_{13} e^{i\delta} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \leftarrow \text{with } c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}$$

$$= \begin{pmatrix} C_{12} C_{13} & S_{12} C_{13} & S_{13} e^{-i\delta} \\ -S_{12} C_{13} - C_{12} S_{23} S_{13} e^{i\delta} & C_{12} C_{23} - S_{12} S_{23} S_{13} e^{i\delta} & S_{23} C_{13} \\ S_{12} S_{23} - C_{12} C_{23} S_{13} e^{i\delta} & -C_{12} S_{23} - S_{12} C_{23} S_{13} e^{i\delta} & C_{23} C_{13} \end{pmatrix}$$

- U is often called "Pontecorvo - Maki - Nakagawa - Sakata" (PMNS) matrix
- For $\delta=0$ or $\delta=\pi$: $U=U^*$

Conventions about mixing: U vs U^*

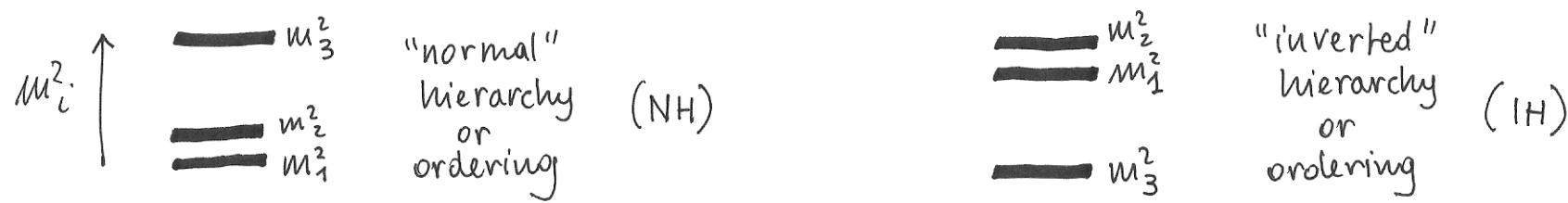
- The PMNS matrix U actually connects quantum fields in the CC lagrangian, $\nu_{\alpha L} = \sum_{i=1}^3 U_{\alpha i} \nu_{iL}$. But, for a field ψ , it is $\bar{\psi}$ (or ψ^+) that creates particles from vacuum $|0\rangle$. Therefore, in terms of created one-particle states (kets $|\nu\rangle$) one has that: $|\nu_{\alpha}\rangle = \sum_{i=1}^3 U_{\alpha i}^* |\nu_i\rangle$ \leftarrow PDG convention in terms of states.
- On the other hand, a generic $|\nu\rangle$ state can be decomposed as:
 $|\nu\rangle = \sum_i r^i |\nu_i\rangle = \sum_{\alpha} r^{\alpha} |\nu_{\alpha}\rangle$, where the components r^i, r^{α} (= complex numbers) transform as: $r^{\alpha} = \sum_i U_{\alpha i} r^i$. Summarizing, for neutrinos:

$$\begin{cases} \nu_{\alpha L} = \sum_i U_{\alpha i} \nu_{iL} & \leftarrow \text{fields} \\ |\nu_{\alpha}\rangle = \sum_i U_{\alpha i}^* |\nu_i\rangle & \leftarrow \text{states} \\ r^{\alpha} = \sum_i U_{\alpha i} r^i & \leftarrow \text{components} \end{cases}, \text{ where } U_{\alpha i} = \langle \nu_{\alpha} | \nu_i \rangle$$
 For antineutrinos, one should change $U \rightarrow U^*$ everywhere.
- Note that neutrino "components" (rather than states or fields) are the numerical objects manipulated in computer calculations. They are often organized into a column vector, e.g.:

$$\begin{pmatrix} r^e \\ r^{\mu} \\ r^{\tau} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
 is the vector of components of a pure $|\nu_e\rangle$ state in flavor basis.

Conventions about mass states

- We consider three mass states ν_1, ν_2, ν_3 with masses m_1, m_2, m_3
- Squared mass splittings: $\Delta m^2_{ij} = m_i^2 - m_j^2$
- Experimentally, one mass² splitting is much smaller than the other; define it as: $\delta m^2 \equiv \Delta m^2_{21} = m_2^2 - m_1^2 > 0$ by convention.
- Then the third mass can be either lighter or heavier than $m_{1,2}$:



- In the following, we shall define the "large" mass² splitting as the average of Δm^2_{31} and Δm^2_{32} :
- $$\Delta m^2 = \frac{1}{2} (\Delta m^2_{31} + \Delta m^2_{32}) \quad \rightarrow \quad \begin{array}{l} \Delta m^2 > 0 \text{ in NH} \\ \Delta m^2 < 0 \text{ in IH} \end{array}$$
- Other people prefer to use as independent splitting Δm^2_{31} , or Δm^2_{32} , or linear combinations of them such as Δm^2_{ee} (see later).

Conventions about flavor indices

- Fixing conventions about flavor index ordering is necessary since, in many cases, (α, β) is not equivalent to (β, α) . Our conventions follow.
- Evolution equation ($t \approx x$, $\partial_t \approx \partial_x$): $\hat{H} |\nu\rangle = i \partial_x |\nu\rangle$, \hat{H} =hamiltonian
- Decomposition in mass basis:
$$\hat{H} = \sum_{ij} |\nu_j\rangle \langle \nu_j| \hat{H} |\nu_i\rangle \langle \nu_i| = \sum_{ij} H_{ji} |\nu_j\rangle \langle \nu_i| \quad \uparrow$$

note (j,i) and (β,α) ordering!
- Decomposition in flavor basis:
$$\hat{H} = \sum_{\alpha\beta} |\nu_\beta\rangle \langle \nu_\beta| \hat{H} |\nu_\alpha\rangle \langle \nu_\alpha| = \sum_{\alpha\beta} H_{\beta\alpha} |\nu_\beta\rangle \langle \nu_\alpha| \quad \leftarrow$$
- Relation among matrix elements:
$$H_{ji} = \langle \nu_j | \hat{H} | \nu_i \rangle ; \quad H_{\beta\alpha} = \langle \nu_\beta | \hat{H} | \nu_\alpha \rangle ; \quad H_{\beta\alpha} = \sum_{ij} U_{\beta j} H_{ji} U_{\alpha i}^*$$

In matrix form: $H_{\text{flavor}} = \begin{bmatrix} & \\ & \end{bmatrix} H_{\text{mass}} \begin{bmatrix} & \\ & \end{bmatrix}^+$
- Formal solution (evolution operator): $|\nu(x)\rangle = \hat{S}(x, 0) |\nu(0)\rangle$
- Flavor oscillation probability : $P(\nu_\alpha \rightarrow \nu_\beta) = |S_{\beta\alpha}|^2$

Exercise : 3ν oscillations in vacuum -general case

- Given the hamiltonian in vacuum

$$H_{\text{mass}} = \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & E_3 \end{pmatrix} \simeq p \delta_{ij} + \frac{m_i^2}{2E} \delta_{ij}$$

prove that:

$$\boxed{P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i < j} \operatorname{Re} J_{\alpha\beta}^{ij} \sin^2 \left(\frac{\Delta m_{ij}^2 x}{4E} \right) - 2 \sum_{i < j} \operatorname{Im} J_{\alpha\beta}^{ij} \sin \left(\frac{\Delta m_{ij}^2 x}{2E} \right)}$$

$$(\Delta m_{ij}^2 = m_i^2 - m_j^2)$$

where $J_{\alpha\beta}^{ij} = U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}$ (Jarlskog invariant)

SOLUTION -

- Any term proportional to $\mathbf{1}$ in H can be dropped in oscillation phenomena
(it just shifts all energies by the same amount \rightarrow unobservable overall phase)
- Since $H = \frac{m_i^2}{2\epsilon} \delta_{ij} = \frac{1}{2\epsilon} \text{diag}(m_1^2, m_2^2, m_3^2)$ is x -independent, the evolution operator is simply obtained by exponentiation: $\hat{S} = \exp(-i\hat{H}x)$
- In mass basis: $S_{ji} = \langle v_j | \hat{S} | v_i \rangle = \delta_{ij} e^{-i \frac{m_i^2}{2\epsilon} x}$
- In flavor basis: $S_{\beta\alpha} = \langle v_\beta | \hat{S} | v_\alpha \rangle = \sum_{ij} U_{\beta j} S_{ji} U_{\alpha i}^* = \sum_i U_{\alpha i}^* U_{\beta i} e^{-i \frac{m_i^2}{2\epsilon} x}$
- Flavor oscillation probability:

$$\begin{aligned}
 P(v_\alpha \rightarrow v_\beta) &= |S_{\beta\alpha}|^2 \\
 &= \left| \sum_i U_{\alpha i}^* U_{\beta i} e^{-i \frac{m_i^2}{2\epsilon} x} \right|^2 \\
 &= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* e^{-i \frac{m_i^2}{2\epsilon} x + i \frac{m_j^2}{2\epsilon} x} \\
 &= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left(e^{i \frac{m_j^2 - m_i^2}{2\epsilon} x} - 1 + 1 \right) \\
 &= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left(e^{i \frac{m_j^2 - m_i^2}{2\epsilon} x} - 1 \right) + \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*
 \end{aligned}$$

$\rightarrow \text{cont'd}$

$$\begin{aligned}
&= \left(\sum_{i < j} + \sum_{i > j} \right) \cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^* \left(e^{i \frac{m_j^2 - m_i^2}{2\epsilon} x} - 1 \right) + \sum_i \cup_{\alpha i}^* \cup_{\beta i} \sum_j \cup_{\alpha j} \cup_{\beta j}^* \\
&= \sum_{i > j} \cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^* \left(e^{i \frac{m_j^2 - m_i^2}{2\epsilon} x} - 1 \right) + \sum_{i > j} \cup_{\alpha i} \cup_{\beta i}^* \cup_{\alpha j}^* \cup_{\beta j} \left(e^{-i \frac{m_j^2 - m_i^2}{2\epsilon} x} - 1 \right) + \delta_{\alpha\beta} \cdot \delta_{\alpha\beta} \\
&= \sum_{i > j} (\cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^* + \cup_{\alpha i} \cup_{\beta i}^* \cup_{\alpha j}^* \cup_{\beta j}) [\cos(\frac{m_j^2 - m_i^2}{2\epsilon} x) - 1] \\
&\quad + \sum_{i > j} (\cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^* - \cup_{\alpha i} \cup_{\beta i}^* \cup_{\alpha j}^* \cup_{\beta j}) [i \sin(\frac{m_j^2 - m_i^2}{2\epsilon} x)] + \delta_{\alpha\beta} \\
&= \delta_{\alpha\beta} - \sum_{i > j} 2 \operatorname{Re} (\cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^*) [\cos(\frac{m_j^2 - m_i^2}{2\epsilon} x) - 1] - \sum_{i > j} 2 \operatorname{Im} (\cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^*) \sin(\frac{m_j^2 - m_i^2}{2\epsilon} x) \\
&= \delta_{\alpha\beta} - 4 \sum_{i > j} \operatorname{Re} (\cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^*) \sin^2(\frac{m_j^2 - m_i^2}{4\epsilon} x) - 2 \sum_{i > j} \operatorname{Im} (\cup_{\alpha i}^* \cup_{\beta i} \cup_{\alpha j} \cup_{\beta j}^*) \sin(\frac{m_j^2 - m_i^2}{2\epsilon} x) \\
&\stackrel{i \leftrightarrow j}{=} \delta_{\alpha\beta} - 4 \sum_{i < j} \operatorname{Re} J_{\alpha\beta}^{ij} \sin^2\left(\frac{\Delta m_{ij}^2}{4\epsilon} x\right) - 2 \sum_{i < j} \operatorname{Im} J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta m_{ij}^2}{2\epsilon} x\right)
\end{aligned}$$

with $J_{\alpha\beta}^{ij} = \cup_{\alpha i} \cup_{\beta i}^* \cup_{\alpha j}^* \cup_{\beta j}$

Exercise : changing units in vacuum

- Prove that : $\frac{\Delta m_{ij}^2}{4E} x = 1.267 \left(\frac{\Delta m_{ij}^2}{eV^2} \right) \left(\frac{x}{m} \right) \left(\frac{MeV}{E} \right)$ $\left[= 1.267 \frac{\Delta m_{ij}^2}{eV^2} \frac{x}{km} \frac{GeV}{E} \right]$

\uparrow
 natural units

- Solution: $\hbar c = 197.327 \text{ MeV} \cdot \text{fm} \equiv 1$ in natural units
 $\rightarrow 1 \text{ MeV} \cdot 1 \text{ m} = 5.0677 \times 10^{12}$

$$\begin{aligned}
 \frac{\Delta m_{ij}^2 x}{4E} &= \frac{1}{4} \left(\frac{\Delta m_{ij}^2}{eV^2} eV^2 \right) \left(\frac{x}{m} \cdot m \right) \left(\frac{MeV}{E} \cdot \frac{1}{MeV} \right) \\
 &= \frac{1}{4} \left(\frac{1 \text{ eV}^2 \cdot 1 \text{ m}}{1 \text{ MeV}} \right) \left(\frac{\Delta m_{ij}^2}{eV^2} \right) \left(\frac{x}{m} \right) \left(\frac{MeV}{E} \right) \\
 &= \frac{10^{-12}}{4} (MeV \cdot m) \left(\frac{\Delta m_{ij}^2}{eV^2} \right) \left(\frac{x}{m} \right) \left(\frac{MeV}{E} \right) \\
 &= 1.267 \left(\frac{\Delta m_{ij}^2}{eV^2} \right) \left(\frac{x}{m} \right) \left(\frac{MeV}{E} \right)
 \end{aligned}$$

Exercise : CP(T) properties of $\nu_{\alpha\beta}$ in vacuum

One of the current frontiers in ν research is to investigate CP.

Try to argue that the general form of $P(\nu_{\alpha} \rightarrow \nu_{\beta})$ is naturally split in a CP-conserving and a CP-violating part, $P = P_{CP} + P_{CP'}$:

$$P(\nu_{\alpha} \rightarrow \nu_{\beta}) = \delta_{\alpha\beta} - 4 \sum_{i < j} \operatorname{Re} J_{\alpha\beta}^{ij} \sin^2 \left(\frac{\Delta m^2_{ij} x}{4E} \right) \quad \leftarrow P_{CP}$$
$$- 2 \sum_{i < j} \operatorname{Im} J_{\alpha\beta}^{ij} \sin \left(\frac{\Delta m^2_{ij} x}{2E} \right) \quad \leftarrow P_{CP'}$$

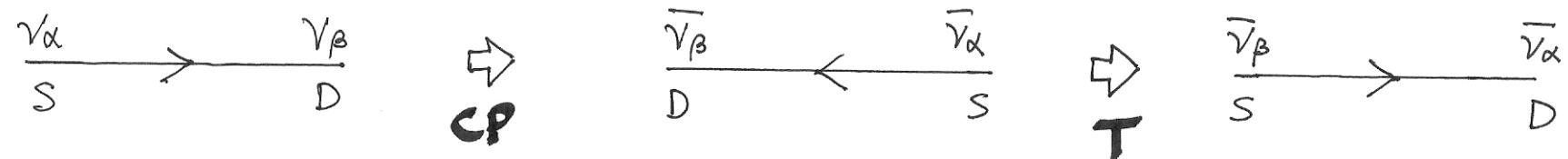
where CP invariance would imply $U = U^*$ and $P(r) = P(\bar{r})$

Prove also that CPT invariance holds, and implies

$$P(\nu_{\alpha} \rightarrow \nu_{\beta}) = P(\bar{\nu}_{\beta} \rightarrow \bar{\nu}_{\alpha})$$

Heuristic argument :

Action of CP and T transformations on $\nu_\alpha \rightarrow \nu_\beta$ process from source (S) to detector (D) amount to :



$$\text{CP invariance} : P(\nu_\alpha \rightarrow \nu_\beta) = P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \quad \Leftarrow (\nu \leftrightarrow \bar{\nu})$$

$$\text{T invariance} : \begin{cases} P(\nu_\alpha \rightarrow \nu_\beta) = P(\nu_\beta \rightarrow \nu_\alpha) \\ P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) = P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha) \end{cases} \quad \Leftarrow (\alpha \leftrightarrow \beta)$$

$$\text{CPT invariance} : P(\nu_\alpha \rightarrow \nu_\beta) = P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha) \quad \Leftarrow (\nu \leftrightarrow \bar{\nu}) \oplus (\alpha \leftrightarrow \beta)$$

For 3ν oscillations in vacuum, in the general form of $P_{\alpha\beta}$, it is easy to check that either $(\alpha \leftrightarrow \beta)$ or $(\nu \leftrightarrow \bar{\nu})$ exchange amount to $(U \leftrightarrow U^*)$, only affecting the P_{CP} part. Therefore, CP invariance requires $U = U^*$ and $\delta = 0, \pi$ in the PDG convention for U , while CPT invariance holds in any case. Experimental evidence for $\delta \neq 0, \pi$ would prove CP violation in the neutrino sector. See next exercise.

Exercise: conditions to observe CP in vacuum

Consider the general 3ν oscillation formula $P = P_{\text{CP}} + P_{\text{CP}}$.

Prove that, in order to have $P_{\text{CP}} \neq 0$, the following conditions must be satisfied:

- $\delta \neq 0, \pi$ $\leftarrow U$ must be complex ($\sin(\delta) \neq 0$)
- $\alpha \neq \beta$ \leftarrow Neutrino appearance experiments $\nu_\alpha \rightarrow \nu_\beta$
- $\theta_{ij} \neq 0$ \leftarrow All mixing angles must be $\neq 0$
- $\Delta m^2_{ij} \neq 0$ \leftarrow Need sensitivity to both Δm^2 and Δm^2

(Need some manipulation of $J_{\alpha\beta}^{ij}$ to solve it.)

Solution-

- For $\delta = 0, \pi$: $U = U^* \rightarrow P_{\text{CP}} = 0$
- For $\alpha = \beta$: $\text{Im}(J_{\alpha\alpha}^{ij}) = 0 \rightarrow P_{\text{CP}} = 0$
- For $\alpha \neq \beta$: It turns out that all $\text{Im}(J_{\alpha\beta}^{ij})$ are equal to each other, up to a sign.

Define: $J \equiv \text{Im}(J_{e\mu}^{12})$. Then: $\text{Im}(J_{\alpha\beta}^{ij}) = \pm J$ for $\alpha \neq \beta$ and $i \neq j$, where:

$$\begin{cases} +J & \text{for } (\alpha, \beta) = (e, \mu), (\mu, \tau), (\tau, e) \\ +J & \text{for } (i, j) = (12), (23), (31) \\ -J & \text{otherwise.} \end{cases} \leftarrow \begin{array}{l} \text{flavor cyclic} \\ \text{generation cyclic} \end{array} \quad (J = \text{Jarlskog invariant})$$

$$\begin{aligned} \text{E.g.: } \text{Im}(J_{e\tau}^{12}) &= \text{Im}(U_{e1} U_{\tau 1}^* U_{e2}^* U_{\tau 2}) \\ &= \text{Im}(U_{e1} U_{e2}^* (-U_{e1}^* U_{e2} - U_{\mu 1}^* U_{\mu 2})) \quad \leftarrow U \text{ unitary} \\ &= \text{Im}(-U_{e1} U_{e2}^* U_{\mu 1}^* U_{\mu 2}) = -\text{Im}(J_{e\mu}^{12}) = -J, \quad \text{etc.} \end{aligned}$$

Note: $\text{Im}(J_{e\tau}^{21}) = +J = (-1)(-1)J$ since both $(e\tau)$ and (21) are anticyclic.

Cyclic rules can be embedded in compact form via antisymmetric tensor ϵ :

$$\text{Im}(J_{\alpha\beta}^{ij}) = J \cdot \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk}$$

Using PDG convention for $U = U(\theta_{ij}, \delta)$, it is:

$$J = \text{Im}(J_{e\mu}^{12}) = \text{Im}(U_{e1} U_{\mu 1}^* U_{e2}^* U_{\mu 2}) = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$

which vanishes for any $\theta_{ij} = 0$ and/or for $\delta = 0, \pi$ ($\sin \delta = 0$).

- P_{CP} has been previously derived in the form of "sum" of oscillating factors.
It can also be recast in "product" form:

$$P_{CP}(\nu_\alpha \rightarrow \nu_\beta) = \pm 8J \sin\left(\frac{\Delta m^2_{12}x}{4E}\right) \sin\left(\frac{\Delta m^2_{23}x}{4E}\right) \sin\left(\frac{\Delta m^2_{31}x}{4E}\right)$$

which vanishes in experimental conditions where the oscillation phase of one factor is $\ll 1$.
The proof makes use of the following trigonometric identity:

If $x+y+z=0$, then $\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z$.

In our case, the identity is applied to $\Delta m^2_{12} + \Delta m^2_{23} + \Delta m^2_{31} = 0$. Indeed:

$$\begin{aligned} P_{CP}(\nu_\alpha \rightarrow \nu_\beta) &= -2 \sum_{i<j} \text{Im } J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta m^2_{ij}x}{2E}\right) \\ &= -2 \left[\text{Im } J_{\alpha\beta}^{12} \sin\left(\frac{\Delta m^2_{12}x}{2E}\right) + \text{Im } J_{\alpha\beta}^{23} \sin\left(\frac{\Delta m^2_{23}x}{2E}\right) + \text{Im } J_{\alpha\beta}^{13} \sin\left(\frac{\Delta m^2_{31}x}{2E}\right) \right] \\ &= -2 \text{Im } J_{\alpha\beta}^{12} \left[\sin\left(\frac{\Delta m^2_{12}x}{2E}\right) + \sin\left(\frac{\Delta m^2_{23}x}{2E}\right) - \sin\left(\frac{\Delta m^2_{31}x}{2E}\right) \right] \quad \text{This is } +\sin\left(\frac{\Delta m^2_{31}x}{2E}\right) \\ &= +8 \text{Im } J_{\alpha\beta}^{12} \sin\left(\frac{\Delta m^2_{12}x}{4E}\right) \sin\left(\frac{\Delta m^2_{23}x}{4E}\right) \sin\left(\frac{\Delta m^2_{31}x}{4E}\right) \\ &= +8 \text{Im } J_{\alpha\beta}^{12} \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta m^2_{ij}x}{4E}\right) \end{aligned}$$

In particular, it would be $P_{CP} \rightarrow 0$ for $\Delta m^2 \rightarrow 0$, i.e. for appearance experiments basically insensitive to the Δm^2 -induced oscillation phase: $\frac{\Delta m^2 x}{4E} \ll 1$.

→ CP violation is a genuine 3ν effect

Exercise : $P_{\alpha\beta}$ in vacuum for $\left(\frac{\Delta m^2 x}{4E}\right) \sim \Theta(1)$ and $\left(\frac{\delta m^2 x}{4E}\right) \ll 1$

Several experiments are mainly sensitive to Δm^2 and largely insensitive to δm^2 , namely : $\frac{\Delta m^2 x}{4E} \sim \Theta(1)$ while $\frac{\delta m^2 x}{4E} \ll 1$ (where $\delta m^2 = \Delta m_{21}^2$).

Prove that, in the limit $\delta m^2 \rightarrow 0$, the oscillation probabilities become :

$$\boxed{\begin{aligned} P_{\alpha\alpha} &= 1 - 4 |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \sin^2 \left(\frac{\Delta m^2 x}{4E} \right) \\ P_{\alpha\beta} &= 4 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \left(\frac{\Delta m^2 x}{4E} \right) \quad \alpha \neq \beta \end{aligned}} \quad (3\nu \text{ vacuum}, \delta m^2 \rightarrow 0)$$

$$\text{where } |U_{e 3}|^2 = S_{13}^2 ; \quad |U_{\mu 3}|^2 = C_{13}^2 S_{23}^2 ; \quad |U_{\tau 3}|^2 = C_{13}^2 C_{23}^2$$

Note that these probabilities do not depend on :

- $\nu/\bar{\nu}$ distinction
- (ν_1, ν_2) parameters $(\delta m^2, \theta_{12})$
- CP violation phase δ
- hierarchy = $\text{sign}(\pm \Delta m^2)$

SOLUTION - Note that for $\Delta m^2 \rightarrow 0$: $\Delta m_{31}^2 \approx \Delta m_{32}^2 \approx \Delta m^2$

$$P_{\alpha\alpha} = 1 - 4 \operatorname{Re} (J_{\alpha\alpha}^{13} + J_{\alpha\alpha}^{23}) \sin^2 \left(\frac{\Delta m^2 x}{4E} \right) - 2 \operatorname{Im} (J_{\alpha\alpha}^{13} + J_{\alpha\alpha}^{23}) \sin \left(\frac{\Delta m^2 x}{2E} \right)$$

$$J_{\alpha\alpha}^{13} + J_{\alpha\alpha}^{23} = |U_{\alpha 1}|^2 |U_{\alpha 3}|^2 + |U_{\alpha 2}|^2 |U_{\alpha 3}|^2 = |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2), \text{ with imaginary part} = 0$$

$$P_{\alpha\alpha} = 1 - 4 |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \sin^2 \left(\frac{\Delta m^2 x}{4E} \right)$$

$$P_{\alpha\beta} = -4 \operatorname{Re} (J_{\alpha\beta}^{13} + J_{\alpha\beta}^{23}) \sin^2 \left(\frac{\Delta m^2 x}{4E} \right) - 2 \operatorname{Im} (J_{\alpha\beta}^{13} + J_{\alpha\beta}^{23}) \sin \left(\frac{\Delta m^2 x}{2E} \right) \quad \alpha \neq \beta$$

$$\begin{aligned} J_{\alpha\beta}^{13} + J_{\alpha\beta}^{23} &= U_{\alpha 1} U_{\beta 1}^* U_{\alpha 3}^* U_{\beta 3} + U_{\alpha 2} U_{\beta 2}^* U_{\alpha 3}^* U_{\beta 3} \\ &= U_{\alpha 3}^* U_{\beta 3} (U_{\alpha 1} U_{\beta 1}^* + U_{\alpha 2} U_{\beta 2}^*) \quad \leftarrow \text{unitarity of } U \\ &= U_{\alpha 3}^* U_{\beta 3} (-U_{\alpha 3} U_{\beta 3}^*) \\ &= -|U_{\alpha 3}|^2 |U_{\beta 3}|^2, \text{ with imaginary part} = 0 \end{aligned}$$

$$P_{\alpha\beta} = 4 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \left(\frac{\Delta m^2 x}{4E} \right), \quad \alpha \neq \beta$$

The previous approximation $\Delta m^2 \rightarrow 0$ is sometimes called "one-dominant mass-scale approximation", implying that Δm^2 is the only relevant mass splitting. In this approximation, it is useful to work out some probabilities in the PDG convention for the PMNS matrix:

$$P_{\mu\tau} = \cos^4 \theta_{13} \sin^2 2\theta_{23} \sin^2 \left(\frac{\Delta m^2 x}{4E} \right)$$

$$P_{\mu\mu} = 1 - 4 C_{13}^2 S_{23}^2 (1 - C_{13}^2 S_{23}^2) \sin^2 \left(\frac{\Delta m^2 x}{4E} \right)$$

$$P_{\mu e} = S_{23}^2 \cdot \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 x}{4E} \right)$$

$$P_{ee} = 1 - \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 x}{4E} \right)$$

The angle θ_{13} is relatively small; in the limit $\theta_{13} \rightarrow 0$ one gets the 2ν limits:

$$\begin{aligned} P_{\mu\tau} &\simeq \sin^2 2\theta_{23} \sin^2 \left(\frac{\Delta m^2 x}{4E} \right) \\ P_{\mu\mu} &\simeq 1 - \sin^2 2\theta_{23} \sin \left(\frac{\Delta m^2 x}{4E} \right) \end{aligned} \quad \left. \right\} \text{"Pontecorvo" formulae for } 2\nu \text{ oscillations}$$

$$\left(\text{also: } \begin{aligned} P_{\mu e} &\simeq 0 \\ P_{ee} &\simeq 0 \end{aligned} \right)$$

Exercise : P_{ee} in vacuum for $(\frac{\Delta m^2 x}{4E}) \gg 1$ and $(\frac{\Delta m^2 x}{4E}) \sim \Theta(1)$

Previously we have considered experiments with sensitivity to Δm^2 in the limit $\delta m^2 \rightarrow 0$. At the other end of the spectrum, there are a few experiments with leading sensitivity to δm^2 , for which one can take the limit $\Delta m^2 \rightarrow \infty$ or better:

$$\frac{\delta m^2 x}{4E} \sim \Theta(1) \quad \text{and} \quad \frac{\Delta m^2 x}{4E} \gg 1.$$

Since Δm^2 is "small", the energy E must also be small to have $\delta m^2 x / 4E \sim \Theta(1)$. In practice, these experiments work at $E \sim$ few MeV, well below the threshold for μ or τ production via charged-current processes. An example is the KamLAND experiment using $\bar{\nu}_e$ from reactors. In this case, the main observable is just P_{ee} . Prove that:

$$P_{ee} \simeq \cos^4 \theta_{13} \left[1 - \sin^2 2\theta_{12} \sin^2 \left(\frac{\delta m^2 x}{4E} \right) \right] + \sin^4 \theta_{13}$$

and note that it does not depend on v/\bar{v} , δ , sign($\pm \Delta m^2$).

Solution -

For $\bar{\nu}_e$ (disappearance) it is $P_{\bar{\nu}e} = 0$

For $\frac{\Delta m^2 x}{4E} \gg 1$, the oscillating factor becomes on average: $\sin^2\left(\frac{\Delta m^2 x}{4E}\right) \approx \frac{1}{2} \approx \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$

Then:

$$\begin{aligned} P(\bar{\nu}_e \rightarrow \bar{\nu}_e) &= 1 - 4 \operatorname{Re}(J_{ee}^{12}) \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 2 \operatorname{Re}(J_{ee}^{13} + J_{ee}^{23}) \\ &= 1 - 4 |U_{e1}|^2 |U_{e2}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 2 |U_{e3}|^2 (|U_{e1}|^2 + |U_{e2}|^2) \\ &= 1 - 4 C_{13}^4 S_{12}^2 C_{12}^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 2 S_{13}^2 C_{13}^2 \\ &= C_{13}^4 + S_{13}^4 - 4 C_{13}^2 S_{12}^2 C_{12}^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) \\ &= C_{13}^4 P_{ee}^{2\nu} + S_{13}^4 \end{aligned}$$

where we have defined the 2ν oscillation limit as $P_{ee}^{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$

Namely, the 3ν probability $P_{ee}^{3\nu}$ is related to the 2ν one (for $\theta_{13}=0$ limit) by the relation:

$$P_{ee}^{3\nu} = \cos^4 \theta_{13} P_{ee}^{2\nu} + \sin^4 \theta_{13}$$

It turns out that this relation between $P_{ee}^{3\nu}$ and $P_{ee}^{2\nu}$ also holds for $\bar{\nu}_e$ propagation in matter at low energy (\sim MeV), e.g., for solar neutrinos -

Exercise : Pee in vacuum, general 3ν case

Calculate Pee in vacuum in the general 3ν case, in terms of θ_{ij} , δm^2 , $\pm \Delta m^2$ and prove that it is not invariant under change of hierarchy : $+\Delta m^2 \rightarrow -\Delta m^2$.

(This implies that precision reactor experiments may be sensitive to the hierarchy)

Solution — Let us consider normal hierarchy ($+\Delta m^2$) for definiteness. Then:

$$m_2^2 - m_3^2 = \delta m^2; \quad m_3^2 - m_2^2 = \Delta m^2 - \delta m^2/2; \quad m_3^2 - m_1^2 = \Delta m^2 + \delta m^2/2. \quad (\Delta m^2 \stackrel{\text{def}}{=} \frac{1}{2}(\Delta m_{31}^2 + \Delta m_{32}^2))$$

$$\text{Im}(J_{ee}^{ii}) = 0; \quad \text{Re}(J_{ee}^{ii}) = |\langle e i | \bar{e} j \rangle|^2 = \begin{cases} S_{12}^2 C_{12}^2 C_{13}^4 & ij = 12 \\ S_{12}^2 S_{13}^2 C_{13}^2 & ij = 23 \\ C_{12}^2 S_{13}^2 C_{13}^2 & ij = 13 \end{cases}; \quad \text{then:}$$

$$\boxed{\begin{aligned} \text{Pee}^{3\nu} = & 1 - \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \left(\frac{\delta m^2 x}{4E} \right) \\ & - \sin^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 - \frac{\delta m^2}{2}}{4E} x \right) \\ & - \cos^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 + \frac{\delta m^2}{2}}{4E} x \right) \end{aligned}}$$

Note that the above $\text{Pee}^{3\nu}$ is not invariant under the replacement $\Delta m^2 \rightarrow -\Delta m^2$. It would be so only for $\theta_{12} = \pi/4$ (i.e. $\sin^2 \theta_{12} = \frac{1}{2} = \cos^2 \theta_{12}$) which, however, is experimentally excluded ($\sin^2 \theta_{12} \approx 0.3 < 1/2$).

Exercise: $P_{ee}^{3\nu}$ in vacuum - general case in alternative formulation

Prove that $P_{ee}^{3\nu}$ can be recast in the following form:

$$P_{ee}^{3\nu} = C_{13}^4 P_{ee}^{2\nu} + S_{13}^4 + 2S_{13}^2 C_{13}^2 \sqrt{P_{ee}^{2\nu}} \cos \left(\frac{\Delta m_{ee}^2 x}{2E} \pm \varphi \right) \quad (+ = NH) \quad (- = IH)$$

$$\text{where } P_{ee}^{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2 \left(\frac{\delta m^2 x}{4E} \right)$$

$$\text{and } \Delta m_{ee}^2 = C_{12}^2 \Delta m_{31}^2 + S_{12}^2 \Delta m_{32}^2 = \Delta m^2 \pm \frac{1}{2} (C_{12}^2 - S_{12}^2) \delta m^2$$

$$\text{with } \begin{cases} \cos \varphi = [C_{12}^2 \cos(2S_{12}^2 \Delta_{21}) + S_{12}^2 \cos(2C_{12}^2 \Delta_{21})] / \sqrt{P_{ee}^{2\nu}} \\ \sin \varphi = [C_{12}^2 \sin(2S_{12}^2 \Delta_{21}) - S_{12}^2 \sin(2C_{12}^2 \Delta_{21})] / \sqrt{P_{ee}^{2\nu}} \end{cases} \quad \leftarrow \Delta_{21} = \frac{\delta m^2 x}{4E}$$

This formulation of $P_{ee}^{3\nu}$ emphasizes the physical effect of the mass hierarchy, namely, the fact that NH (IH) induces an advancement (retardation) of phase φ , with respect to the dominant "phase" induced by the effective mass parameter Δm_{ee}^2 . It is particularly useful in the discussion of future medium-baseline reactor experiments sensitive to the hierarchy, see e.g. eq.(2.1) of arXiv 1507.05613.

Solution -

Assume NH for the moment. (For IH, just flip the relative sign of Δm^2 and δm^2).

Definitions : $\Delta m^2_{ij} = m_i^2 - m_j^2$; $\Delta_{ij} = \Delta m^2_{ij} \times /4E$

$$\Delta m^2_{ee} = c_{12}^2 \Delta m^2_{31} + s_{12}^2 \Delta m^2_{32}; \Delta_{ee} = \Delta m^2_{ee} \times /4E$$

$$\rightarrow \begin{cases} \Delta m^2_{31} = \Delta m^2_{ee} + s_{12}^2 \delta m^2 = \Delta m^2 + \delta m^2/2 \\ \Delta m^2_{32} = \Delta m^2_{ee} - c_{12}^2 \delta m^2 = \Delta m^2 - \delta m^2/2 \\ \Delta m^2 = \Delta m^2_{ee} - \frac{1}{2} (c_{12}^2 - s_{12}^2) \delta m^2 \end{cases}$$

Then the $P_{ee}^{3\nu}$ obtained in the previous exercise can be re-written as:

$$\begin{aligned} P_{ee}^{3\nu} &= 1 - c_{13}^4 (1 - P_{ee}^{2\nu}) - \sin^2 2\theta_{13} [s_{12}^2 \sin^2 (\Delta_{ee} - c_{12}^2 \Delta_{21}) + c_{12}^2 \sin^2 (\Delta_{ee} + s_{12}^2 \Delta_{21})] \\ &= 1 - c_{13}^4 + c_{13}^4 P_{ee}^{2\nu} + \frac{1}{2} \sin^2 2\theta_{13} [s_{12}^2 \cos (2\Delta_{ee} - 2c_{12}^2 \Delta_{21}) + c_{12}^2 \cos (2\Delta_{ee} + 2s_{12}^2 \Delta_{21}) - 1] \\ &= c_{13}^4 P_{ee}^{2\nu} + s_{13}^4 + 2s_{13}^2 c_{13}^2 [s_{12}^2 \cos (2\Delta_{ee} - 2c_{12}^2 \Delta_{21}) + c_{12}^2 \cos (2\Delta_{ee} + 2s_{12}^2 \Delta_{21})] \end{aligned}$$

Let us recast the last term in [...] in the following form:

$$s_{12}^2 \cos (2\Delta_{ee} - 2c_{12}^2 \Delta_{21}) + c_{12}^2 \cos (2\Delta_{ee} + 2s_{12}^2 \Delta_{21}) = \eta \cos (2\Delta_{ee} + \varphi)$$

with the amplitude η and phase φ to be determined.

In other words, we are summing two "waves" with oscillating phases slightly different from $2\Delta_{ee}$, into a "single wave" with a phase which is also slightly different from $2\Delta_{ee}$.

In order to fulfill the previous eq. in (η, φ) it must be:

$$\begin{aligned} & s_{12}^2 \cos(2\Delta_{ee}) \cos(2c_{12}^2 \Delta_{21}) + s_{12}^2 \sin(2\Delta_{ee}) \sin(2c_{12}^2 \Delta_{21}) \\ & + c_{12}^2 \cos(2\Delta_{ee}) \cos(2s_{12}^2 \Delta_{21}) - c_{12}^2 \sin(2\Delta_{ee}) \sin(2s_{12}^2 \Delta_{21}) \\ & = \eta \cos(2\Delta_{ee}) \cos \varphi - \eta \sin(2\Delta_{ee}) \sin \varphi \end{aligned}$$

and thus:

$$\begin{aligned} & \cos(2\Delta_{ee}) [s_{12}^2 \cos(2c_{12}^2 \Delta_{21}) + c_{12}^2 \cos(2s_{12}^2 \Delta_{21}) - \eta \cos \varphi] \\ & = \sin(2\Delta_{ee}) [-s_{12}^2 \sin(2c_{12}^2 \Delta_{21}) + c_{12}^2 \sin(2s_{12}^2 \Delta_{21}) - \eta \sin \varphi] \end{aligned}$$

which, in general, can be solved only if the terms in [...] are both vanishing:

$$\rightarrow \begin{cases} s_{12}^2 \cos(2c_{12}^2 \Delta_{21}) + c_{12}^2 (2s_{12}^2 \Delta_{21}) = \eta \cos \varphi \\ s_{12}^2 \sin(2c_{12}^2 \Delta_{21}) - c_{12}^2 (2s_{12}^2 \Delta_{21}) = -\eta \sin \varphi \end{cases}$$

If we square and sum, we get:

$$\eta^2 = s_{12}^4 + c_{12}^4 + 2s_{12}^2 c_{12}^2 [\cos(2\Delta_{12} c_{12}^2) \cos(2\Delta_{21} s_{12}^2) - \sin(2\Delta_{12} c_{12}^2) \sin(2\Delta_{21} s_{12}^2)]$$

where the term in [...] can be simplified by noticing that:

$$\begin{aligned} \sin^2(\Delta_{21}) &= \sin^2(\Delta_{21} (c_{12}^2 + s_{12}^2)) = \frac{1}{2} (1 - \cos 2(\Delta_{21} (c_{12}^2 + s_{12}^2))) \\ &= \frac{1}{2} - \frac{1}{2} [\cos(2\Delta_{21} c_{12}^2) \cos(2\Delta_{21} s_{12}^2) - \sin(2\Delta_{21} c_{12}^2) \sin(2\Delta_{21} s_{12}^2)] \end{aligned}$$

$$\begin{aligned} \rightarrow \eta^2 &= s_{12}^4 + c_{12}^4 + 4s_{12}^2 c_{12}^2 \left[\frac{1}{2} - \sin^2(\Delta_{21}) \right] \\ &= 1 - 4s_{12}^2 c_{12}^2 \sin^2(\Delta_{21}) \equiv P_{ee}^{vv} \end{aligned}$$

Therefore, it is also:

$$\begin{cases} \cos \varphi = \frac{1}{\sqrt{P_{ee}^{2\nu}}} (c_{12}^2 \cos(2s_{12}^2 \Delta_{21}) + s_{12}^2 \cos(2c_{12}^2 \Delta_{21})) \\ \sin \varphi = \frac{1}{\sqrt{P_{ee}^{2\nu}}} (c_{12}^2 \sin(2s_{12}^2 \Delta_{21}) - s_{12}^2 \sin(2c_{12}^2 \Delta_{21})) \end{cases}$$

which completes the proof.

For IH: $\varphi \rightarrow -\varphi$.

Sketchy proof of $V = \sqrt{2} G_F N_e$ in matter

From the CC hamiltonian of the Standard Model,

$$\mathcal{H}_{CC} = \frac{G_F}{\sqrt{2}} \bar{e} \gamma^\mu (1 - \gamma_5) \nu_e \cdot \bar{\nu}_e \gamma_\mu (1 - \gamma_5) e \quad \leftarrow J_{CC} \otimes J_{CC}$$

$$= \frac{G_F}{\sqrt{2}} \bar{e} \gamma^\mu (1 - \gamma_5) e \cdot \bar{\nu}_e \gamma_\mu (1 - \gamma_5) \nu_e \quad \leftarrow J_e \otimes J_\nu$$

From the neutrino viewpoint, the e^- is \sim nonrelativistic and \sim unpolarized ($\langle \vec{\sigma} \rangle = 0$) in ordinary matter. It is then useful to use the Dirac representation for the electron, where only the upper components of the field e survive: $e \simeq \begin{bmatrix} \xi \\ 0 \end{bmatrix}$ where ξ = Pauli spinor.

$$\text{Then: } \bar{e} \gamma^\mu (1 - \gamma_5) e \simeq (\underbrace{\xi^+ \xi}_\text{density N_e}, \underbrace{\xi^+ \vec{\sigma} \xi}_\text{polarization} \sim 0) \simeq N_e \delta_{\mu 0}$$

$$\text{and } \mathcal{H}_{CC} \simeq \frac{G_F}{\sqrt{2}} N_e \bar{\nu}_e \gamma_0 (1 - \gamma_5) \nu_e = [\underbrace{\sqrt{2} G_F N_e}_\text{Coupling}] \cdot \underbrace{\bar{\nu}_e L \gamma_0 \nu_e L}_\text{}$$

which adds $\sqrt{2} G_F N_e$ to the ν_e energy in the evolution equation.

Exercise: changing units in matter

For $A = 2\sqrt{2} G_F N_e E$ (N_e = electron number density)

Prove that:

$$\boxed{\frac{A}{\Delta m_{ij}^2} = 1.526 \times 10^{-7} \left(\frac{N_e}{\text{mol/cm}^3} \right) \left(\frac{E}{\text{MeV}} \right) \left(\frac{eV^2}{\Delta m_{ij}^2} \right)}$$

Solution -

$$1 \text{ mol} = 6.022 \times 10^{23} \text{ particles.}$$

$$1 \frac{\text{mol}}{\text{cm}^3} = \frac{6.022 \times 10^{23}}{10^{-6} \text{ m}^3} \frac{\text{MeV}^3}{\text{MeV}^3} = 6.022 \times 10^{29} \frac{1}{(\text{m} \cdot \text{MeV})^3} \text{ MeV}^3$$

$$= \frac{6.022 \times 10^{29}}{(5.0677 \times 10^{12})^3} \text{ MeV}^3 = 4.627 \times 10^{-9} \text{ MeV}^3 \quad \begin{cases} \text{in natural units } \hbar c = 1 \\ = 197.327 \text{ MeV} \cdot \text{fm} \end{cases}$$

$$G_F = 1.16637 \times 10^{-5} \text{ GeV}^{-2} = 1.16637 \times 10^{-11} \text{ MeV}^{-2}$$

$$\frac{2\sqrt{2} G_F N_e E}{\Delta m_{ij}^2} = 2\sqrt{2} (1.1664 \times 10^{-11} \text{ MeV}^{-2}) \left(\frac{N_e}{\text{mol/cm}^3} \text{ mol/cm}^3 \right) \left(\frac{E}{\text{MeV} \cdot \text{MeV}} \right) \left(\frac{eV^2}{\Delta m_{ij}^2} \cdot \frac{1}{eV^2} \right)$$

$$= 3.299 \times 10^{-11} \frac{\text{MeV}^{-2} \text{ MeV}}{eV^2} \frac{\text{mol}}{\text{cm}^3} \left(\frac{N_e}{\text{mol/cm}^3} \right) \left(\frac{E}{\text{MeV}} \right) \left(\frac{eV^2}{\Delta m_{ij}^2} \right)$$

$$3.299 \times 10^{-11} \frac{\text{MeV}^{-2} \text{ MeV}}{eV^2} \frac{\text{mol}}{\text{cm}^3} = 3.299 \times 10^{-11} \frac{10^{12}}{\text{MeV}^3} \times 4.627 \times 10^{-9} \text{ MeV}^3 = 1.526 \times 10^{-7}$$

Exercise : 2ν oscillations in matter with constant density

Let us consider only the (ν_1, ν_2) states with oscillation parameters $(\tilde{\delta m^2}, \theta_{12}) \neq 0$ and $\theta_{13} \approx 0$ (2ν limit).

Prove that, in this limit, the $\bar{\nu}_e \rightarrow \nu_e$ probability in matter \tilde{P} takes the form:

$$\tilde{P}_{\bar{\nu}e}^{2\nu} = 1 - \sin^2 2\tilde{\theta}_{12} \sin^2 \left(\frac{\tilde{\delta m^2} x}{4E} \right) \quad \text{for } N_e = \text{constant}$$

i.e., it has the same vacuum-like structure, but with the replacement:

$$\sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}} , \quad \tilde{\delta m^2} = \delta m^2 \frac{\sin 2\theta_{12}}{\sin 2\tilde{\theta}_{12}}$$

where $A = \pm 2\sqrt{2} G_F N_e E$ (+ for ν , - for $\bar{\nu}$)

In the 2ν limit governed by the oscillation parameters $(\delta m^2, \theta_{12})$,
the hamiltonian of γ propagation in matter is, in flavor basis:

$$\begin{aligned}\tilde{H} = H + \begin{pmatrix} V_0 \\ 0 \end{pmatrix} &= \frac{1}{2E} U \begin{pmatrix} m_1^2 & m_2^2 \\ m_2^2 & m_1^2 \end{pmatrix} U^\top + \begin{pmatrix} V_0 \\ 0 \end{pmatrix} \quad \leftarrow U = U^* = \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \\ \text{matter} \quad \text{vac.} & \\ &= \frac{1}{2E} \left[\begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right]\end{aligned}$$

It is convenient to extract the part proportional to the trace ($\propto 1$) and make \tilde{H} traceless:

$$\tilde{H} = \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{12} \delta m^2 & \sin 2\theta_{12} \delta m^2 \\ \sin 2\theta_{12} \delta m^2 & -A + \cos 2\theta_{12} \delta m^2 \end{bmatrix}$$

which has eigenvalues $\pm \frac{\tilde{\delta m}^2}{4E}$, with $\tilde{\delta m}^2 = \delta m^2 \sqrt{(\cos 2\theta_{12} - \frac{A}{\delta m^2})^2 + \sin^2 2\theta_{12}}$.

The diagonalizing rotation is:

$$\tilde{H} = \begin{pmatrix} \cos \tilde{\theta}_{12} & \sin \tilde{\theta}_{12} \\ -\sin \tilde{\theta}_{12} & \cos \tilde{\theta}_{12} \end{pmatrix} \begin{pmatrix} -\frac{\tilde{\delta m}^2}{4E} & 0 \\ 0 & +\frac{\tilde{\delta m}^2}{4E} \end{pmatrix} \begin{pmatrix} \cos \tilde{\theta}_{12} & -\sin \tilde{\theta}_{12} \\ +\sin \tilde{\theta}_{12} & \cos \tilde{\theta}_{12} \end{pmatrix} = \tilde{U} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \tilde{U}^\top$$

$$\text{where } \sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - \frac{A}{\delta m^2})^2 + \sin^2 2\theta_{12}}} ; \quad \cos 2\tilde{\theta}_{12} = \frac{\cos 2\theta_{12} - A/\delta m^2}{\sqrt{(\cos 2\theta_{12} - \frac{A}{\delta m^2})^2 + \sin^2 2\theta_{12}}}$$

so that: $\tilde{\delta m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$

The evolution operator in matter is obtained by exponentiating \tilde{H} :

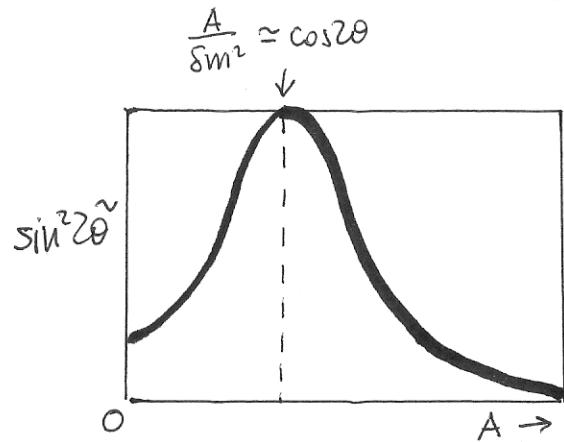
$$\tilde{S} = e^{-i\tilde{H}x} = \tilde{U} \begin{pmatrix} e^{i\frac{\delta\tilde{m}^2 x}{4\varepsilon}} & \\ & e^{-i\frac{\delta\tilde{m}^2 x}{4\varepsilon}} \end{pmatrix} \tilde{U}^\dagger = \cos\left(\frac{\delta\tilde{m}^2 x}{4\varepsilon}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin\left(\frac{\delta\tilde{m}^2 x}{4\varepsilon}\right) \begin{pmatrix} -\cos 2\tilde{\theta}_{12} & \sin 2\tilde{\theta}_{12} \\ \sin 2\tilde{\theta}_{12} & \cos 2\tilde{\theta}_{12} \end{pmatrix}$$

By squaring the diagonal element of \tilde{S} one gets the survival probability in matter:

$$\tilde{P}_{ee}^{vv} = |\tilde{S}_{\text{diag}}|^2 = 1 - \sin^2 2\tilde{\theta}_{12} \sin^2\left(\frac{\delta\tilde{m}^2 x}{4\varepsilon}\right)$$

By squaring the off-diagonal element one would get the complementary transition probability.

QUALITATIVE BEHAVIOR OF $\tilde{m}_{1,2}^2$ and $\tilde{\theta} = \tilde{\theta}_{12}$:



Mykheev-Smirnov-Wolfenstein (MSW) resonance:

For $A/\delta m^2 > 0$, the effective parameters have a resonant behavior around $A/\delta m^2 \approx \cos 2\theta$

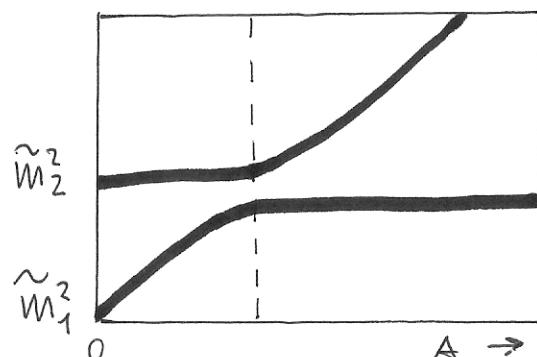
(only for ν : no resonance for $\bar{\nu}$, since $A < 0$ for $\bar{\nu}$)

Limiting cases:

$$A/\delta m^2 \ll 1 \rightarrow (\delta\tilde{m}^2, \tilde{\theta}) \approx (\delta m^2, \theta) \quad \leftarrow \text{vacuum-like}$$

$$A/\delta m^2 \approx \cos 2\theta \rightarrow (\delta\tilde{m}^2, \tilde{\theta}) \approx (\delta m^2 \sin 2\theta, \pi/4) \quad \leftarrow \text{resonance}$$

$$A/\delta m^2 \gg 1 \rightarrow (\delta\tilde{m}^2, \tilde{\theta}) \approx (A, \pi/2) \quad \leftarrow \text{matter dominance}$$



Generally: expect large matter effects for $A/\delta m^2 \sim \mathcal{O}(1)$.

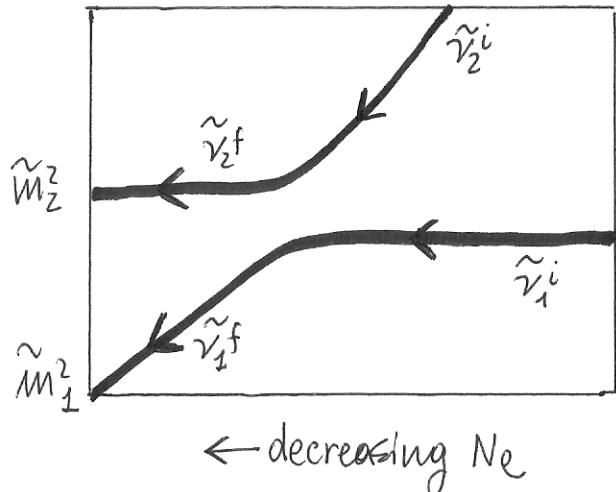
Exercise: 2ν oscillations in matter with slowly varying density

If $N_e = N_e(x)$ changes slowly from $x=x_i$ (with $\tilde{\theta}=\tilde{\theta}_i$) to $x=x_f$ (with $\tilde{\theta}=\tilde{\theta}_f$) while the \tilde{m}^2 -driven oscillations are fast, the averaged \tilde{P}_{ee} probability takes the form:

$$\tilde{P}_{ee} \approx \cos^2 \tilde{\theta}_i \cos^2 \tilde{\theta}_f + \sin^2 \tilde{\theta}_i \sin^2 \tilde{\theta}_f \quad \leftarrow \text{"adiabatic" approximation}$$

Solution. For a quasi-constant hamiltonian, one can solve the evolution equation "one x " at a time, and then patch the solutions from x_i to x_f . Namely, given the initial state $|\nu_e^i\rangle = \cos \tilde{\theta}_i |\nu_1^i\rangle + \sin \tilde{\theta}_i |\nu_2^i\rangle$, the effective mass eigenstates $|\tilde{\nu}_1^i\rangle$ and $|\tilde{\nu}_2^i\rangle$ at x_i slowly transform into $|\tilde{\nu}_1^f\rangle$ and $|\tilde{\nu}_2^f\rangle$ at x_f , respectively.

The mass² eigenstate trajectories for decreasing A or N_e are then:



$$\leftarrow |\langle \tilde{\nu}_1^f | \nu_1^i \rangle| = 1 = |\langle \tilde{\nu}_2^f | \nu_2^i \rangle|$$

and $|\langle \tilde{\nu}_{2,1}^f | \tilde{\nu}_{1,2}^i \rangle| = 0$ (no "level crossing")

Therefore:

$$\tilde{P}_{ee} = |\langle \tilde{\nu}_e^f | \tilde{\nu}_e^i \rangle|^2 = |\cos \tilde{\theta}_i \cos \tilde{\theta}_f \langle \tilde{\nu}_1^f | \tilde{\nu}_1^i \rangle + \sin \tilde{\theta}_i \sin \tilde{\theta}_f \langle \tilde{\nu}_2^f | \tilde{\nu}_2^i \rangle|^2$$

Averaging out interference terms after many oscillations:

$$\begin{aligned} \tilde{P}_{ee} &\approx \cos^2 \tilde{\theta}_i \cos^2 \tilde{\theta}_f |\langle \tilde{\nu}_1^f | \tilde{\nu}_1^i \rangle|^2 + \sin^2 \tilde{\theta}_i \sin^2 \tilde{\theta}_f |\langle \tilde{\nu}_2^f | \tilde{\nu}_2^i \rangle|^2 \\ &= \cos^2 \tilde{\theta}_i \cos^2 \tilde{\theta}_f + \sin^2 \tilde{\theta}_i \sin^2 \tilde{\theta}_f \end{aligned}$$

→ Application to solar ν_e

It turns out that, for the $(\delta m^2, \theta_{12})$ values chosen by nature, the adiabatic approximation can be applied to solar ν_e . In this case, $\tilde{\theta}_{12}(x_f) = \theta_{12}$ (vacuum value at the exit from the Sun) while $\tilde{\theta}_{12}(x_i)$ must be evaluated at the production point x_i . Limiting cases:

- Vacuum dominance, $E \lesssim \text{few MeV}$

In this case $A/\delta m^2 \lesssim 1$ and $\tilde{\theta}_{12}(x_i) \approx \theta_{12}$, so that:

$$\tilde{P}_{ee}^{2\nu} \approx C_{12}^4 + S_{12}^4 = 1 - \frac{1}{2} \sin^2 2\theta_{12}$$

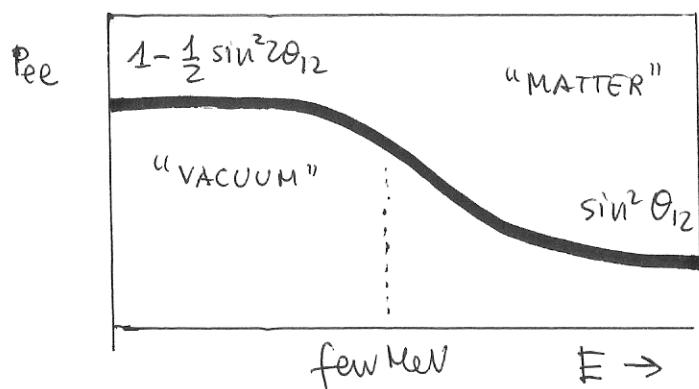
which is the usual averaged vacuum probability (octant-symmetric in θ_{12})

- Matter dominance, $E \gtrsim \text{few MeV}$

In this case $A/\delta m^2 \gtrsim 1$ and $\tilde{\theta}_{12}(x_i) \approx \pi/2$, so that:

$$\tilde{P}_{ee}^{2\nu} \approx \sin^2 \theta_{12}$$

which is the matter-dominated limit, octant-asymmetric in θ_{12} .



← General behaviour of $\tilde{P}_{ee}^{2\nu}(E)$

The transition from "low" to "high" E is a signature of matter effects in the Sun.

Thanks to matter effects we can determine the octant of the mixing angle θ_{12} .

3ν oscillations in matter: general reduction tools

- 3ν Hamiltonian in flavor basis for generic $N_e(x)$ density profile:

$$\tilde{H} = \frac{1}{2E} U M^2 U^\dagger + V \quad \text{where}$$

$$M^2 = \text{diag}(m_1^2, m_2^2, m_3^2) \quad \text{and}$$

$$U = O_{23} \Gamma_\delta O_{13} \Gamma_\delta^\dagger O_{12} \quad \text{with } \Gamma_\delta = \text{diag}(1, 1, e^{i\delta}) \quad \text{and } O_{ij}^\dagger = O_{ij}^T$$

$$V(x) = \text{diag}(\sqrt{2} G_F N_e(x), 0, 0)$$

- It is easy to prove that:

$$(O_{23} \Gamma_\delta)^\dagger V (O_{23} \Gamma_\delta) = V$$

$$\Gamma_\delta^\dagger O_{12} M^2 O_{12}^T \Gamma_\delta = O_{12} M^2 O_{12}^T$$

- Let's go from the flavor basis to a new "primed flavor basis" defined as:

$$\begin{bmatrix} (\nu^e)' \\ (\nu^u)' \\ (\nu^d)' \end{bmatrix} = (O_{23} \Gamma_\delta)^\dagger \begin{bmatrix} \nu^e \\ \nu^u \\ \nu^d \end{bmatrix} \leftarrow \text{components, with } O_{23} \Gamma_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} & c_{23} e^{i\delta} \end{pmatrix} \quad \begin{pmatrix} \text{(note:} \\ (\nu^e)' = (\nu^e) \end{pmatrix}$$

- Hamiltonian in the primed basis:

$$\tilde{H}' = (O_{23} \Gamma_\delta)^\dagger \tilde{H} (O_{23} \Gamma_\delta) = O_{13} O_{12} \frac{M^2}{2E} (O_{13} O_{12})^T + V$$

Such \tilde{H}' does not depend on δ (and is thus real symmetric), nor on O_{23} .

It is thus simpler to find the evolution operator \tilde{S}' in the primed basis.

- Given \tilde{S}' in the primed basis, the evolution operator \tilde{S} in flavor basis is:
- $\tilde{S}(x_f, x_i) = (O_{23} \Gamma_\delta) \tilde{S}'(x_f, x_i) (O_{23} \Gamma_\delta)^+$
- In terms of matrix components:

if $\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & \tilde{S}'_{e\mu} & \tilde{S}'_{e\tau} \\ \tilde{S}'_{\mu e} & \tilde{S}'_{\mu\mu} & \tilde{S}'_{\mu\tau} \\ \tilde{S}'_{\tau e} & \tilde{S}'_{\tau\mu} & \tilde{S}'_{\tau\tau} \end{pmatrix}$, then $\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} & c_{23} e^{i\delta} \end{pmatrix} \tilde{S}' \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} e^{-i\delta} & c_{23} e^{-i\delta} \end{pmatrix}$

$$\tilde{S}_{ee} = \tilde{S}'_{ee}$$

$$\tilde{S}_{e\mu} = \tilde{S}'_{e\mu} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}$$

$$\tilde{S}_{\tau e} = -\tilde{S}'_{e\mu} s_{23} + \tilde{S}'_{\tau e} c_{23} e^{i\delta}$$

$$\tilde{S}_{\mu\mu} = \tilde{S}'_{\mu\mu} c_{23}^2 + \tilde{S}'_{\mu\tau} c_{23} s_{23} e^{-i\delta} + \tilde{S}'_{\tau\mu} c_{23} s_{23} e^{i\delta} + \tilde{S}'_{\tau\tau} s_{23}^2$$

$$\tilde{S}_{\tau\mu} = -\tilde{S}'_{\mu\mu} c_{23} s_{23} - \tilde{S}'_{\mu\tau} s_{23}^2 e^{-i\delta} + \tilde{S}'_{\tau\mu} c_{23}^2 e^{i\delta} + \tilde{S}'_{\tau\tau} c_{23} s_{23}$$

$$\tilde{S}_{\tau\tau} = \tilde{S}'_{\mu\mu} s_{23}^2 - \tilde{S}'_{\mu\tau} c_{23} s_{23} - \tilde{S}'_{\tau\mu} c_{23} s_{23} e^{i\delta} + \tilde{S}'_{\tau\tau} c_{23}^2$$

with $\tilde{S}_{e\mu}, \tilde{S}_{e\tau}, \tilde{S}_{\mu\tau}$ obtained by $\tilde{S}'_{\alpha\beta} \leftrightarrow \tilde{S}'_{\beta\alpha}$ and $\delta \leftrightarrow -\delta$

- In general, $\tilde{S}'_{\alpha\beta} \neq \tilde{S}'_{\beta\alpha}$, even if $\tilde{H}'_{\alpha\beta} = \tilde{H}'_{\beta\alpha}$ (real symmetric).
Indeed, let us divide a generic $N_e(x)$ profile into steps $\{\Delta x_i\}_{i=1,\dots,N}$ with constant N_e in each step. Then:

$$\tilde{S}' = e^{-i\tilde{H}'_N \Delta x_N} e^{-i\tilde{H}'_{N-1} \Delta x_{N-1}} \dots e^{-i\tilde{H}'_2 \Delta x_2} e^{-i\tilde{H}'_1 \Delta x_1}.$$

Then, although $(H'_i)^T = H'_i$, the transpose of \tilde{S}' is not equal to \tilde{S}' , since the ordering of the steps is reversed from $1, \dots, N$ to $N, \dots, 1$ ("reversed" profile):
 $(\tilde{S}')^T = e^{-i\tilde{H}_1 \Delta x_1} \dots e^{-i\tilde{H}_N \Delta x_N}$. In other words:

$$\tilde{S}'_{\alpha\beta} [\text{direct profile}] = \tilde{S}'_{\beta\alpha} [\text{reverse profile}] \neq \tilde{S}'_{\beta\alpha} [\text{direct profile}].$$

Only if the direct and reverse profile are symmetrical (= coincide upon reflection) then it is: $\tilde{S}'_{\alpha\beta} [\text{symmetric profile}] = \tilde{S}'_{\beta\alpha} [\text{symmetric profile}]$.

This is true, in particular, for constant density N_e :

$$\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha} \text{ for } N_e = \text{const.}$$

In this case: $\tilde{S}_{\alpha\beta} = \tilde{S}_{\beta\alpha} (\delta \rightarrow -\delta)$ and $P_{\alpha\beta} = P_{\beta\alpha} (\delta \rightarrow -\delta)$
since $P_{\alpha\beta} = |\tilde{S}_{\beta\alpha}|^2 = P(\gamma_\alpha \rightarrow \gamma_\beta)$.

- Further reductions come from some peculiar symmetries of $\tilde{S}_{\alpha\beta}$ under the substitutions $s_{23} \rightarrow \pm c_{23}$ and $c_{23} \rightarrow \mp s_{23}$:

$$\tilde{S}_{\tau\mu} = \pm \tilde{S}_{\mu\tau} \left| \begin{array}{l} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{array} \right. \Rightarrow P_{\tau\mu} = P_{\mu\tau} \left| \begin{array}{l} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{array} \right. \stackrel{\text{def}}{=} P'_{\mu\tau}$$

$$\tilde{S}_{\mu\tau} = \mp \tilde{S}_{\tau\mu} \left| \begin{array}{l} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{array} \right. \Rightarrow P_{\tau\mu} = P_{\mu\tau} \left| \begin{array}{l} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{array} \right. \stackrel{\text{def}}{=} P'_{\mu\tau}$$

$$\tilde{S}_{\mu\mu} = \pm \tilde{S}_{\tau\tau} \left| \begin{array}{l} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{array} \right. \Rightarrow P_{\mu\mu} = P_{\tau\tau} \left| \begin{array}{l} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{array} \right. \stackrel{\text{def}}{=} P'_{\tau\tau}$$

The previous relations, together with the unitarity of $P_{\alpha\beta}$, allow to express all the probabilities in terms of just two, e.g., $P_{e\mu}$ and $P_{\mu\tau}$, and their transformed $P'_{e\mu} \Big|_{\substack{S_{23} \rightarrow \pm c_{23} \\ C_{23} \rightarrow \mp S_{23}}} \quad \text{and} \quad P'_{\mu\tau} \Big|_{\substack{S_{23} \rightarrow \pm c_{23} \\ C_{23} \rightarrow \mp S_{23}}}$

[It is equivalent to choose the upper or the lower substitution.]

Explicitly :

$$P_{ee} = 1 - P_{e\mu} - P_{e\tau} = 1 - P_{e\mu} - P'_{e\mu}$$

$$P_{e\tau} = P'_{e\mu}$$

$$P_{\mu e} = 1 - P_{\mu\mu} - P_{\mu\tau} = 1 - P_{\mu\mu} - P_{e\mu} + P_{e\mu} - P_{\mu\tau} = P_{e\mu} + P_{\tau\mu} - P_{\mu\tau} = P_{e\mu} + P'_{\mu\tau} - P_{\mu\tau}$$

$$P_{\mu\mu} = 1 - P_{e\mu} - P_{\tau\mu} = 1 - P_{e\mu} - P'_{\mu\tau}$$

$$P_{\tau\mu} = P'_{\mu\tau}$$

$$P_{\tau\tau} = 1 - P_{e\tau} - P_{\mu\tau} = 1 - P'_{e\mu} - P_{\mu\tau}$$

$$\text{We also have : } P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) = P(\nu_\alpha \rightarrow \nu_\beta \mid V \rightarrow -V; \delta \rightarrow -\delta)$$

$$\text{and, in constant density : } P_{\alpha\beta} = P_{\beta\alpha} (\delta \rightarrow -\delta).$$

Such relations allow to reduce the calculation of $P_{\alpha\beta}(v, \bar{v})$ to a few independent probabilities. In the following we shall calculate one of them, $P_{\alpha\beta}$ for $(\alpha_\beta) = (e\mu)$, at second order in the small parameters Δm^2 and Θ_{13} .

Calculation of $P_{e\mu}$ in constant density and 2nd order in Δm^2 , θ_{13}

- The so-called "golden channel" $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$ is particularly important in the context of future long-baseline accelerator experiments ($E \gtrsim 1$ GeV). In this context:
- Let us show that, for constant density N_e , and at 2nd order in the small parameters Δm^2 and θ_{13} , $P_{e\mu} = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu)$ takes the form:

$$\left\{ \begin{array}{l} P_{e\mu} = X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos \left(\delta - \frac{\Delta m^2 x}{4E} \right) + Z, \quad \text{where :} \\ X = \sin^2 \theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2} \right)^2 \sin^2 \left(\frac{A - \Delta m^2}{4E} x \right) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\Delta m^2}{A} \right) \left(\frac{\Delta m^2}{A - \Delta m^2} \right) \sin \left(\frac{Ax}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \\ Z = \cos^2 \theta_{23} \sin^2 2\theta_{12} \left(\frac{\Delta m^2}{A} \right)^2 \sin^2 \left(\frac{Ax}{4E} \right) \end{array} \right.$$

(Note that, sometimes, a further $\cos \theta_{13}$ factor is inserted in Y . This, however, is irrelevant at the stated 2nd order approximation.)

- Note that, given $P_{e\mu} = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu)$ one can get :

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu | A \rightarrow -A, \delta \rightarrow -\delta)$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu | \delta \rightarrow -\delta)$$

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_\tau) = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) | \begin{array}{l} c_{23} \rightarrow \mp s_{23} \\ s_{23} \rightarrow \pm c_{23} \end{array}$$

while changing mass hierarchy is equivalent to $\Delta m^2 \rightarrow -\Delta m^2$.

- We start the calculation by reminding that:

$$P(\nu_e \rightarrow \nu_\mu) = P_{\mu\mu} = |\tilde{S}'_{\mu e}|^2 \text{ with } \tilde{S}'_{\mu e} = \tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}, \text{ so that:}$$

$$P_{\mu\mu} = |\tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}|^2 = A_{\mu\mu} \cos \delta + B_{\mu\mu} \sin \delta + C_{\mu\mu} \text{ with:}$$

$$A_{\mu\mu} = 2 \operatorname{Re} [\tilde{S}'_{\mu e}^* \tilde{S}'_{\tau e}] c_{23} s_{23}$$

$$B_{\mu\mu} = -2 \operatorname{Im} [\tilde{S}'_{\mu e}^* \tilde{S}'_{\tau e}] c_{23} s_{23}$$

$$C_{\mu\mu} = |\tilde{S}'_{\mu e}|^2 c_{23}^2 + |\tilde{S}'_{\tau e}|^2 s_{23}^2$$

- The next "trick" is to reduce the evolution from 3ν to approximately $2\nu \oplus 1\nu$, by exploiting the expansion in two small parameters: δm^2 and θ_{13} (or better s_{13}). A term T will be called "of 1st order" if proportional to s_{13} or δm^2 :

$T \sim O_1$ if $T \propto s_{13}$ or $T \propto \delta m^2$. Analogously:

$T \sim O_2$ if $T \propto s_{13}^2$ or $T \propto (\delta m^2)^2$ or $T \propto s_{13} \delta m^2$ etc.

- We shall show that $\tilde{S}'_{\mu e} \sim O_1$ and $\tilde{S}'_{\tau e} \sim O_1$. Therefore, since $P_{\mu\mu}$ is quadratic in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$, it is $P_{\mu\mu} \sim O_2$ as desired.

- Let's remind that, in primed basis and for normal hierarchy:

$$\tilde{H}' = O_{13} O_{12} \frac{\tilde{m}^2}{2E} (O_{13} O_{12})^T + V$$

$$\tilde{m}^2 = \operatorname{diag} \left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \Delta m^2 \right) \quad \leftarrow \text{up to terms} \propto 1$$

$$V = \operatorname{diag} (\sqrt{2} G_F N_e, 0, 0)$$

- In the primed basis, the evolution decouples as $3\nu = 2\nu \oplus 1\nu$ in two limits:

$$S_{13} \rightarrow 0 \Rightarrow O_{13} = 1$$

$$\delta m^2 \rightarrow 0 \Rightarrow O_{12} M^2 O_{12}^T = M^2$$

It is then convenient to define:

$$\tilde{H}^l = \lim_{S_{13} \rightarrow 0} \tilde{H}'$$

(← "l" and "h" refer to "low" and "high")
 2ν subcases in literature jargon.

$$\tilde{H}^h = \lim_{\delta m^2 \rightarrow 0} \tilde{H}'$$

and to study the evolution operator components $\tilde{S}'_{\tau e}$ and $\tilde{S}'_{\tau e}$ in \tilde{H}^l and \tilde{H}^h .
The task is simpler since both \tilde{H}^l and \tilde{H}^h have only 1 nontrivial 2×2 submatrix.

- \tilde{H}^l in primed basis ($S_{13} \rightarrow 0$ limit):

$$\tilde{H}^l = \lim_{S_{13} \rightarrow 0} \tilde{H}' = \frac{1}{2E} \left[O_{12} \begin{pmatrix} -\delta m^2/2 & +\delta m^2/2 \\ \Delta m^2 & \end{pmatrix} O_{12}^T + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right]$$

$$= \frac{A}{4E} \mathbb{1} + \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{12} \delta m^2 & \sin 2\theta_{12} \delta m^2 & 0 \\ \sin 2\theta_{12} \delta m^2 & \cos 2\theta_{12} \delta m^2 - A & 0 \\ 0 & 0 & 2\Delta m^2 - A \end{bmatrix}. \text{ Given this structure:}$$

In the primed basis, for $S_{13} \rightarrow 0$, the (e, μ') flavors evolve separately from the τ' one →

$$\tilde{S}'_{\tau e} = \lim_{S_{13} \rightarrow 0} \tilde{S}'_{\tau e} = 0 \quad (\text{i.e., no } \nu_e \rightarrow \nu_{\tau'} \text{ transitions}), \text{ thus:}$$

$$\tilde{S}'_{\tau e} = O(S_{13}) = O_1 \text{ at least.}$$

Instead, $\tilde{S}'_{\mu e}$ is nonzero. From the 2ν case in matter (already worked out) we get:

$$\tilde{S}'_{\mu e} = e^{-i \frac{A}{4E} x} \left[-i \sin 2\theta_{12} \sin \left(\frac{\delta m^2 x}{4E} \right) \right] \text{ by exponentiation of } \tilde{H}^l, \text{ with:}$$

$\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} / \sqrt{(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12}}$ and $\tilde{\Delta m}^2 = \Delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$, implying:

$$\tilde{S}_{\mu e}^l = \mathcal{O}(\delta m^2) = O_1$$

- \tilde{H}^h in primed basis ($\delta m^2 \rightarrow 0$ limit):

$$\begin{aligned}\tilde{H}^h &= \lim_{\delta m^2 \rightarrow 0} \tilde{H}' = \frac{1}{2\epsilon} \left(O_{13} \begin{bmatrix} 0 & 0 \\ 0 & \Delta m^2 \end{bmatrix} O_{13}^T + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= \left(\frac{\Delta m^2}{4\epsilon} + \frac{A}{4\epsilon} \right) \mathbb{1} + \frac{1}{4\epsilon} \begin{bmatrix} A - \cos 2\theta_{13} \Delta m^2 & 0 & \sin 2\theta_{13} \Delta m^2 \\ 0 & -\Delta m^2 - A & 0 \\ \sin 2\theta_{13} \Delta m^2 & 0 & \cos 2\theta_{13} \Delta m^2 - A \end{bmatrix}.\end{aligned}$$

Given this structure:

In the primed basis, for $\delta m^2 \rightarrow 0$, the (e, τ') flavors evolve separately from the μ' one \rightarrow

$$\tilde{S}_{\mu e}^h = \lim_{\delta m^2 \rightarrow 0} \tilde{S}'_{\mu e} = 0 \quad (\text{no } \nu_e \rightarrow \nu_{\mu'} \text{ transitions}), \text{ thus:}$$

$$\tilde{S}'_{\mu e} = \mathcal{O}(\delta m^2) = O_1 \text{ at least.}$$

Instead, $\tilde{S}_{\tau e}^h$ is nonzero. From the 2ν case in matter (already worked out) we get:

$$\tilde{S}_{\tau e}^h = e^{-i\frac{A}{4\epsilon}x} e^{-i\frac{\Delta m^2 x}{4\epsilon}} \left[-i \sin 2\tilde{\theta}_{13} \sin \left(\frac{\Delta \tilde{m}^2 x}{4\epsilon} \right) \right] \text{ by exponentiation of } \tilde{H}^h,$$

with:

$\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} / \sqrt{(\cos 2\theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13}}$ and $\Delta \tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13}$, implying:

$$\tilde{S}_{\tau e}^h = \mathcal{O}(s_{13}) = O_1$$

- Summarizing, at O_1 we have that:

$$S'_{\mu e} = \mathcal{O}(\delta m^2) \simeq \tilde{S}_{\mu e}^l = e^{-i\frac{A}{4\epsilon}x} \left[-i \sin 2\tilde{\theta}_{12} \sin \left(\frac{\Delta \tilde{m}^2 x}{4\epsilon} \right) \right]$$

$$S'_{\tau e} = \mathcal{O}(s_{13}) \simeq \tilde{S}_{\tau e}^h = e^{-i\frac{A}{4\epsilon}x} e^{-i\frac{\Delta m^2}{4\epsilon}x} \left[-i \sin 2\tilde{\theta}_{13} \sin \left(\frac{\Delta \tilde{m}^2 x}{4\epsilon} \right) \right]$$

- One can drop the overall phase $e^{-i\frac{A}{4\epsilon}x}$ and get:

$$\tilde{S}'_{\mu e} = [-i \sin 2\tilde{\theta}_{12} \sin(\frac{\delta\tilde{m}^2 x}{4\epsilon})] + O_2$$

$$\tilde{S}'_{\tau e} = e^{-i\frac{\Delta m^2}{4\epsilon}x} [-i \sin 2\tilde{\theta}_{13} \sin(\frac{\Delta\tilde{m}^2 x}{4\epsilon})] + O_2$$

which provide all that is needed to get $P_{\mu\mu}$ as a quadratic form in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$. Indeed:

- $A_{\mu\mu} = 2 \operatorname{Re} [\tilde{S}'_{\mu e}^* \tilde{S}'_{\tau e}] c_{23} s_{23} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{13} \sin 2\theta_{23} \sin(\frac{\Delta\tilde{m}^2 x}{4\epsilon}) \sin(\frac{\delta\tilde{m}^2 x}{4\epsilon}) \cos(\frac{\Delta\tilde{m}^2 x}{4\epsilon})$
- $B_{\mu\mu} = -2 \operatorname{Im} [\tilde{S}'_{\mu e}^* \tilde{S}'_{\tau e}] c_{23} s_{23} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{13} \sin 2\theta_{23} \sin(\frac{\Delta\tilde{m}^2 x}{4\epsilon}) \sin(\frac{\delta\tilde{m}^2 x}{4\epsilon}) \sin(\frac{\Delta\tilde{m}^2 x}{4\epsilon})$
- $C_{\mu\mu} = |\tilde{S}'_{\mu e}|^2 c_{23}^2 + |\tilde{S}'_{\tau e}|^2 s_{23}^2 = \cos^2 \theta_{23} \sin^2 2\tilde{\theta}_{12} \sin^2(\frac{\delta\tilde{m}^2 x}{4\epsilon}) + \sin^2 \theta_{23} \sin^2 2\tilde{\theta}_{13} \sin^2(\frac{\Delta\tilde{m}^2 x}{4\epsilon})$

$$P_{\mu\mu} = A_{\mu\mu} \cos \delta + B_{\mu\mu} \sin \delta + C_{\mu\mu}$$

- The solution will now be further reduced by a proper organization of terms, as well as by an expansion in the small parameter:

$$\frac{\delta m^2}{A} = \frac{\delta m^2}{2\pi G_F N_e E} \ll 1 \quad (\text{valid for } E \gtrsim 1 \text{ GeV and } N_e \text{ in the crust or mantle}).$$

In particular, in this "high-energy approximation" (useful for accelerator neutrino experiments) one can express the matter parameters $\delta\tilde{m}^2$, $\Delta\tilde{m}^2$, $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ in terms of the vacuum parameters δm^2 , Δm^2 , θ_{12} and θ_{13} (together with an expansion in the small parameter s_{13}).

- For $\delta m^2/A \ll 1$:

$$\begin{aligned}
 \sin 2\tilde{\theta}_{12} &= \sin 2\theta_{12} / \left[(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12} \right]^{1/2} \\
 &= \sin 2\theta_{12} / \left[\cos^2 2\theta_{12} - \frac{2A}{\delta m^2} \cos 2\theta_{12} + \left(\frac{A}{\delta m^2} \right)^2 + \sin^2 2\theta_{12} \right]^{1/2} \\
 &= \sin 2\theta_{12} / \left[\left(\frac{A}{\delta m^2} \right)^2 \left(1 - 2 \frac{\delta m^2}{A} \cos 2\theta_{12} + \dots \right) \right]^{1/2} \\
 &\simeq \sin 2\theta_{12} / \left[\frac{|A|}{\delta m^2} \left(1 - \frac{\delta m^2}{A} \cos 2\theta_{12} \right) \right] \simeq \sin 2\theta_{12} \frac{\delta m^2}{|A|} + O_2
 \end{aligned}$$

$$\begin{aligned}
 \delta m^2 / \tilde{\delta m}^2 &= \sin 2\tilde{\theta}_{12} / \sin 2\theta_{12} \\
 &= \delta m^2 / |A| + O_2 \quad \Rightarrow \quad \tilde{\delta m}^2 = |A| + O_2, \text{ thus:}
 \end{aligned}$$

$$\sin \left(\frac{\tilde{\delta m}^2 x}{4\epsilon} \right) \simeq \sin \left(\frac{|A|x}{4\epsilon} \right) + O_2$$

- For $S_{13} \ll 1$:

$$\begin{aligned}
 \sin 2\tilde{\theta}_{13} &= \sin 2\theta_{13} / \left[(\cos 2\theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13} \right]^{1/2} \\
 &\simeq \sin 2\theta_{13} / \left[\left(1 - \frac{A}{\Delta m^2} \right)^2 \right]^{1/2} + O_2 = \sin 2\theta_{13} / \left| 1 - \frac{A}{\Delta m^2} \right| + O_2
 \end{aligned}$$

$$\sin 2\tilde{\theta}_{13} = \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin 2\theta_{13} + O_2$$

$$\tilde{\Delta m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13} \simeq \Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right|$$

- We have then:

$$A_{\text{eff}} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|} \right) \sin 2\theta_{13} \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin 2\theta_{23} \sin \left(\frac{|A|x}{4E} \right) \sin \left(\Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right| \frac{x}{4E} \right) \cos \left(\frac{\Delta m^2 x}{4E} \right)$$

$$B_{\text{eff}} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|} \right) \sin 2\theta_{13} \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin 2\theta_{23} \sin \left(\frac{|A|x}{4E} \right) \sin \left(\Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right| \frac{x}{4E} \right) \sin \left(\frac{\Delta m^2 x}{4E} \right)$$

$$C_{\text{eff}} \simeq \cos^2 \theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A} \right)^2 \sin^2 \left(\frac{Ax}{4E} \right) + \sin^2 \theta_{23} \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A} \right)^2 \sin^2 \left(\frac{|\Delta m^2 - A| x}{4E} \right)$$

- Absolute values can be eliminated by inspection of all relevant \pm cases.

E.g.: by changing sign of $(\Delta m^2 - A)$: A_{eff} , B_{eff} and C_{eff} do not change.

By changing sign of Δm^2 : only A_{eff} changes. Etc...

Then we have:

$$A_{\text{eff}} \simeq \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A} \right) \frac{\Delta m^2}{A - \Delta m^2} \sin \left(\frac{Ax}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \cos \left(\frac{\Delta m^2 x}{4E} \right)$$

$$B_{\text{eff}} \simeq \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A} \right) \frac{\Delta m^2}{A - \Delta m^2} \sin \left(\frac{Ax}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \sin \left(\frac{\Delta m^2 x}{4E} \right)$$

$$C_{\text{eff}} \simeq \cos^2 \theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A} \right)^2 \sin^2 \left(\frac{Ax}{4E} \right) + \sin^2 \theta_{23} \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A} \right)^2 \sin^2 \left(\frac{\Delta m^2 - A}{4E} x \right)$$

$$P_{\text{eff}} = A_{\text{eff}} \cos \delta + B_{\text{eff}} \sin \delta + C_{\text{eff}}$$

- Finally, the terms in P_{eff} can be organized as:

$$P_{\text{eff}} = X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos \left(\delta - \frac{\Delta m^2 x}{4E} \right) + Z \quad \text{where}$$

$$X = \sin^2 \theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2} \right)^2 \sin^2 \left(\frac{A - \Delta m^2}{4E} x \right)$$

$$Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\Delta m^2}{A} \right) \left(\frac{\Delta m^2}{A - \Delta m^2} \right) \sin \left(\frac{Ax}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right)$$

$$Z = \cos^2 \theta_{23} \sin^2 2\theta_{12} \left(\frac{\Delta m^2}{A} \right)^2 \sin^2 \left(\frac{Ax}{4E} \right)$$

as desired

- In the literature, one can also find the following way to organize terms:

$$P_{\text{eff}} = x^2 f^2 + 2xy fg \cos(\Delta - \delta) + y^2 g^2, \quad \text{where}$$

$$x = \sin \theta_{23} \sin 2\theta_{13}$$

$$y = \frac{\Delta m^2}{\Delta m^2} \cos \theta_{23} \sin 2\theta_{12}$$

$$\Delta = \Delta m^2 x / 4E$$

$$f = \sin \left(\frac{\Delta m^2 - A}{4E} x \right) \frac{\Delta m^2}{\Delta m^2 - A}$$

$$g = \sin \left(\frac{Ax}{4E} \right) \frac{\Delta m^2}{A}$$