An Introduction to the Standard Model of Fundamental Interacions

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Giovanni Ridolfi
Università di Genova and INFN Sezione di Genova, Italy

Lay-out:

1. **Construction of the standard model**
2. **Electromagnetic interactions**
3. **Electroweak interactions: Spontaneous breaking of the gauge symmetry**
4. **Electroweak interactions: Explicit breaking of accidental symmetries**
5. **Strong interactions**
6. **Open questions**

A few basic facts I assume you are familiar with:

- **The relativistic version of Quantum Mechanics is Quantum Field Theory (non-conservation of the particle number in reactions)**
- **QFT's are completely determined by symmetry properties (hinted by experiments)**
- **Gauge invariance plays a special role.**

1. Construction of the standard model: phenomenological inputs

- Fundamental matter particles are spin-$\frac{1}{2}$ fermions: charged leptons, neutrinos, quarks.
- Parity invariance is respected by electromagnetic and strong interactions, but (maximally) violated by weak interactions: positive- and negative-chirality fermions behave differently wrt weak interactions.
- Weak interactions are short-range interactions.
- Color confinement (more on this later).

Theoretical constraints

- **Renormalizability: consistent perturbation theory with no restrictions on the energy scale. In practice, a restriction on the mass dimension of parameters:** $d \geq 0$
- **A necessary, but not sufficient, condition for unitarity (conservation of probability): if the theory contains vector bosons, a gauge invariance is also needed.**

Our attitude towards renormalizability has changed in time.

2. Electromagnetic interactions

The lagrangian density of a collection of non-interacting fermions (labelled by an index i) with masses m_i:

$$\mathcal{L}_0 = \sum_i \bar{\psi}_i (i\partial\!\!\!/ - m_i)\psi_i$$

is invariant under constant phase multiplication ($U(1)$)

$$\psi_i \to e^{ieQ_i\alpha}\psi_i$$

As a consequence

$$\partial^\mu J_\mu = 0; \qquad J_\mu = -e\sum_i Q_i \bar{\psi}_i \gamma_\mu \psi_i$$

which can be interpreted as the em current if Q_i is the electric charges of fermion i in units of $-e$.

Electromagnetic interactions can be taken into account by analogy with classical physics:

$$H_{\rm em} = \int d^3x \, J_\mu(x) A^\mu(x)$$

where A^μ is the 4-potential. It follows that

$$\mathcal{L}_{\rm em} = -J_\mu(x) A^\mu(x) = e A^\mu \sum_i Q_i \bar{\psi}_i \gamma_\mu \psi_i$$

or

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm em} \\ &= \sum_i \bar{\psi}_i \left[i\gamma_\mu (\partial^\mu - ieQ_i A^\mu) - m_i \right] \psi_i \end{aligned}$$

The lagrangian density has now a local invariance under the transformations

$$\psi_i \to e^{ieQ_i\alpha(x)}\psi_i$$

provided A^μ is also transformed:

$$A^\mu \to A^\mu + \partial^\mu\alpha$$

(the usual gauge invariance of classical electrodynamics). Indeed

$$(\partial^\mu - ieQ_i A^\mu)\psi_i \to e^{ieQ_i\alpha}(\partial^\mu - ieQ_i A^\mu)\psi_i$$

The operator

$$D_\mu = \partial_\mu - ieQ_i A_\mu$$

is called a covariant derivative.

In order to describe the dynamics of the electromagnetic fields, we need a term in the lagrangian which contains derivatives of A_μ, with the following requirements:

- **no more than first derivatives;**
- **Lorentz and partity invariant;**
- **gauge invariant;**
- **dimension 4**

Only candidate:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Two important facts:

1.

$$F_{\mu\nu} \to F_{\mu\nu}$$

under a gauge transformation

2. **a mass term** $m^2 A_\mu A^\mu$ **not compatible with gauge invariance.**

So finally

$$\mathcal{L}_{QED} = \sum_i \bar{\psi}_i (i \not{D} - m) \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Field equations:

$$\partial_\mu F^{\mu\nu} = -J^\nu$$

Maxwell equations involving sources.

The two Maxwell equations without sources

$$\epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0$$

are automatically solved by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$**.**

The argument can be reversed: require local gauge invariance (with given fermion charges) and renormalizability. The field theory is uniquely fixed.

In a sense, gauge invariance is the origin of interactions.

Quantization

Quantization of gauge field theories is a difficult subject. Just to give you some ideas:

The gauge vector propagator does not exist:

$$(g_{\nu\mu}\partial^2 - \partial_\nu\partial_\mu)\Delta^{\mu\rho}(x) = -\delta_\nu^\rho\delta(x)$$

$$(k^2 g_{\nu\mu} - k_\nu k_\mu)\tilde{\Delta}^{\mu\rho}(k) = \delta_\nu^\rho$$

but

$$k^2 g_{\nu\mu} - k_\nu k_\mu$$

has no inverse: k^μ **is an eigenvector with zero eigenvalue!**

A consequence of gauge invariance. A gauge choice is needed.

- **A gauge choice can be performed by adding a suitable gauge-fixing term to the lagrangian density;**
- **Lorentz invariance and renormalizability restrict the possible choices to $-(\partial_\mu A^\mu)^2/(2\xi)$; no harm: $\partial_\mu A^\mu$ is a free field.**
- **with $\xi = 1$ (Feynman gauge) the photon propagator is**

$$\tilde{\Delta}_{\mu\nu} = \frac{g_{\mu\nu}}{k^2 - i\epsilon}$$

  which respects power counting, and the theory is renormalizable and unitary.
- **A useful tool in this context: the generating functional formalism.**

3. Weak interactions

The Fermi theory of weak interactions is non-renormalizable ($G_F \sim m^{-2}$) and non-unitary (cross sections grow as s^2).

The idea of interpreting the Fermi four-fermion interaction vertex as originated by vector boson exchange (in order to have a dimensionless coupling constant) dates back to Fermi himself.

There is only one way to build a unitary and renormalizable field theory of vectors: a gauge theory.

An endless list of experimental confirmations of this fact. The most striking one: the pattern of vector couplings.

Non abelian gauge theories:

$$\psi(x) \to U(x)\psi(x); \qquad U(x) \in \mathcal{G}; \qquad U(x) = e^{ig\alpha_A(x)t_A}$$

Then

$$\mathcal{L} = \bar{\psi}\gamma_\mu iD^\mu\psi - \frac{1}{4}F^A_{\mu\nu}F^{\mu\nu\,A}$$

is invariant, provided

$$D_\mu = \partial_\mu - igA^\mu = \partial_\mu - igA^\mu_A t_A$$

$$A_\mu \to UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1}$$

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + gf^{ABC}A^B_\mu A^C_\nu$$

with

$$[t_A, t_B] = if_{ABC}t_C$$

The structure constants f_{ABC} are completely antisymmetric, if

$$\mathrm{Tr}\, t_A t_B = T\delta_{AB}$$

Conserved currents:

$$J_\mu^A = -g\bar{\psi}\gamma_\mu t_A \psi; \qquad \partial^\mu J_\mu^A = 0$$

which imply

$$\frac{dQ^A}{dt} = 0; \qquad Q^A = \int d^3x\, J_0^A(t, \vec{x})$$

with

$$\left[Q^A, Q^B\right] = i f_{ABC} Q^C$$

Also note that

$$F_{\mu\nu} = F_{\mu\nu}^A t_A \to U F_{\mu\nu} U^{-1}$$

Warning: quantization much less trivial than in the abelian case.

Which gauge theory for weak interactions? Specifically,

- **Which gauge group?**
- **Which assignments of matter fields to representations?**
- **Which realization of symmetries?**

The first indication of early data (β decays of nuclei, muon decay) is that weak interactions distinguish left helicity from right helicity states. Only left-handed fermions are involved in the Fermi Lagrangian.

The structure of the interaction suggests a symmetry based on the group $SU(2)$.

Unification with electromagnetism requires an extension of the gauge group. The minimal extension is

$$SU(2)_L \otimes U(1)_Y$$

which requires four gauge vector bosons:

$$W_\mu^a, a = 1, 2, 3 \qquad \text{for } \; SU(2)_L$$

$$B_\mu \qquad \text{for } \; U(1)_Y$$

Next, we must assign fermion matter fields to representations of the gauge group.

Six flavours of quarks:

$$u \quad d \quad s \quad c \quad b \quad t$$

Three charged leptons:

$$e \quad \mu \quad \tau$$

and three neutrinos:

$$\nu_e \quad \nu_\mu \quad \nu_\tau$$

(I'm making a long story VERY short!)

Data are consistent with the following scheme:

$$
\begin{array}{ccccccc}
Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} & u_R^i & d_R^i & L_L^i = \begin{pmatrix} \nu_L^i \\ \ell_L^i \end{pmatrix} & \ell_R^i \\
\psi_1^i & \psi_2^i & \psi_3^i & \psi_4^i & \psi_5^i
\end{array}
$$

A family structure emerges:

$$\psi_r{}^i$$

The index i labels fermion generations: $i = 1, \ldots, 3$ (as far as we know).

The index r labels group representations.

Comments:

- **Fermion fields with different chiralities transform differently:**

$$\psi_R = \frac{1+\gamma_5}{2}\psi \qquad \psi_L = \frac{1-\gamma_5}{2}\psi$$

Parity is not conserved by weak interacions.

- **Left-handed quarks Q_L and leptons L_L transform as $SU(2)$ doublets ($r = 1$ and $r = 4$), right-handed fermions as $SU(2)$ singlets ($r = 2, 3, 5$). Right-handed fermion do not participate in charged-current interactions.**

- **Different representations have different values of the hypercharge quantum number (more on this later).**

- **neutrinos are massless: no right-handed neutrinos around. Much more on this later.**

A unique *gauge-invariant lagrangian density* **can now be written:**

$$\mathcal{L}_{SM} = \mathcal{L}_{\mathrm{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i$$

with

$$D_r^\mu = \partial^\mu - igT_r^a W_a^\mu - ig' \frac{Y_r}{2} B^\mu$$

$$T_r^a = \frac{\tau^a}{2} \quad \text{for SU(2) doublets} \quad (r = 1, 4)$$

$$T_r^a = 0 \quad \text{for SU(2) singlets} \quad (r = 2, 3, 5)$$

- **Hypercharge values undetermined so far**
- **Axial anomaly cancelled if $n_q = n_\ell = N$ (a prediction of the standard model.)**

The interaction lagrangian density includes a charged-current term which can be written as

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}\left[\overline{L}_L\gamma^\mu\tau^+L_L\,W_\mu^+ + \overline{L}_L\gamma^\mu\tau^-L_L\,W_\mu^- + \overline{Q}_L\gamma^\mu\tau^+Q_L\,W_\mu^+ + \overline{Q}_L\gamma^\mu\tau^-Q_L\,W_\mu^-\right]$$

with the definitions

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2); \qquad \tau^\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

This interaction term accounts for all processes described by the Fermi theory. For example

$$\mathcal{L}_{cc} = \frac{g}{2\sqrt{2}}\bar{u}\gamma^\mu(1-\gamma_5)d\,W_\mu^- + \frac{g}{2\sqrt{2}}\bar{e}\gamma^\mu(1-\gamma_5)\nu_e\,W_\mu^- + \ldots$$

Electro-weak unification

The neutral-current interaction term is

$$\mathcal{L}_{nc} = g\,\bar{\psi}\,\gamma_\mu\,T_3\,\psi\,W_3^\mu + g'\,\bar{\psi}\,\gamma_\mu\,\frac{Y}{2}\,\psi\,B^\mu$$

where

$$\psi = \psi_r^i, \quad r = 1, \ldots, 5$$

$$T_3 = (T_3)_r$$

$$(T_3)_r = \frac{\tau_3}{2} \qquad \text{for doublets} \;\; (r = 1, 4)$$

$$(T_3)_r = 0 \qquad \text{for singlets} \;\; (r = 2, 3, 5)$$

$$Y = Y_r \;\; (r = 1, \ldots, 5)$$

Neither W_μ^3 nor B_μ can be identified with A_μ.

Reparametrization of the neutral sector:

$$W_3^\mu = A^\mu \, \sin\theta_W + Z^\mu \, \cos\theta_W$$
$$B^\mu = A^\mu \, \cos\theta_W - Z^\mu \, \sin\theta_W$$

(an orthogonal transformation, in order to keep kinetic terms diagonal in the vector fields).

$$\begin{aligned}\mathcal{L}_{nc} &= \bar\psi\,\gamma_\mu \left[g\sin\theta_W\, T_3 + g'\cos\theta_W \frac{Y}{2}\right] \psi\, A^\mu \\ &+ \bar\psi\,\gamma_\mu \left[g\cos\theta_W\, T_3 - g'\sin\theta_W \frac{Y}{2}\right] \psi\, Z^\mu\end{aligned}$$

We may identify A_μ with the photon field provided

$$g\sin\theta_W\, T_3 + g'\cos\theta_W \frac{Y}{2} = eQ$$

Choosing g and g' so that

$$g \sin\theta_W = g' \cos\theta_W = e$$

we obtain

$$Q = T_3 + \frac{Y}{2}$$

for all fermions:

$$Y(u_L) = 2\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{1}{3}$$

$$Y(d_L) = 2\left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{3}$$

$$Y(e_L) = 2\left(-1 + \frac{1}{2}\right) = -1$$

$$\ldots$$

[Alternatively: choose e.g. $Y = -1$ for the lepton doublets, and solve

$$+\frac{1}{2}g\sin\theta_W - \frac{1}{2}g'\cos\theta_W = 0$$
$$-\frac{1}{2}g\sin\theta_W - \frac{1}{2}g'\cos\theta_W = -e$$

with respect to $g\sin\theta_W, g'\cos\theta_W$. You get $g\sin\theta_W = g'\cos\theta_W = e$, and $Q = T_3 + Y/2$ follows.]

The value of $\sin\theta_W$ can only be extracted from the observation of weak neutral-current phenomena, induced by interactions with the Z^0 boson:

$$\begin{aligned}
\mathcal{L}_{nc} &= e\bar{\psi}\,\gamma_\mu\,Q\,\psi\,A^\mu \\
&+ e\bar{\psi}\,\gamma_\mu\left[\frac{\cos\theta_W}{\sin\theta_W}\,T_3 - \frac{\sin\theta_W}{\cos\theta_W}\,\frac{Y}{2}\right]\psi\,Z^\mu
\end{aligned}$$

Historical example: neutral-current deep inelastic scattering.

$$\begin{aligned}
\mathrm{NC}: &\quad \nu_\mu + H \to \nu_\mu + X \\
\mathrm{CC}: &\quad \nu_\mu + H \to \mu^- + X
\end{aligned}$$

$$R = \frac{\sigma_{\bar{\nu}}^{(\mathrm{NC})} - \sigma_{\nu}^{(\mathrm{NC})}}{\sigma_{\bar{\nu}}^{(\mathrm{CC})} - \sigma_{\nu}^{(\mathrm{CC})}} \simeq \frac{1 - 2\sin^2\theta_W}{2}$$

The most precise determinations of $\sin\theta_W$ come from forward-backward asymmetries measured in e^+e^- collisions:

$$A_{FB}(f) = \frac{\int_{\cos\theta>0} d\sigma(e^+e^- \to f\bar{f}) - \int_{\cos\theta<0} d\sigma(e^+e^- \to f\bar{f})}{\sigma(e^+e^- \to f\bar{f})}$$

where f is any charged fermion. Present result:

$$\sin^2\theta_W = 0.23116(13)$$

It follows that g, g' are of the same order of magnitude as e:

$$g = \frac{e}{\sin\theta_W} \simeq 2.1\, e; \qquad g' = \frac{e}{\cos\theta_W} \simeq 1.1\, e$$

$$\frac{g^2}{4\pi} \simeq 4\alpha_{\rm em}; \qquad \frac{{g'}^2}{4\pi} \simeq \alpha_{\rm em}$$

Not yet a realistic theory

- **The gauge symmetry must be (spontaneously) broken,**

$$SU(2)_L \otimes U(1)_Y \to U(1)_{\rm em}$$

because weak vector bosons are observed to be massive (short range of weak interactions).

- **The fermionic sector has a large global symmetry which is not observed. An explicit breaking**

$$[U(N)]^5 \to U(1)_B \otimes U(1)_e \otimes U(1)_\mu \otimes U(1)_\tau$$

or

$$[U(N)]^5 \to U(1)_B \otimes U(1)_L$$

is needed.

Addendum n. 1

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W_a^{\mu\nu}$$

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

$$W_i^{\mu\nu} = \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g\epsilon_{ijk} W_j^\mu W_k^\nu$$

The corresponding expressions in terms of $W_\mu^\pm$, Z_μ and A_μ can be easily worked out:

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-)$$

$$W_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-)$$

$$W_\mu^3 = A_\mu \sin\theta_W + Z_\mu \cos\theta_W$$

$$B_\mu = A_\mu \cos\theta_W - Z_\mu \sin\theta_W$$

We get

$$
\begin{aligned}
W^1_{\mu\nu} &= \frac{1}{\sqrt{2}}\left[W^+_{\mu\nu} + ig\,\sin\theta_W(W^+_\mu A_\nu - W^+_\nu A_\mu)\right. \\
&\qquad \left. + \,ig\,\cos\theta_W(W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right] + \text{h.c.} \\
W^2_{\mu\nu} &= \frac{i}{\sqrt{2}}\left[W^+_{\mu\nu} + ig\,\sin\theta_W(W^+_\mu A_\nu - W^+_\nu A_\mu)\right. \\
&\qquad \left. + \,ig\,\cos\theta_W(W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right] + \text{h.c.} \\
W^3_{\mu\nu} &= F_{\mu\nu}\sin\theta_W + Z_{\mu\nu}\cos\theta_W - ig(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu) \\
B_{\mu\nu} &= F_{\mu\nu}\cos\theta_W - Z_{\mu\nu}\sin\theta_W
\end{aligned}
$$

where

$$
\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
Z^{\mu\nu} &= \partial^\mu Z^\nu - \partial^\nu Z^\mu \\
W^{\mu\nu}_\pm &= \partial^\mu W^\nu_\pm - \partial^\nu W^\mu_\pm
\end{aligned}
$$

It follows that

$$\begin{aligned}\mathcal{L}_{\text{Yang-Mills}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W^{+}_{\mu\nu}W^{\mu\nu}_{-} \\ &\quad +ig\sin\theta_W\left(W^{+}_{\mu\nu}W^{\mu}_{-}A^{\nu} - W^{-}_{\mu\nu}W^{\mu}_{+}A^{\nu} + F_{\mu\nu}W^{\mu}_{+}W^{\nu}_{-}\right) \\ &\quad +ig\cos\theta_W\left(W^{+}_{\mu\nu}W^{\mu}_{-}Z^{\nu} - W^{-}_{\mu\nu}W^{\mu}_{+}Z^{\nu} + Z_{\mu\nu}W^{\mu}_{+}W^{\nu}_{-}\right) \\ &\quad +\frac{g^2}{2}\left(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\Bigg[\frac{1}{2}W^{+}_{\mu}W^{+}_{\nu}W^{-}_{\rho}W^{-}_{\sigma} \\ &\quad -W^{+}_{\mu}W^{-}_{\nu}\left(A_\rho A_\sigma\sin^2\theta_W + Z_\rho Z_\sigma\cos^2\theta_W + 2A_\rho Z_\sigma\sin\theta_W\cos\theta_W\right)\Bigg]\end{aligned}$$

Addendum n. 2

Why did I say that

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i$$

has a global $[U(N)]^5$ invariance?

For each $r = 1, \ldots, 5$ consider the transformation

$$\psi_r^{i\,\prime} = \sum_{j=1}^{N} U_r^{ij} \psi_r^j$$

where U^r is a constant unitary $N \times N$ matrix. This transformation leaves $\mathcal{L}_{SM}$ unchanged. There are 5 symmetries of this kind, one for each representation r. The full global symmetry is therefore $[U(N)]^5$, as announced.

3. Spontaneous breaking of the gauge symmetry

A simple argument shows that the W boson must be massive. The amplitude for β decay in the Fermi theory is given by

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} \overline{u} \gamma^\mu (1-\gamma_5) d \, \overline{e} \gamma_\mu (1-\gamma_5) \nu_e$$

In the standard model, the same process is induced by the exchange of a W boson:

$$\mathcal{M}^{\mathrm{SM}} = \left(\frac{g}{\sqrt{2}} \overline{u}_L \gamma^\mu d_L \right) \frac{1}{q^2 - m_W^2} \left(\frac{g}{\sqrt{2}} \overline{e}_L \gamma_\mu \nu_{e\,L} \right)$$

We have

$$q^2 \leq (m_N - m_P)^2 \sim (1.3 \text{ MeV})^2$$

Hence, the two amplitudes coincide in the limit $m_W^2 \gg q^2$ **if**

$$\frac{G_F}{\sqrt{2}} = \left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2}.$$

A lower bound on the W **mass can be set: since**

$$g = \frac{e}{\sin \theta_W}$$

we obtain

$$m_W^2 = \frac{\sqrt{2}}{G_F} \frac{g^2}{8} \geq \frac{\sqrt{2}}{G_F} \frac{e^2}{8} \sim (40 \text{ GeV})^2$$

quite a large value, compared to the nucleon mass, and an enormous number, compared to the present upper bound on the photon mass

$$m_\gamma \leq 2 \cdot 10^{-16} \text{ eV}.$$

Breaking gauge invariance explicitly with a mass term

$$m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu$$

leads to a non-renormalizable and non-unitary theory.

The gauge symmetry of the standard model must be spontaneously broken, in order to introduce masses for the W and Z vector bosons without spoiling unitarity and renormalizability.

A flavour of the argument: **a mass term inserted by hand (explicit breaking) leads to a massive gauge boson propagator**

$$\Delta^{\mu\nu}(k) = \frac{i}{k^2 - m^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)$$

For large k, the term proportional to $k^\mu k^\nu$ dominates, and $\Delta(k) \sim k^0$ rather than k^{-2}: the behaviour of this propagator at large k is much worse than that of the scalar propagator. This suggests a worse UV behaviour of the Feynman integrals, which leads to a non-renormalizable theory.

A related problem: unitarity of the scattering matrix. The amplitude for a generic physical process with the emission or the absorption of a vector boson with four-momentum k and polarization vector $\epsilon(k)$ has the form

$$\mathcal{M} = \mathcal{M}^{\mu}\epsilon_{\mu}(k).$$

A massive vector (contrary to a massless one) may be polarized longitudinally. In this case, choosing the z axis along the direction of the 3-momentum of the vector boson, the polarization is given by

$$\epsilon_L = \left(\frac{|\vec{k}|}{m}, 0, 0, \frac{E}{m}\right) = \frac{k}{m} + \mathcal{O}\left(\frac{m^2}{E^2}\right),$$

(because $k \cdot \epsilon = 0$ and $\epsilon^2 = -1$).

The amplitude $\mathcal{M}$ grows indefinitely with the energy E, and eventually violates the unitarity bound.

Both sources of power-counting violation are rendered harmless if the vector particles are coupled to conserved currents, so that

$$k^\mu \mathcal{M}_\mu = 0$$

Gauge invariance provides such conservation relations.

Spontaneous symmetry breaking is not really a way of breaking a symmetry: rather, it is a different realization of the symmetry itself.

More precisely, SSB takes place whenever the ground state is not invariant under symmetry transformations. As a consequence, the lagrangian density is symmetric, but the spectrum of physical states is not.

The prototype: ferromagnetism.

In quantum field theory, SSB takes place when some operator with non-trivial transformation properties under the gauge group has non-vanishing vacuum expectation value:

$$\langle 0|\phi_j|0\rangle = v_j \neq 0$$

Easy to prove: after an infinitesimal transformation

$$\phi_i \to \phi_i + i\alpha^a \, t^a_{ij}\phi_j = \phi_i + i\alpha^a \, [Q^a, \phi_i]$$

$$t^a_{ij}\langle 0|\phi_j|0\rangle = \langle 0| \, [Q^a, \phi_i] \, |0\rangle \neq 0 \;\Leftrightarrow\; Q^a|0\rangle \neq 0$$

which is the condition for spontaneous symmetry breaking, i.e. non-invariance of the vacuum state.

Observations:

- $\langle 0|\phi_i|0\rangle$ **is constant over space-time if the vacuum is invariant under translations:**

$$\langle 0|\phi_i(x)|0\rangle = \langle 0|e^{iPx}\phi_i(0)e^{-iPx}|0\rangle = \langle 0|\phi_i(0)|0\rangle$$

- ϕ **must be a scalar, otherwise its vacuum expectation value is frame-dependent.**
- ϕ **is not necessarily an elementary field**

The simplest realization: the Higgs mechanism

$$\mathcal{L}_{SM} = \mathcal{L}_{\mathrm{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i + \mathcal{L}_{\mathrm{Higgs}}$$

$$\mathcal{L}_{\mathrm{Higgs}} = (D_\mu \phi)^\dagger D^\mu \phi - V(\phi)$$

where ϕ is a set of one or more fundamental scalar fields.

The simplest among simplest: ϕ is an $SU(2)_L$ doublet:

$$D_\mu \phi = \partial_\mu \phi - \frac{ig}{2} W_\mu^a \tau^a \phi - \frac{ig'}{2} Y_\phi B_\mu \phi; \qquad V(\phi) = m^2 \left|\phi\right|^2 + \lambda \left|\phi\right|^4$$

If $m^2 < 0$ the scalar potential has a minimum at

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; \qquad v_1^2 + v_2^2 = -\frac{m^2}{\lambda} \equiv v^2.$$

The value of the hypercharge Y_ϕ is dictated by the requirement that $U_{\rm em}(1)$ remains unbroken:

$$e^{ieQ}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

This amounts to

$$Q\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

with

$$Q = \frac{1}{2}\left(\tau^3 + Y_\phi\, \mathbb{I}_2\right) = \frac{1}{2}\begin{pmatrix} 1+Y_\phi & 0 \\ 0 & -1+Y_\phi \end{pmatrix}$$

which has nonzero solutions only if

$$(Y_\phi + 1)(Y_\phi - 1) = 0$$

Two solutions:

$$Y_\phi = 1, \quad v_1 = 0, v_2 = v \qquad\qquad Y_\phi = -1, \quad v_1 = v, v_2 = 0$$

(related to each other by charge conjugation). We choose $Y_\phi = 1$, so that

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

The $|D\phi|^2$ term contains a term

$$
\begin{aligned}
\mathcal{L}_{\phi\phi VV} &= \frac{1}{4}(g^2 W_a^\mu W_\mu^a + g'^2 B^\mu B_\mu)\phi^\dagger\phi + \frac{1}{2} g g' B^\mu W_\mu^i \phi^\dagger \tau^i \phi \\
&= \frac{1}{4} g^2 v^2 W_1^{+\mu} W_\mu^- + \frac{1}{4} v^2 (W_3^\mu \;\; B^\mu) \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix} \\
&+ \quad \ldots
\end{aligned}
$$

The first term is a mass term for the W:

$$m_W^2 = \frac{1}{4} g^2 v^2$$

The matrix

$$\frac{1}{4}v^2\begin{pmatrix} g^2 & -gg' \\ -gg' & {g'}^2 \end{pmatrix}$$

has zero determinant and trace $\frac{1}{4}\left(g^2+{g'}^2\right)v^2$**. Hence the two neutral mass eigenvectors**

$$A^3_\mu = W^3_\mu \sin\theta_W + B_\mu \cos\theta_W$$
$$Z_\mu = W^3_\mu \cos\theta_W - B_\mu \sin\theta_W$$

have masses

$$m_Z^2 = \frac{1}{4}\left(g^2+{g'}^2\right)v^2 \qquad m_\gamma^2 = 0$$

respectively.

The value of the order parameter v^2 is obtained from matching with the Fermi theory of β decay: from

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}; \qquad m_W^2 = \frac{1}{4}g^2v^2$$

we get

$$v = (\sqrt{2}G_F)^{-1/2} \sim (247\ \mathrm{GeV})^2$$

where we have used the measured value $G_F \sim 1.1 \times 10^{-5}\ \mathrm{GeV}^{-2}$.

Weak interactions have a characteristic energy scale of the order of a few hundred GeV.

Three of the four scalar degrees of freedom in ϕ are unphysical: they can be eliminated from the spectrum by a gauge choice.

An easy (but slightly deceptive) way to see it: parametrize ϕ by

$$\phi = \frac{1}{\sqrt{2}} e^{\frac{i\tau^i \theta^i(x)}{v}} \left(\begin{array}{c} 0 \\ v + H(x) \end{array} \right) \rightarrow \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ v + H(x) \end{array} \right)$$

after a suitable gauge transformation. The massive gauge boson propagators take the form

$$\Delta^{\mu\nu}(k) = \frac{i}{k^2 - m^2} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right)$$

It looks like we are in troubles again with renormalization!

This is not true: we are working with a renormalizable theory, so renormalizability must arise in calculations, even though it is not manifest (the propagator does not respect the usual power-counting rule).

This is called the unitary gauge: unitarity is manifest, in the sense that unphysical degrees of freedom are removed from the spectrum, but manifest renormalizability is lost.

Useful in tree-level calculations.

When loop corrections become relevant, it is advisable to adopt a renormalizable gauge. The starting point is a linear parametrization of the scalar field:

$$\phi = \phi_1 + \phi_2,$$

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} G_1(x) + iG_2(x) \\ H(x) + iG_3(x) \end{pmatrix}$$

A convenient gauge-fixing term (suggested by 't Hooft) is

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \left[\partial^\mu W^i_\mu - \xi f^i(\phi)\right]^2 - \frac{1}{2\xi} \left[\partial^\mu B_\mu - \xi f(\phi)\right]^2$$

with

$$f^i(\phi) = \frac{ig}{2}(\phi_1^\dagger \tau^i \phi_2 - \phi_2^\dagger \tau^i \phi_1) \qquad f(\phi) = \frac{ig'}{2}(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1)$$

Two main advantages:

- **No mixing between vector fields and derivative of the scalar field**
- **Manifest renormalizability:**

$$\Delta_\xi^{\mu\nu}(k) = \frac{i}{k^2 - m^2}\left[-g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m^2}\right]$$

(the unitary gauge is formally recovered in the limit $\xi \to \infty$).

Draw-back: **the unphysical scalars G_1, G_2, G_3 are in the game. They cannot appear as asymptotic states (external lines in Feynman diagrams), and their contributions as internal lines is cancelled by the unphysical singularity in the vector boson propagators (not easy to prove).**

This is an additional difficulty in spontaneously broken non abelian gauge theories.

The usual ones (cancellation of unphysical longitudinal vector degrees of freedom by Faddeev-Popov ghosts) are there as usual.

A difficult subject; functional methods are most effective in proving the renormalizability of spontaneously broken gauge theories.

Physical interpretation: **massless vector bosons have two physical degrees of freedom: the two helicity states (no longitudinal polarization).**

After SSB, vector bosons become massive: the longitudinal modes are provided by the three would-be Goldstone bosons, which disappear from the spectrum.

The existence of longitudinally polarized W and Z is the most striking evidence of spontaneous gauge symmetry breaking.

The scalar potential simplifies considerably in the unitary gauge:

$$\begin{aligned} V(\phi) &= m^2 |\phi|^2 + \lambda |\phi|^4 \\ &= \frac{m^2}{2}(v+H)^2 + \frac{\lambda}{4}(v+H)^4 \\ &= H(m^2 v + \lambda v^3) + \frac{1}{2} H^2 (m^2 + 3\lambda v^2) + \lambda v H^3 + \frac{\lambda}{4} H^4 \end{aligned}$$

Since $m^2 = -\lambda v^2$, the linear term vanishes, and the quadratic term has a coefficient

$$\frac{1}{2} 2\lambda v^2$$

and can be interpreted as a true mass term for the scalar field H.

The Higgs boson has been observed at the LHC on July 4$^{\text{th}}$, 2012

- **The simplest realization of SSB is the one which is realized in nature: not obvious.**
- **The tree-level relation $m_H^2 = 2\lambda v^2$, together with the measured value $m_H = 125$ GeV, imply $\lambda \sim 0.13$. The Standard Model is weakly coupled even in the Higgs sector. Also not obvious.**
- **m_H much smaller than the unitarity upper bound of about ~ 1 TeV (perturbative unitarity of $W_L W_L \to W_L W_L$, similar to the upper bound on m_W in the Fermi theory).**

A word on perturbation theory beyond leading order:

- **An expansion in powers of some small dimensionless parameter, typically a coupling constant;**
- **Perturbative coefficients usually display powers of** $\log \frac{E}{M}$**, which may spoil convergence at large energies;**
- **Large logarithms resummed by changing the espansion parameter**

  $$\lambda \to \lambda(E)$$

  the *running coupling constant***;**
- **The so-called beta functions**

  $$\beta(E) = E \frac{d\lambda(E)}{dE}$$

  are computable in perturbation theory.

Many important consequences:

1) The present Universe may not be in a stable quantum state.

A sketch of the argument: the tree-level scalar potential of the standard model

$$V(h) = \frac{1}{2}m^2h^2 + \frac{1}{4}\lambda h^4; \qquad |\phi|^2 = \frac{1}{2}h^2$$

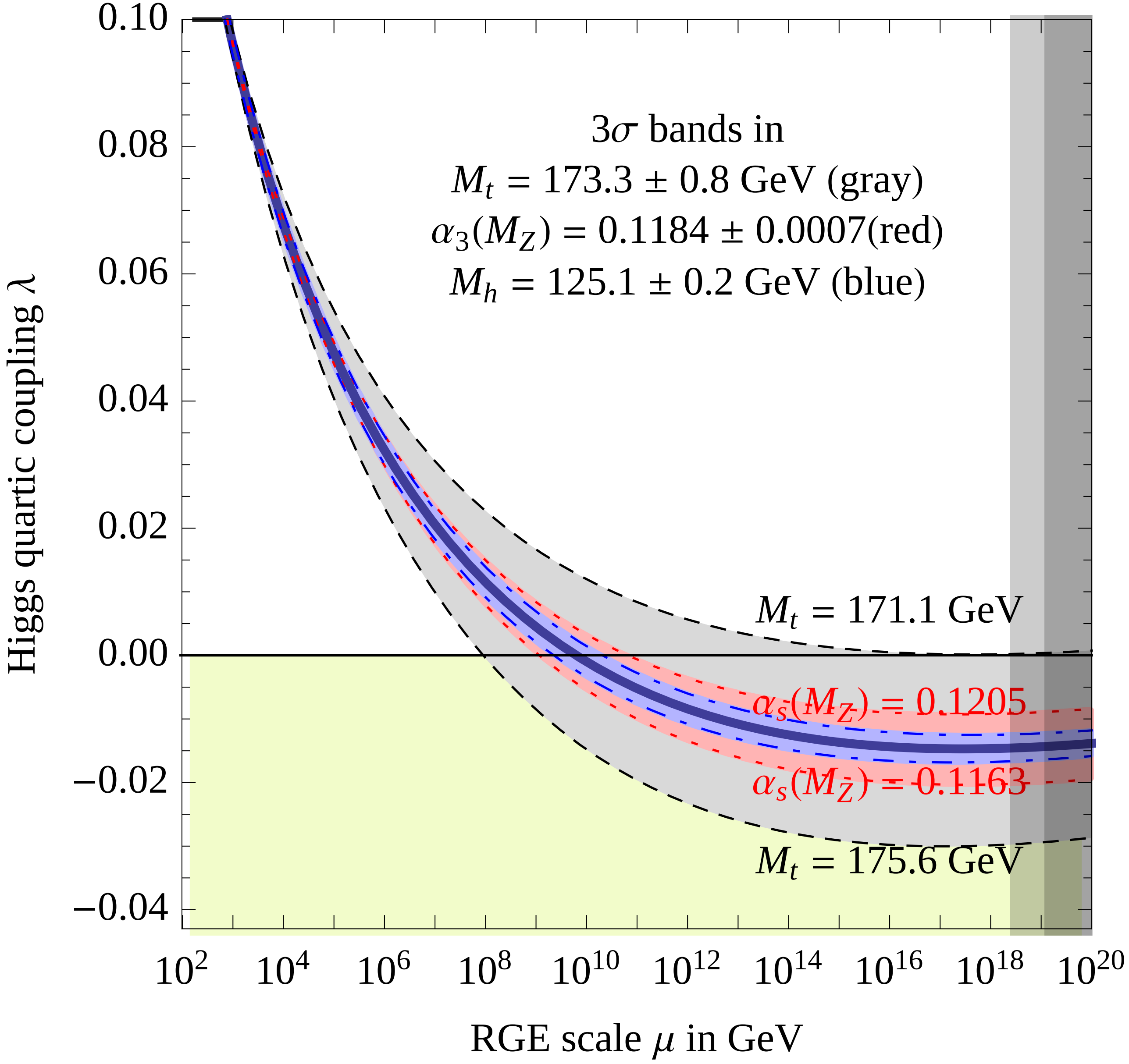
has a minimum at

$$h^2 = v^2 = -\frac{m^2}{\lambda}; \qquad v \simeq 246 \text{ GeV},$$

the present ground state. However, beyond tree level

$$V(h) \to V_{\text{eff}}(h) \sim \frac{1}{4}\lambda(h)h^4$$

and $\lambda(\mu) < 0$ for μ sufficiently large [+ m_{Higgs} sufficiently small], because of top quark loops.



[Buttazzo et al, JHEP 1312(2013)089]

Two relevant parameters: the Higgs boson mass, which sets the initial condition through

$$m^2_{\text{Higgs}} \sim 2\lambda(v)v^2$$

and the top quark mass, which determines the slope at $\mu \sim v$.

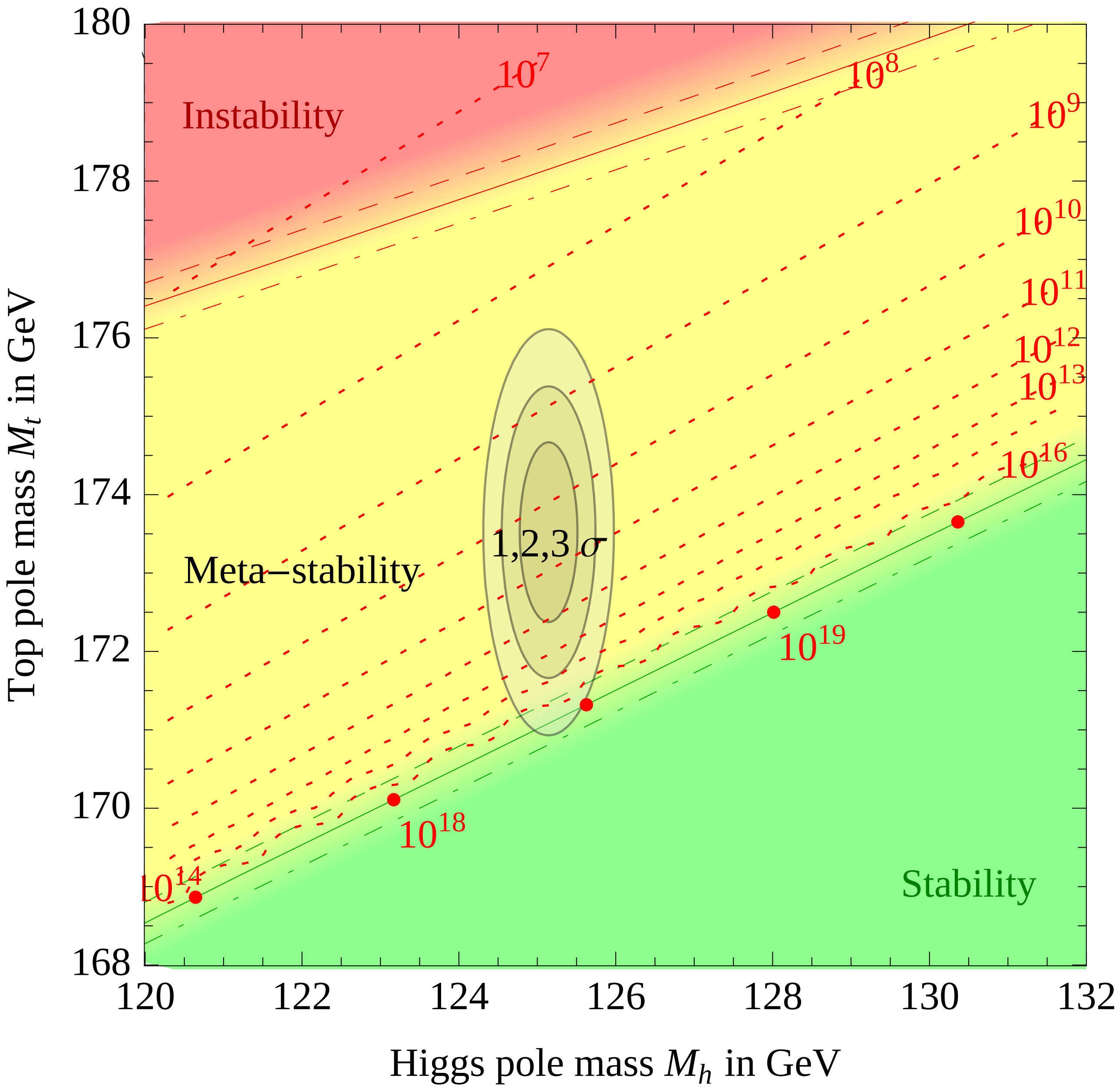
Historically, a tool to set a lower bound m_{Higgs}: $\lambda(\mu)$ is positive-definite if $\lambda(v)$ (and therefore m_{Higgs}) is large enough.

Today, m_{Higgs} and m_{top} are measured to less that 1% accuracy. The same argument can be used to answer different questions:

- **Do we need non-standard physics to restore stability?**
- **At what energy scale?**

Furthermore, it is not necessary to require absolute stability: an unstable vacuum with lifetime $\tau > \tau_{\text{universe}} \sim 1.3 \times 10^{10}$ y also acceptable.

The decay rate of the unstable vacuum state can be computed as a function of ew parameters.



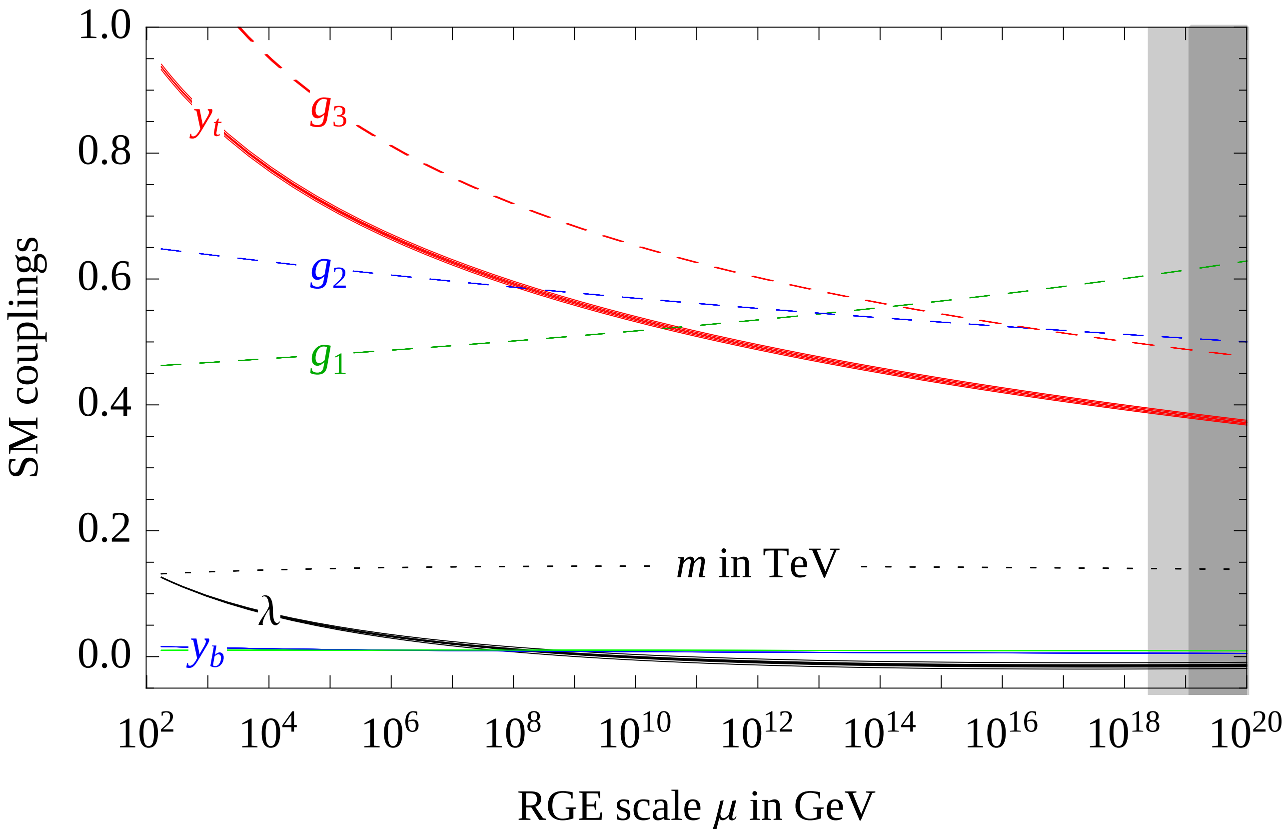
[Buttazzo et al, JHEP 1312(2013)089]

With present values of the relevant parameters (m_{top} and m_{Higgs}) and no new physics the ground state turns out to be in the metastability region:

$$\tau_U < \tau < +\infty$$

No need for new physics to stabilize the ew ground state (which doesn't mean there isn't any...)

2) Gauge running couplings:



[Buttazzo et al, JHEP 1312(2013)089]

A question arises:

Does a large value of λ generate large radiative corrections on observables?

The answer is no: the typical example is the ratio

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W}$$

which is 1 at tree level, and receives a one-loop correction

$$\Delta\rho = -\frac{11g^2}{48\pi^3}\tan^2\theta_W \log\frac{m_H^2}{m_W^2}$$

which grows only logarithmically with m_H^2.

This arises from a symmetry property of the scalar potential, called the custodial symmetry.

A different way to state the problem: after the inclusion of radiative corrections,

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m_W^2 (W^{1^\mu} W^1_\mu + W^{2^\mu} W^2_\mu) + \frac{1}{2} (W_3^\mu \;\; B^\mu) \begin{bmatrix} M^2 & M'^2 \\ M'^2 & M''^2 \end{bmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix}.$$

with $M'^2 = MM''$**,** $M^2 + M''^2 = m_Z^2$**. Hence**

$$\tan\theta_W = \frac{\sqrt{m_Z^2 - M^2}}{M}.$$

and

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = \frac{m_W^2}{M^2},$$

that is, $\rho = 1$ **only if** $M^2 = m_W^2$**.**

The reason is that the scalar potential posesses an $O(4)$ invariance, larger than the gauge symmetry. Indeed, the scalar field

$$\tilde{\phi} = \left(\begin{array}{c} \phi_0^* \\ -\phi^- \end{array} \right) = \epsilon \phi^*$$

can be shown to be an $SU(2)_L$ doublet with $Y_{\tilde{\phi}} = -1$. Hence the matrix

$$\mathcal{H} = \left[\tilde{\phi} \;\; \phi \right] = \left[\begin{array}{cc} \phi^{0\,*} & \phi^+ \\ -\phi^- & \phi^0 \end{array} \right]$$

transforms as

$$\mathcal{H}(x) \longrightarrow \exp\left[\frac{ig}{2} \tau^a \alpha^a(x) \right] \mathcal{H}(x)$$

$$\mathcal{H}(x) \longrightarrow \mathcal{H}(x) \exp\left[-\frac{ig'}{2} \beta(x) \tau_3 \right].$$

under gauge transformations.

The scalar potential can be written

$$V(\phi) = \frac{1}{2} m^2 \,\mathrm{Tr}\left(\mathcal{H}^\dagger \mathcal{H}\right) + \frac{1}{4}\lambda \left[\mathrm{Tr}\left(\mathcal{H}^\dagger \mathcal{H}\right)\right]^2$$

which is invariant under the $SU(2)_L \times SU_R(2)$ **transformations**

$$\mathcal{H} \to U \mathcal{H} V^\dagger$$

where $V \in SU(2)$**.**

We also have

$$(D_\mu \phi)^\dagger D^\mu \phi = \frac{1}{2} \mathrm{Tr}\left[(D_\mu \mathcal{H})^\dagger D^\mu \mathcal{H}\right],$$

with

$$D_\mu \mathcal{H} = \partial_\mu \mathcal{H} - \frac{ig}{2} W_\mu^a \tau^a \mathcal{H} + \frac{ig'}{2} B_\mu \mathcal{H} \tau_3$$

Clearly

$$
\begin{aligned}
D_\mu \mathcal{H} &\rightarrow (U D_\mu U^\dagger)(U \mathcal{H} V^\dagger) \\
&= U(\partial_\mu \mathcal{H} - \frac{ig}{2} W_\mu^a \tau^a \mathcal{H}) V^\dagger + \frac{ig'}{2} B_\mu U \mathcal{H} V^\dagger \tau_3 \\
&\neq U(D_\mu \mathcal{H}) V^\dagger
\end{aligned}
$$

because of the last term.

The full lagrangian has a custodial symmetry only for $g' = 0$.

Due to spontaneous breaking of $SU(2)_L$, the vacuum expectation value

$$\langle 0|\mathcal{H}|0\rangle = \frac{v}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is not $O(4)$ invariant. However, there is a residual $O(3) \sim SU(2)$ symmetry

$$\mathcal{H} \to U\mathcal{H}U^\dagger$$

which leaves the vacuum expectation value unchanged. The only mass term for the W_μ^i fields allowed by this residual symmetry is proportional to $W_\mu^i W_i^\mu$, which in turn implies $M^2 = m_w^2$ and $\rho = 1$.

4. Breaking of accidental symmetries

Consider $n_q = n_\ell = 1$ for simplicity. Then

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{r=1}^{5} \bar{\psi}_r \, i\not{D}_r \, \psi_r + \mathcal{L}_{\text{Higgs}}$$

has a $[U(1)]^5$ global invariance:

$$\psi_r \to e^{i\alpha_r} \, \psi_r$$

The corresponding conserved currents are

$$\begin{array}{llcl}
J_1^\mu = \bar{u}_L \gamma^\mu u_L + \bar{d}_L \gamma^\mu d_L & & & J_Y^\mu = \sum_{r=1}^{5} \frac{Y_r}{2} J_r^\mu \\
J_2^\mu = \bar{u}_R \gamma^\mu u_R & & & J_\ell^\mu = J_4^\mu + J_5^\mu \equiv \bar{\nu}\gamma^\mu \nu + \bar{e}\gamma^\mu e \\
J_3^\mu = \bar{d}_R \gamma^\mu d_R & & \to & J_{\ell 5}^\mu = J_5^\mu - J_4^\mu \equiv \bar{\nu}\gamma^\mu \gamma_5 \nu + \bar{e}\gamma^\mu \gamma_5 e \\
J_4^\mu = \bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L & & & J_b^\mu = \frac{1}{3}(J_1^\mu + J_2^\mu + J_3^\mu) \equiv \frac{1}{3}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d) \\
J_5^\mu = \bar{e}_R \gamma^\mu e_R & & & J_{b5}^\mu = J_2^\mu + J_3^\mu - J_1^\mu \equiv \bar{u}\gamma^\mu \gamma_5 u + \bar{d}\gamma^\mu \gamma_5 d.
\end{array}$$

Conserved charges:

Y	**local symmetry**
$N_L - N_{\bar{L}}$	**OK**
$N_B - N_{\bar{B}}$	**OK**
$N_L + N_{\bar{L}}$	**not observed**
$N_B + N_{\bar{B}}$	**not observed**

With N families, the symmetry is much larger: generation mixings also allowed,

$$\psi_r^i \to U_r^{ij}\, \psi_r^j; \quad U_r^\dagger U_r = I$$

A global $[U(N)]^5$ symmetry which is not present in observed phenomena. This is called an accidental symmetry.

Accidental symmetries are an accidental consequence of gauge invariance and renormalizability. For example, fermion mass terms would break accidental symmetries explicitly:

$$-m\bar{\psi}\psi = -m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$$

but they are forbidden by gauge invariance.

Accidental symmetries can be broken (while preserving gauge invariance) by fermion couplings to ϕ:

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}$$

$$\psi_1^i \equiv Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}; \quad \psi_2^i \equiv u_R^i; \quad \psi_3^i \equiv d_R^i; \quad \psi_4^i \equiv L_L^i = \begin{pmatrix} \nu_L^i \\ \ell_L^i \end{pmatrix}; \quad \psi_5^i \equiv \ell_R^i$$

$$\mathcal{L}_{\text{Yukawa}} = -\bar{Q}_L^i \, h_u^{ij} \, u_R^j \, \tilde{\phi} - \bar{Q}_L^i \, h_d^{ij} \, d_R^j \, \phi - \bar{L}_L^i \, h_\ell^{ij} \, \ell_R^j \, \phi + \text{h.c.}$$

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \tilde{\phi} = \begin{pmatrix} \phi_0^* \\ -\phi^- \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

- $\mathcal{L}_{\mathrm{Yukawa}}$ **is allowed by Lorentz invariance, gauge symmetry and renormalizability.**
- **Each term in** $\mathcal{L}_{\mathrm{Yukawa}}$ **breaks part of the accidental symmetry explicitly; for example, the first term is not invariant under independent** $U(N)$ **rotation of right-handed up quarks and left-handed quark doublets**

$$Q_L \to U\, Q_L; \qquad u_R \to V\, u_R$$

(although still invariant under the subgroup $U = V$**).**

- **Important remark: For the same reason, removing one or more term from** $\mathcal{L}_{\mathrm{Yukawa}}$ **increases the symmetry of the theory. Yukawa couplings are protected from receiving large radiative corrections.**

In matrix notation

$$\mathcal{L}_{\text{Yukawa}} = -\bar{Q}_L\, h_u\, u_R\, \tilde{\phi} - \bar{Q}_L\, h_d\, d_R\, \phi - \bar{L}_L\, h_\ell\, \ell_R\, \phi + \text{h.c.}$$

h_u, h_d, h_ℓ **are generic complex** $N \times N$ **matrices.**

A theorem in linear algebra: Any generic complex squared matrix h **can be diagonalized by a bi-unitary transformation**

$$\hat{h} = U^\dagger\, h\, V$$

where U, V **are unitary matrices, and** $\hat{h}$ **is diagonal with real positive entries.**

Thus, for example, we may redefine the lepton fields by

$$L_L \to U\, L_L; \qquad \ell_R \to V\, \ell_R$$

with U, V **such that**

$$\hat{h}_\ell = U^\dagger\, h_\ell\, V$$

is diagonal with real and positive entries.

The theory is otherwise unaffected, because this operation leaves the rest of $\mathcal{L}_{SM}$ unchanged. Hence, in the leptonic sector,

$$\begin{aligned}
\mathcal{L}_{\rm Yukawa}^{\rm lept} &= -\bar{L}_L\,\hat{h}_\ell\,\ell_R\,\phi + \text{h.c.} \to -\frac{1}{\sqrt{2}}\bar{\ell}_L\,\hat{h}_\ell\,\ell_R\,(v+H) + \text{h.c.} \\
&= -m_e\bar{e}e - m_\mu\bar{\mu}\mu - m_\tau\bar{\tau}\tau - \frac{H}{\sqrt{2}}(\hat{h}_e\bar{e}e + \hat{h}_\mu\bar{\mu}\mu + \hat{h}_\tau\bar{\tau}\tau)
\end{aligned}$$

- **Lepton masses $m_\ell^i = \frac{\hat{h}_\ell^i\, v}{\sqrt{2}}$ are generated**
- **The original global symmetry is broken, but a residual $[U(1)]^3$ invariance**

  $$\ell^i \to e^{i\alpha_i}\,\ell^i$$

  is still present. This symmetry corresponds to the conservation of individual (e, μ, τ) leptonic numbers.

The same argument does not apply to the hadron sector:

$$\mathcal{L}^{\mathrm{hadr}}_{\mathrm{Yukawa}} = -\bar{Q}_L\, h_u\, u_R\, \tilde{\phi} - \bar{Q}_L\, h_d\, d_R\, \phi + \mathrm{h.c.}$$

We may transform the quark fields

$$u_L \longrightarrow U_L\, u_L; \quad d_L \longrightarrow V_L\, d_L; \quad u_R \longrightarrow U_R\, u_R; \quad u_R \longrightarrow V_R\, u_R$$

with $U_{L,R}, V_{L,R}$ chosen so that

$$\hat{h}_u = U_L^\dagger\, h_u\, U_R; \qquad \hat{h}_d = V_L^\dagger\, h_d\, V_R$$

are diagonal, but this is *not* a symmetry for the rest of the Lagrangian.

Only one term is affected by such a rotation: the charged-current interaction term in the hadron sector

$$
\begin{aligned}
\mathcal{L}_{cc} &= \frac{g}{\sqrt{2}} \left[\bar{u}_L \, \gamma^\mu \, d_L \, W^+_\mu + \bar{d}_L \, \gamma^\mu \, u_L \, W^-_\mu \right] \\
&\to \frac{g}{\sqrt{2}} \left[\bar{u}_L \, \gamma^\mu \, (U_L^\dagger V_L) d_L \, W^+_\mu + \bar{d}_L \, \gamma^\mu \, (V_L^\dagger U_L) u_L \, W^-_\mu \right]
\end{aligned}
$$

The matrix

$$V = U_L^\dagger V_L$$

is a unitary $N \times N$ matrix, usually called the Cabibbo-Kobayashi-Maskawa matrix.

With the Yukawa couplings in diagonal form we have

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}}^{\text{hadr}} &= -\bar{Q}_L\,\hat{h}_u\,u_R\,\tilde{\phi} - \bar{Q}_L\,\hat{h}_d\,d_R\,\phi + \text{h.c.}\\ &\to -\frac{1}{\sqrt{2}}(v+H)\left[\bar{u}_L\,\hat{h}_u\,u_R + \bar{d}_L\,\hat{h}_d\,d_R\right] + \text{h.c.}\end{aligned}$$

- **Quark mass terms appear:**

$$m_u^i = \frac{\hat{h}_u^i v}{\sqrt{2}} \qquad m_d^i = \frac{\hat{h}_d^i v}{\sqrt{2}}$$

- **The original global symmetry is lost; the residual symmetry is now a $U(1)$ symmetry**

$$u_L^i \to e^{i\alpha}\,u_L^i; \qquad d_L^i \to e^{i\alpha}\,d_L^i; \qquad u_R^i \to e^{i\alpha}\,u_R^i; \qquad d_R^i \to e^{i\alpha}\,d_R^i$$

with a common phase α for all flavours, because of the CKM mixing matrix. Baryon number conservation.

The entries of the CKM matrix are fundamental parameters of the theory: they must be extracted from experiments.

How many independent numbers does V contain? A generic $N \times N$ unitary matrix depends on N^2 independent real parameters. Some (N_A) of them can be thought of as rotation angles in the N-dimensional space of generations, and they are as many as the coordinate planes in N dimensions:

$$N_A = \begin{pmatrix} N \\ 2 \end{pmatrix} = \frac{1}{2} N (N-1).$$

The remaining

$$\hat{N}_P = N^2 - N_A = \frac{1}{2}\,N\,(N+1)$$

parameters are complex phases. Some can be removed by a redefinition of left-handed quarks:

$$u_L^f \to e^{i\alpha_f}\,u_L^f; \qquad d_L^g \to e^{i\beta_g}\,d_L^g$$

which leaves all terms in $\mathcal{L}_{\rm SM}$ unchanged except $\mathcal{L}_c^{\rm hadr}$, and therefore amount to a redefinition of the CKM matrix:

$$V_{fg} \to e^{i(\beta_g - \alpha_f)}\,V_{fg}$$

The $2N$ constants α_f, β_g can be chosen so that $2N-1$ phases are eliminated from the matrix V, since there are $2n-1$ independent differences $\beta_g - \alpha_f$. The number of really independent complex phases in V is therefore

$$N_P = \hat{N}_P - (2N-1) = \frac{1}{2}(N-1)(N-2)$$

To summarize, the total number of independent parameters in the CKM matrix is

$$N_A + N_P = (N-1)^2; \qquad N_P = \frac{1}{2}(N-1)(N-2)$$

Comments:

- **with $N = 1$ or $N = 2$ the CKM matrix can be made real. In particular, for $N = 2$ it is fixed by one rotation angle, the Cabibbo angle.**
- **The first case with non-trivial phases is $N = 3$, which corresponds to $N_P = 1$.**
- **The presence of complex coupling constants implies violation of the CP symmetry.**

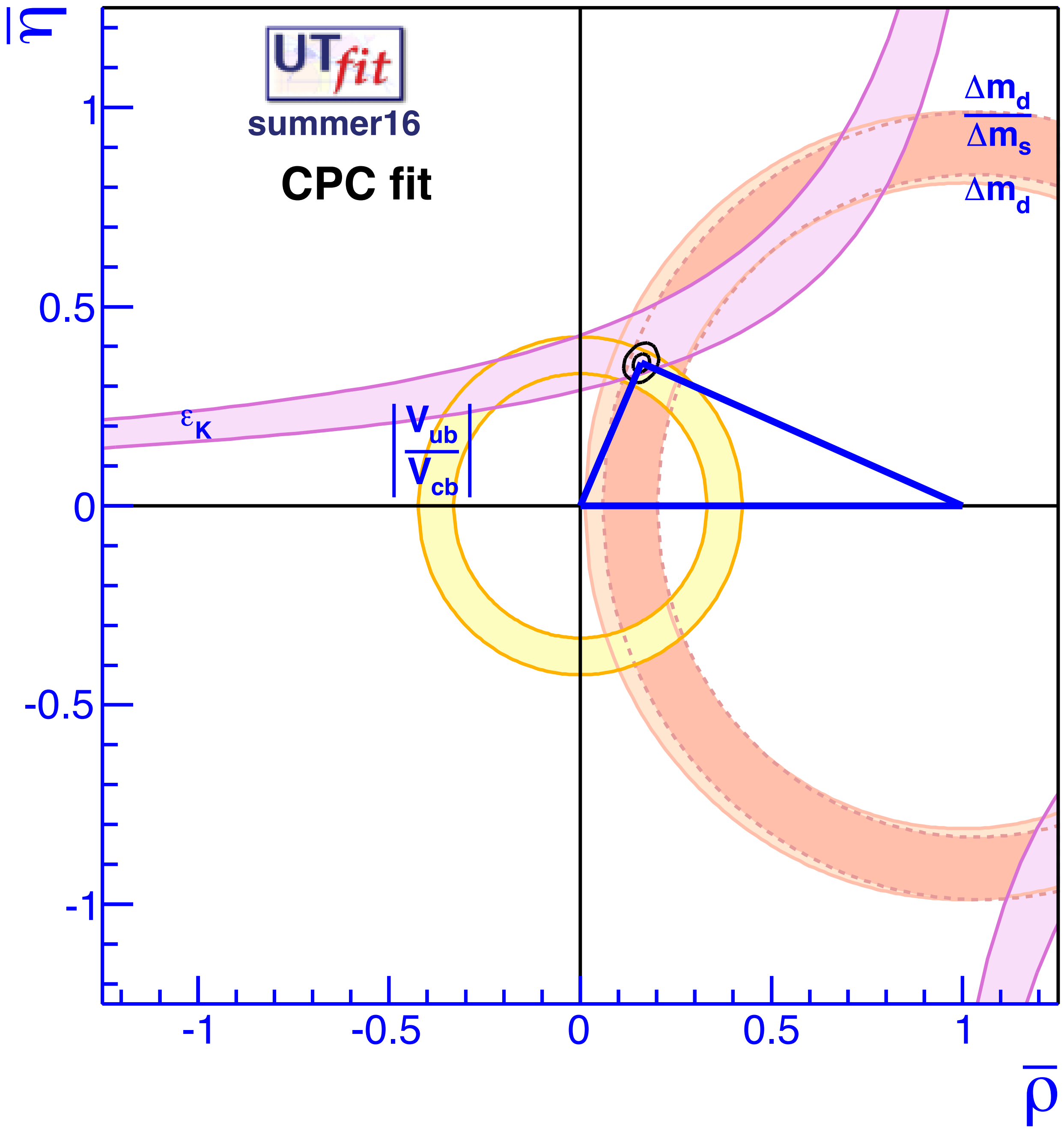
Much effort devoted to investigations in the flavour sector. A subject of special interest: CP violation in B systems and the unitarity relation

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

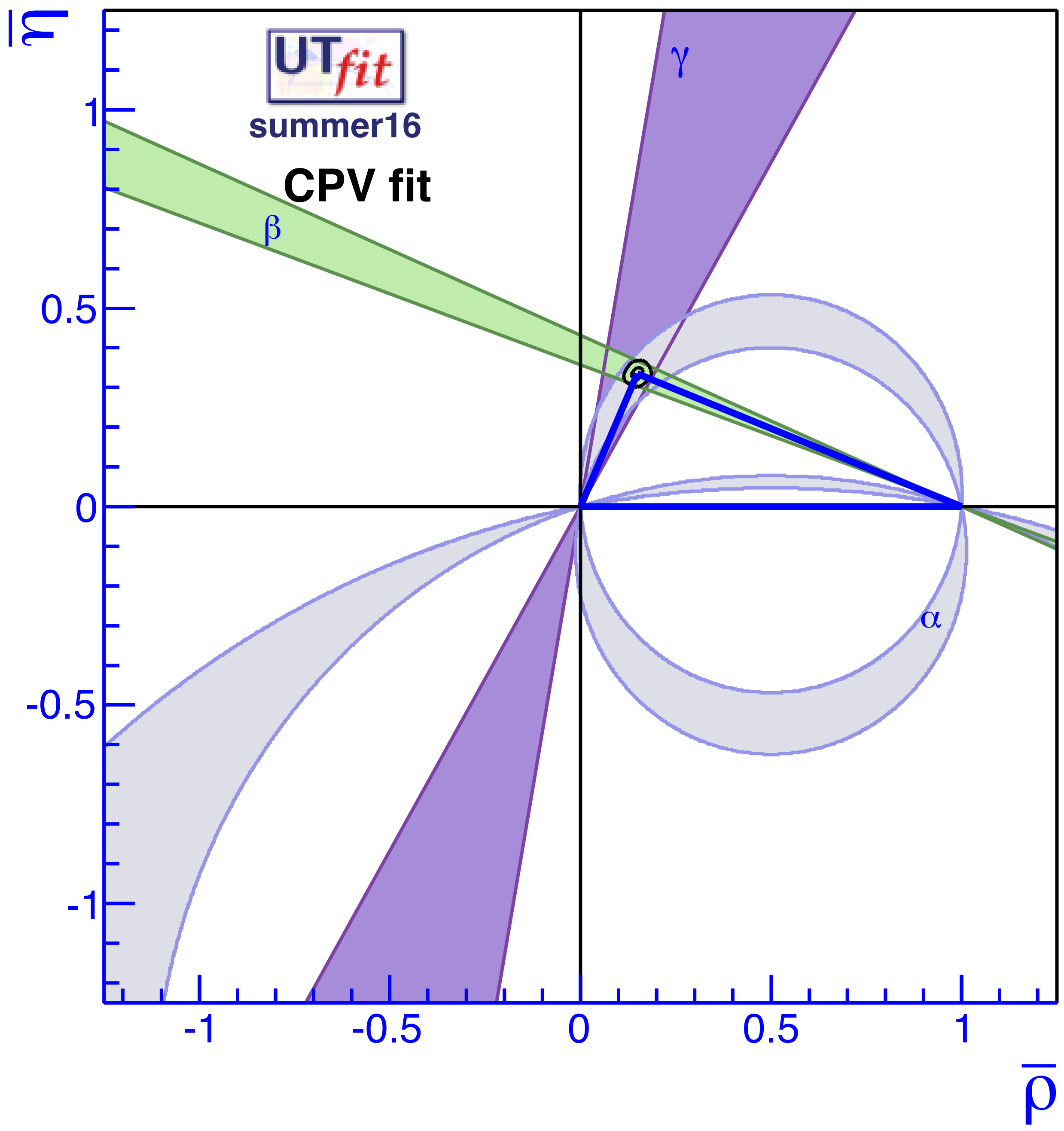
A useful parametrization (by L. Wolfenstein:)

$$V = \begin{pmatrix} 1-\lambda^2/2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

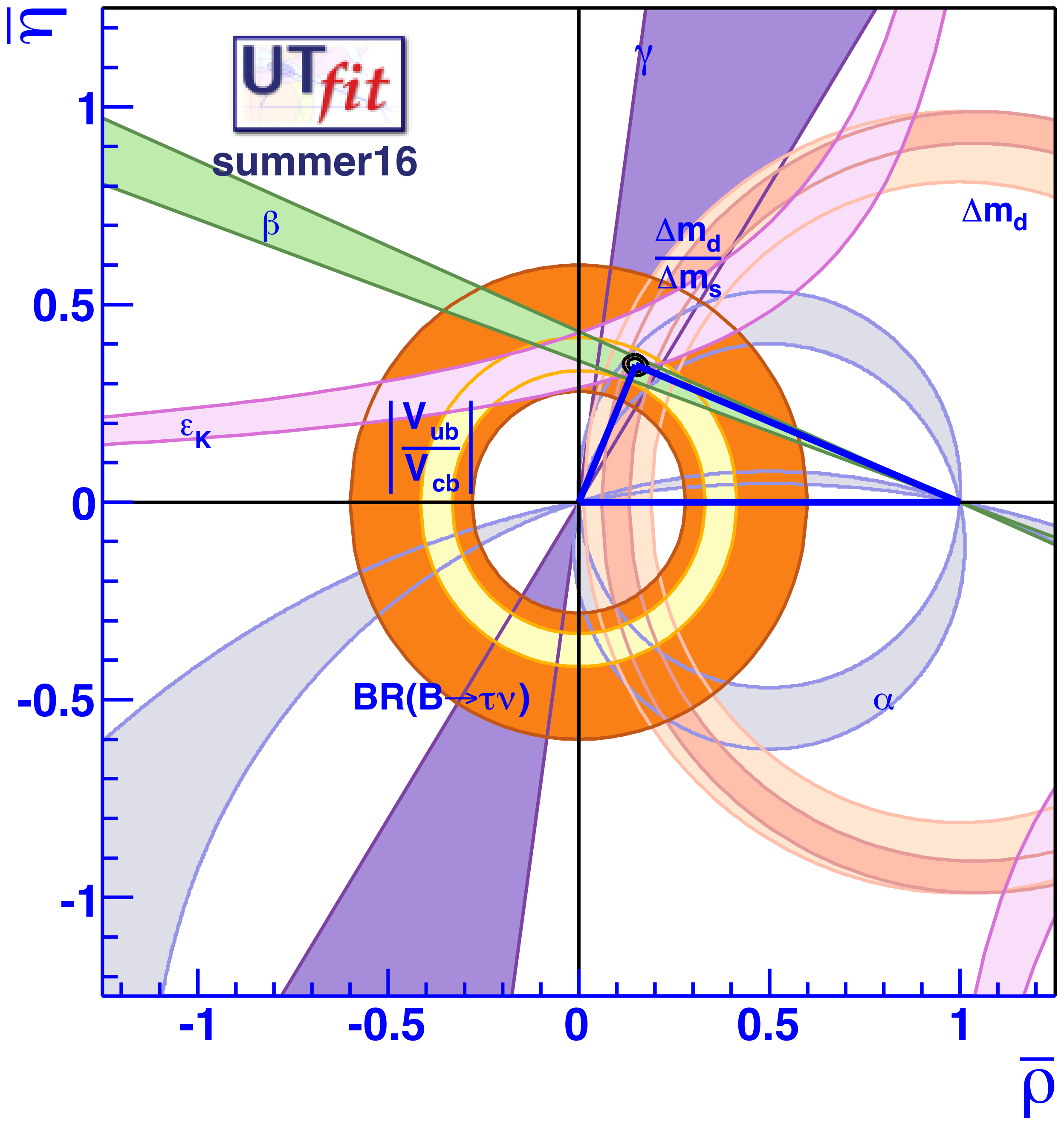
$$\bar{\rho} + i\bar{\eta} = \left(1 - \frac{\lambda^2}{2}\right)(\rho + i\eta)$$



UTfit
summer16
CPC fit
$\bar{\eta}$
$\bar{\rho}$
ε_K
$\left|\frac{V_{ub}}{V_{cb}}\right|$
$\frac{\Delta m_d}{\Delta m_s}$
Δm_d
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1



UT fit
summer16
CPV fit
$\bar{\eta}$
$\bar{\rho}$
β
γ
α
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1


UTfit
summer16
β
γ
ε_K
$\left|\frac{V_{ub}}{V_{cb}}\right|$
$\frac{\Delta m_d}{\Delta m_s}$
Δm_d
α
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$\bar{\eta}$
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1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Unitary triangle fit in the SM in the $\bar{\rho}$ - $\bar{\eta}$ plane

UTfit Collaboration M. Bona, M. Ciuchini, E. Franco, V. Lubicz, G. Martinelli, F. Parodi, M. Pierini, C. Schiavi, L. Silvestrini, V. Sordini, A. Stocchi, C. Tarantino, V. Vagnoni

Very nice:

- **Fermion masses generated**
- **All global symmetries broken except global baryon number and individual lepton numbers (no right-handed neutrinos)**
- **FCNC effects suppressed (flavour mixing confined in the charged-current sector)**
- **CP violated for $N \geq 3$**

Not easy to achieve the same result in different contexts.

(Slightly) beyond the Standard Model: neutrino masses

In the standard model, neutrinos are massless. With good reasons: the experimental upper bounds on neutrino masses are

$$m_{\nu_e} \leq 3\ \text{eV}; \qquad m_{\nu_\mu} \leq 0.19\ \text{MeV}; \qquad m_{\nu_\tau} \leq 18.2\ \text{MeV},$$

so $m_\nu \ll m_f$, and $m_\nu = 0$ is an excellent approximation.

However: non-zero neutrino masses are now a solid experimental evidence, thus we must ask how to modify the standard model in order to keep this evidence into account.

Standard neutrinos are massless because right-handed neutrinos do not exist (more precisely, they transform trivially under the gauge group, and therefore undergo no interaction: they are sterile objects).

Let us now assume right-handed neutrinos do exist (only one generation, for simplicity) with a covariant derivative term

$$\bar{\nu}_R \, i\not{D} \nu_R \equiv \bar{\nu}_R \, i\not{\partial} \, \nu_R .$$

A Dirac mass term can be generated as in the case of up-type quarks:

$$\mathcal{L}_{\text{Yukawa}} \to \mathcal{L}_{\text{Yukawa}} - h_\nu \left[\bar{\ell}_L \, \tilde{\phi} \, \nu_R + \bar{\nu}_R \, \tilde{\phi}^\dagger \, \ell_L \right]$$

contains a term

$$-m \, (\bar{\nu}_L \, \nu_R + \bar{\nu}_R \, \nu_L) \, ; \qquad m = \frac{h_\nu v}{\sqrt{2}}$$

The see-saw mechanism

Why are neutrino masses so much smaller than other fermions' masses? Indeed, the experimental bound implies

$$\frac{h_\nu}{h_e} = \frac{m}{m_e} \lesssim 10^{-6}$$

difficult to understand.

However, right-handed neutrinos also admit a Majorana mass term:

$$-\frac{1}{2}\, M \; \left(\bar{\nu}_R^c\, \nu_R + \bar{\nu}_R\, \nu_R^c\right)$$

where $\nu_R^c = \gamma_0\gamma_2\bar{\nu}_R^T$ is the charge-conjugated spinor (not true for other fermions, e.g. ν_L, because of gauge invariance).

Majorana mass terms induce violation of lepton number conservation, typically suppressed by inverse powers of M. It is natural to assume that M is of the order of the energy scale characteristic of the unknown phenomena (e.g. the effects of grand unification) experienced by right-handed neutrinos.

The most general neutrino mass term:

$$\mathcal{L}_{\nu\,\mathrm{mass}} = -\frac{1}{2}\,(\bar{\nu}_L^c\;\bar{\nu}_R)\left(\begin{array}{cc} 0 & m \\ m & M \end{array}\right)\left(\begin{array}{c} \nu_L \\ \nu_R^c \end{array}\right) + \mathrm{h.c.}$$

diagonalized by a linear transformation

$$\left(\begin{array}{cc} 0 & m \\ m & M \end{array}\right) = \mathcal{U}^T \left(\begin{array}{cc} m_1 & 0 \\ 0 & m_2 \end{array}\right) \mathcal{U}$$

with $\mathcal{U}$ unitary, and m_1, m_2 real and positive:

$$\mathcal{U} = \left(\begin{array}{cc} i\cos\theta & -i\sin\theta \\ \sin\theta & \cos\theta \end{array}\right); \qquad \tan 2\theta = \frac{2m}{M}$$

and

$$m_1 = \frac{1}{2}\left(\sqrt{M^2 + 4m^2} + M\right); \qquad m_2 = \frac{1}{2}\left(\sqrt{M^2 + 4m^2} - M\right).$$

For $m \ll M$, $\theta \simeq m/M$, and

$$m_1 \simeq M; \qquad m_2 \simeq \frac{m^2}{M}$$

One eigenstate not observed at low energy; the other is lighter than ordinary fermions by a factor m/M.

This is the *see-saw* mechanism.

The mass term takes the form

$$\mathcal{L}_{\nu\,\text{mass}} = -\frac{1}{2}\, m_1\, \left(\bar{\nu}_1^c\, \nu_1 + \bar{\nu}_1\, \nu_1^c\right) - \frac{1}{2}\, m_2\, \left(\bar{\nu}_2^c\, \nu_2 + \bar{\nu}_2\, \nu_2^c\right),$$

where

$$\nu_1 = \nu_L\, \sin\theta + \nu_R^c\, \cos\theta$$
$$\nu_2 = -i\nu_L\, \cos\theta + i\nu_R^c\, \sin\theta$$

General case: N species of left-handed neutrinos, ($N = 3$ as far as we know), plus K right-handed neutrinos (not necessarily $N = K$).

m is a $K \times N$ matrix, and M a $K \times K$ matrix.

Choosing $K = N$ we have

$$\begin{aligned}\mathcal{L}_Y^{\text{lept}} &= -\left[\overline{\ell}_L\,\phi\,h_{\text{E}}\,e_R + \overline{e}_R\,\phi^\dagger\,h_{\text{E}}^\dagger\,\ell_L\right] \\ &\quad -\left[\overline{\ell}_L\,\tilde{\phi}\,h_{\text{N}}\,\nu_R + \bar{\nu}_R\,\tilde{\phi}^\dagger\,h_{\text{N}}^\dagger\,\ell_L\right]\end{aligned}$$

The Majorana mass terms for right-handed neutrinos are

$$-\frac{1}{2}\,\left(\bar{\nu}_R'^c\,M\,\nu_R + \bar{\nu}_R\,M^\dagger\,\nu_R'^c\right)$$

Lepton flavour eigenstates are linear combinations of mass eigenstates. Neutrinos produced with a definite flavour (e.g. nuclear β decays in the Sun produce electron neutrinos)

A neutrino beam of definite flavour, is a linear combination of mass eigenstates:

$$|\nu_\alpha\rangle = \sum_{i=1}^{n} U^*_{\alpha i} \, |\nu_i\rangle$$

with U a unitary matrix. Time evolution in the rest frame is given by

$$|\nu_i(\tau)\rangle = e^{-im_i\tau} \, |\nu_i(0)\rangle$$

or, in the laboratory frame,

$$|\nu_i(t)\rangle = e^{-i(E_i t - p_i L)} \, |\nu_i(0)\rangle$$

where L is the distance travelled in the time interval t.

Since neutrinos are almost massless,

$$L \simeq t; \qquad E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2E}$$

where $E \simeq p_i \simeq p_j$. Hence,

$$|\nu_\alpha(L)\rangle \simeq \sum_{i=1}^{n} U_{\alpha i}^* \exp\left(-i\frac{m_i^2}{2E}L\right) |\nu_i(0)\rangle$$

The probability amplitude of observing the flavour β at distance L is given by

$$\begin{aligned} \langle\nu_\beta|\nu_\alpha(L)\rangle &= \sum_{i=1}^{n} U_{\alpha i}^* \exp\left(-i\frac{m_i^2}{2E}L\right) \sum_{j=1}^{n} U_{\beta j} \langle\nu_j|\nu_i\rangle \\ &= \sum_{i=1}^{n} \xi_i^{\alpha\beta} \exp\left(-i\epsilon_i L\right), \end{aligned}$$

where we have used the unitarity of U, and we have defined

$$\xi_i^{\alpha\beta} = U_{\alpha i}^* U_{\beta i}; \qquad \epsilon_i = \frac{m_i^2}{2E}.$$

The corresponding probability is given by

$$P_{\alpha\beta}(L) = |\langle \nu_\beta | \nu_\alpha(L) \rangle|^2 = \delta\alpha\beta - 4 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{Re}\left(\xi_i^{\alpha\beta}\, \xi_j^{*\alpha\beta} \right) \sin^2 \frac{1}{2}(\epsilon_j - \epsilon_i) L$$
$$-2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{Im}\left(\xi_i^{\alpha\beta}\, \xi_j^{*\alpha\beta} \right) \sin(\epsilon_j - \epsilon_i) L.$$

Not very rigorous: Quantum states with definite momentum have an infinite uncertainty in position, and therefore it makes no sense to talk about observation at distance L. We should introduce wave packets, and check that a sizable overlap among packets survives at distance L from the source.

If this is the case, the oscillation probability is correctly given by the above formula.

A simple case: CP invariance + mixing between two flavours. In this case

$$\xi_1^{12} = \xi_1^{21} = -\cos\theta_{12}\sin\theta_{12}$$
$$\xi_2^{12} = \xi_2^{21} = +\cos\theta_{12}\sin\theta_{12}.$$

and therefore

$$P_{12}(L) = P_{21}(L) = \sin^2 2\theta_{12}\ \sin^2\frac{L\Delta m_{12}^2}{4E}.$$

The results of neutrino oscillation experiments are usually displayed in the form of allowed regions in the $(\Delta m_{12}^2, \theta_{12})$ **plane. The following units are often adopted:**

$$\frac{L\Delta m^2}{4E} \simeq 1.27\,\frac{\Delta m^2(\mathrm{eV}^2)\,L(\mathrm{km})}{E(\mathrm{GeV})}.$$

To summarize:

- **Massive neutrinos bring us out of the Standard Model**
- **Heavy sterile neutrinos + see-saw mechanism: a satisfactory scenario**
- **New parameters needed, but no radical modification**

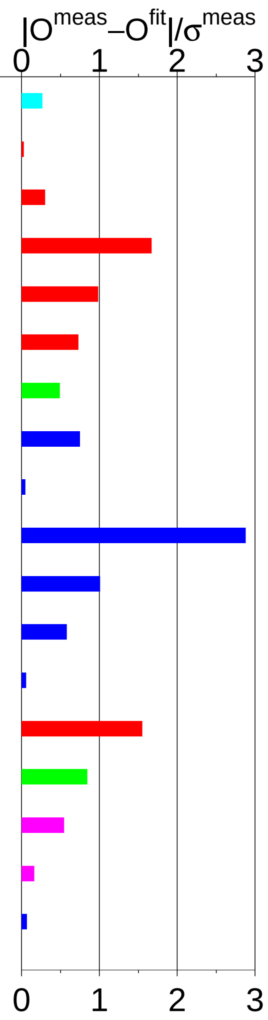
5. Experimental confirmations

- **1974 charm quark and weak neutral currents observed**
- **1977 bottom quark**
- **1983 W and Z vector bosons**
- **1994 top quark**
- **2012 Higgs boson**

Precision data in excellent agreement with SM predictions. The standard model tested at the level of one-loop corrections.

$$N_\nu = 2.9841 \pm .0083$$

	Measurement	Fit
$\Delta\alpha_{had}^{(5)}(m_Z)$	0.02750 ± 0.00033	0.02759
m_Z [GeV]	91.1875 ± 0.0021	91.1874
Γ_Z [GeV]	2.4952 ± 0.0023	2.4959
σ_{had}^{0} [nb]	41.540 ± 0.037	41.478
R_l	20.767 ± 0.025	20.742
$A_{fb}^{0,l}$	0.01714 ± 0.00095	0.01645
$A_l(P_\tau)$	0.1465 ± 0.0032	0.1481
R_b	0.21629 ± 0.00066	0.21579
R_c	0.1721 ± 0.0030	0.1723
$A_{fb}^{0,b}$	0.0992 ± 0.0016	0.1038
$A_{fb}^{0,c}$	0.0707 ± 0.0035	0.0742
A_b	0.923 ± 0.020	0.935
A_c	0.670 ± 0.027	0.668
A_l(SLD)	0.1513 ± 0.0021	0.1481
$\sin^2\theta_{eff}^{lept}(Q_{fb})$	0.2324 ± 0.0012	0.2314
m_W [GeV]	80.385 ± 0.015	80.377
Γ_W [GeV]	2.085 ± 0.042	2.092
m_t [GeV]	173.20 ± 0.90	173.26



March 2012

The Lagrangian density of the Standard Model in the Unitary gauge

$$\mathcal{L}_{\rm SM} = \mathcal{L}_0 + \mathcal{L}_{\rm em} + \mathcal{L}_c + \mathcal{L}_n + \mathcal{L}_{\rm V} + \mathcal{L}_{\rm Higgs}$$

where

$$\begin{aligned}
\mathcal{L}_0 \quad = \quad & \sum_{i=1}^{n} \left[\bar{\nu}^i \, i\partial\!\!\!/ \, \nu^i + \bar{e}^i \, (i\partial\!\!\!/ - m_{\rm E}^i) \, e^i + \bar{u}^i \, (i\partial\!\!\!/ - m_{\rm U}^i) \, u^i + \bar{d}^i \, (i\partial\!\!\!/ - m_{\rm D}^i) \, d^i\right] \\
& - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{2} \, m_{\rm Z}^2 \, Z^\mu Z_\mu - \frac{1}{2} W^+_{\mu\nu} W^{\mu\nu}_- + m_{\rm W}^2 \, W^{\mu +} W^-_\mu - \frac{1}{2} \partial_\mu A_\nu \, \partial^\mu A^\nu \\
& + \frac{1}{2} \, \partial^\mu H \, \partial_\mu H - \frac{1}{2} \, m_H^2 \, H^2,
\end{aligned}$$

with

$$Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu; \qquad W^{\mu\nu}_\pm = \partial^\mu W^\nu_\pm - \partial^\nu W^\mu_\pm .$$

The index i labels the N fermion families ($N = 3$ according to our present knowledge).

$\mathcal{L}_{\rm em}$ **is the electromagnetic interaction term for charged fermions:**

$$\mathcal{L}_{\rm em} = e \sum_{i=1}^{n} \left(-\bar{e}^i \, \gamma_\mu \, e^i + \frac{2}{3} \bar{u}^i \, \gamma_\mu \, u^i - \frac{1}{3} \bar{d}^i \, \gamma_\mu \, d^i \right) A^\mu ,$$

where e **is the proton charge.**

$\mathcal{L}_c$ **is the weak charged-current interaction term:**

$$\begin{aligned} \mathcal{L}_c \quad &= \quad \frac{g}{2\sqrt{2}} \left[\sum_{i=1}^{n} \bar{\nu}^i \, \gamma^\mu (1-\gamma_5) \, e^i + \sum_{i,j=1}^{n} \bar{u}^i \, \gamma^\mu (1-\gamma_5) \, V_{ij} \, d^j \right] W_\mu^+ \\ &+ \quad \frac{g}{2\sqrt{2}} \left[\sum_{i=1}^{n} \bar{e}^i \, \gamma^\mu (1-\gamma_5) \, \nu^i + \sum_{i,j=1}^{n} \bar{d}^j \, \gamma^\mu (1-\gamma_5) \, V_{ij}^* \, u^i \right] W_\mu^- , \end{aligned}$$

where g **is the** $SU_L(2)$ **coupling constant.**

$\mathcal{L}_n$ is the weak neutral-current interaction term:

$$\begin{aligned}\mathcal{L}_n &= \frac{g}{4\cos\theta_{\mathrm{W}}}\sum_{i=1}^{n}\Bigg[\bar{\nu}^i\,\gamma_\mu(1-\gamma_5)\,\nu^i + \bar{e}^i\,\gamma_\mu\left(-1+4\sin^2\theta_{\mathrm{W}}+\gamma_5\right)\,e^i \\ &+ \bar{u}^i\,\gamma_\mu\left(1-\frac{8}{3}\sin^2\theta_{\mathrm{W}}-\gamma_5\right)\,u^i + \bar{d}^i\,\gamma_\mu\left(-1+\frac{4}{3}\sin^2\theta_{\mathrm{W}}+\gamma_5\right)\,d^i\Bigg]Z^\mu.\end{aligned}$$

$\mathcal{L}_{\mathrm{V}}$ **is the interaction term among vector bosons:**

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} \quad = \quad & +ig\sin\theta_{\mathrm{W}}\left(W^{+}_{\mu\nu}W^{\mu}_{-}A^{\nu} - W^{-}_{\mu\nu}W^{\mu}_{+}A^{\nu} + F_{\mu\nu}W^{\mu}_{+}W^{\nu}_{-}\right) \\
& +ig\cos\theta_{\mathrm{W}}\left(W^{+}_{\mu\nu}W^{\mu}_{-}Z^{\nu} - W^{-}_{\mu\nu}W^{\mu}_{+}Z^{\nu} + Z_{\mu\nu}W^{\mu}_{+}W^{\nu}_{-}\right) \\
& +\frac{g^2}{2}\left(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\left[\frac{1}{2}W^{+}_{\mu}W^{+}_{\nu}W^{-}_{\rho}W^{-}_{\sigma}\right. \\
& \left. -W^{+}_{\mu}W^{-}_{\nu}\left(A_{\rho}A_{\sigma}\sin^2\theta_{\mathrm{W}} + Z_{\rho}Z_{\sigma}\cos^2\theta_{\mathrm{W}} + 2A_{\rho}Z_{\sigma}\sin\theta_{\mathrm{W}}\cos\theta_{\mathrm{W}}\right)\right],
\end{aligned}$$

where

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}.$$

$\mathcal{L}_{\mathrm{Higgs}}$ is the interaction term between the Higgs scalar $H(x)$ and fermions and vectors:

$$\begin{aligned}\mathcal{L}_{\mathrm{Higgs}} \quad = \quad & \left(m_{\mathrm{W}}^2\, W^{\mu +} W_\mu^- + \frac{1}{2}\, m_{\mathrm{Z}}^2\, Z^\mu Z_\mu\right)\left(\frac{H^2}{v^2} + \frac{2H}{v}\right) \\ & - \frac{H}{v}\sum_{i=1}^{n}(m_{\mathrm{D}}^i\, \bar{d}^i d^i + m_{\mathrm{U}}^i\, \bar{u}^i u^i + m_{\mathrm{E}}^i\, \bar{e}^i e^i) \\ & - \lambda v H^3 - \frac{1}{4}\lambda H^4 .\end{aligned}$$

Parameters of the Standard Model

The parameters appearing in $\mathcal{L}_{\mathrm{SM}}$ are not all independent. The gauge-Higgs sector is entirely defined by four quantities:

$$g, \quad g', \quad v, \quad m_H.$$

because

$$m_{\mathrm{W}}^2 = \frac{1}{4} g^2 v^2 \quad m_{\mathrm{Z}}^2 = \frac{1}{4}(g^2 + {g'}^2) v^2$$

and

$$\lambda = \frac{m_H^2}{2v^2}; \qquad \tan\theta_{\mathrm{W}} = \frac{g'}{g},$$

where we have used $g \sin\theta_{\mathrm{W}} = g' \cos\theta_{\mathrm{W}} = e$.

In pratice, v, g, g' are usually traded for G_F, $\alpha_{\rm em}$, $m_{\rm Z}$, which are known with high accuracy:

$$G_F = \frac{g^2\sqrt{2}}{8m_{\rm W}^2} = \frac{1}{\sqrt{2}v^2}$$

$$\alpha_{\rm em} = \frac{e^2}{4\pi} = \frac{g^2 {g'}^2}{4\pi(g^2 + {g'}^2)}$$

$$m_{\rm Z}^2 = \frac{1}{4}(g^2 + {g'}^2)v^2,$$

which give

$$v^2 = \frac{1}{\sqrt{2}G_F}$$

$$g^2 = 2\sqrt{2}m_{\rm Z}^2 G_F \left(1 + \sqrt{1 - \frac{4\pi\alpha_{\rm em}}{\sqrt{2}m_{\rm Z}^2 G_F}}\right)$$

$$g'^2 = 2\sqrt{2}m_{\rm Z}^2 G_F \left(1 - \sqrt{1 - \frac{4\pi\alpha_{\rm em}}{\sqrt{2}m_{\rm Z}^2 G_F}}\right)$$

(assuming $\tan\theta_{\rm W} < 1$**).**

The present measured values are

$$G_F = 1.16637(1)\times 10^{-5}\ \mathrm{GeV}^{-2}; \qquad \alpha_{\rm em} = \frac{1}{128}; \qquad m_{\rm Z} = 91.1876\pm 0.0021\ \mathrm{GeV}$$

which give

$$v = 246.2\ \mathrm{GeV} \qquad \alpha_2 = \frac{g^2}{4\pi} = 0.033 \qquad \alpha_1 = \frac{{g'}^2}{4\pi} = 0.010.$$

Finally,

$$m_H = 125\ \mathrm{GeV}.$$

Fermionic sector:

$$m_u = 1.7 - 3.1 \text{ MeV}; \qquad m_d = 4.1 - 5.7 \text{ MeV}; \qquad m_s = 100^{+30}_{-20} \text{ MeV}.$$

Because of features of the strong interactions, heavy quark masses are known to a better level of accuracy:

$$m_c = 1.29^{+0.05}_{-0.11} \text{ GeV}; \qquad m_b = 4.19^{+0.18}_{-0.06} \text{ GeV}; \qquad m_t = 172.9 \pm 0.6 \pm 0.9 \text{ GeV}.$$

The Cabibbo-Kobayashi-Maskawa matrix V is specified by a total of $(N-1)^2$ real parameters for N fermion families. The absolute values of V_{ij} are

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix}$$

$$= \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & (3.89 \pm 0.44) \times 10^{-3} \\ 0.230 \pm 0.011 & 1.023 \pm 0.036 & (40.6 \pm 1.3) \times 10^{-3} \\ (8.4 \pm 0.6) \times 10^{-3} & (38.7 \pm 2.1) \times 10^{-3} & 0.88 \pm 0.07 \end{pmatrix}.$$

This gives a total of $3N$ masses (three for each fermion family) and $(N-1)^2$ phases and angles in V.

Total number of parameters in the standard model:

$$4 + 1 + 3N + (N-1)^2 = N^2 + N + 6$$

which is 18 for $N = 3$.

5. Strong interactions

It took some time (if compared the history with electromagnetic and weak interactions) to recognize that a local field theory for strong interactions exists.

The main steps along this path:

- **Hadrons are not pointlike objects;**
- **The hadron spectrum shows some approximate global symmetries which are useful in many respects;**
- **Quark model;**
- **A new quantum number (in addition to flavor and spin) is needed for a consistent description of hadron spectroscopy;**
- **The constituents of hadrons behave as almost non-interacting when hadrons are observed at high momentum transfer.**

The above considerations lead (almost univocally) to the formulation of Quantum Chromodynamics (QCD) as the field theory underlying strong interactions.

- **QCD is a local non-abelian gauge theory based on the gauge group $SU(3)$; quarks are spin-1/2 particles, and their Dirac fields are in the fundamental representation of $SU(3)$.**
- **The strong interaction is mediated by $N^2 - 1 = 8$ massless vector bosons, called gluons;**
- **QCD (with less then 16 flavors) is asymptotically free.**

One additional assumption is needed: only $SU(3)$ singlets are observable (in particular, quarks and gluons are not). This is called color confinement. A wonderfully simpler theory with a surprising wealth of physics content.

How is it possible to make predictions with QCD, given that its fundamental degrees of freedom (quarks and gluons) are not directly observable?

Factorization of collinear singularities: the parton model survives QCD radiative corrections:

$$d\sigma(p_1,p_2,Q^2) = \sum_{i,j}\int dz_1 \int dz_2\, f_i(z_1,\mu^2) f_j(z_2,\mu^2) d\hat{\sigma}_{ij}\left(z_1p_1, z_2p_2, \alpha_{\mathrm{S}}(\mu^2), \frac{\mu^2}{Q^2}\right)$$

where

- $d\hat{\sigma}_{ij}$ **(the partonic cross sections) are computable in perturbation theory, provided** Q^2 **is large enough;**
- $f_i(z,\mu^2)$ **(the parton distribution functions) cannot be computed from first principles, but are universal: they do not depend on the process, but only on the initial state.**

The scale dependence of parton distribution functions

$$\mu^2 \frac{\partial f_i(z, \mu^2)}{\partial \mu^2}$$

can also be computed perturbatively (Altarelli-Parisi evolution equations). This allows us (i.e. large collaborations) to extract them consistently from data taken at different scales, and to perform global fits.

An important source of uncertainty in theoretical predictions.

What does "large enough" mean? The QCD β function:

$$\mu^2 \frac{d\alpha_{\rm S}(\mu^2)}{d\mu^2} = -\beta_0 \alpha_{\rm S}^2(\mu^2) + O(\alpha_{\rm S}^3)$$

with

$$\beta_0 = \frac{33 - 2n_f}{12\pi} < 0$$

Leading log solution:

$$\alpha_{\rm S}(\mu^2) = \frac{\alpha_{\rm S}(\mu_0^2)}{1 + \beta_0 \alpha_{\rm S}(\mu_0^2) \log \frac{\mu^2}{\mu_0^2}}$$

which goes to zero as $\frac{\mu^2}{\mu_0^2} \to \infty$ (asymptotic freedom).

Sometimes useful to define a parameter Λ by

$$1 + \beta_0 \alpha_{\mathrm{S}}(\mu_0^2) \log \frac{\Lambda^2}{\mu_0^2} = 0$$

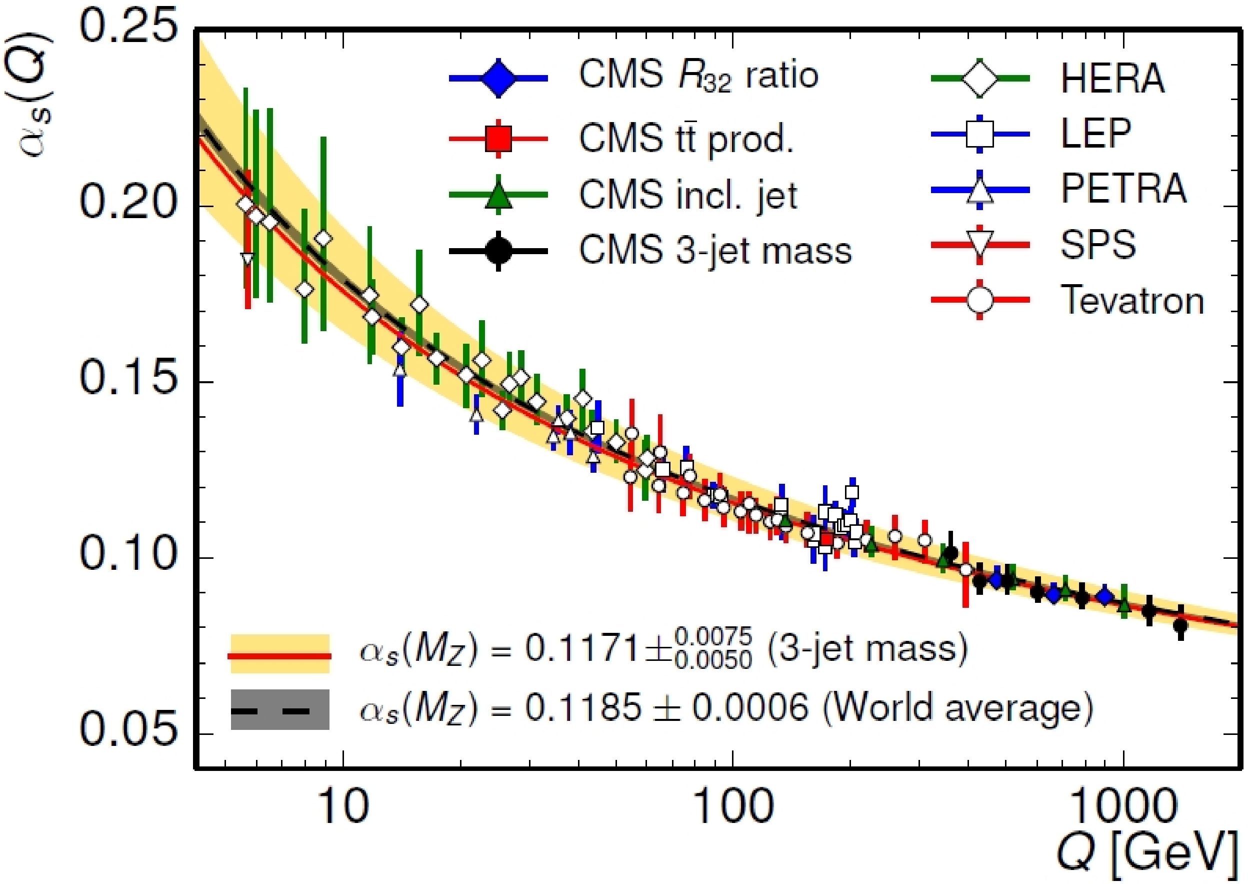
so that

$$\alpha_{\mathrm{S}}(\mu^2) = \frac{\alpha_{\mathrm{S}}(\mu_0^2)}{1 + \beta_0 \alpha_{\mathrm{S}}(\mu_0^2) \log \frac{\Lambda^2}{\mu_0^2} + \beta_0 \alpha_{\mathrm{S}}(\mu_0^2) \log \frac{\mu^2}{\Lambda^2}} = \frac{1}{\beta_0 \log \frac{\mu^2}{\Lambda^2}}$$

(dimensional transmutation). It turns out that

$$\Lambda \sim 150 \text{ MeV}$$

QCD enters the perturbative regime at scales $Q^2 \gg \Lambda^2$.



$\alpha_s(Q)$
0.25
0.20
0.15
0.10
0.05
CMS R_{32} ratio
CMS $t\bar{t}$ prod.
CMS incl. jet
CMS 3-jet mass
HERA
LEP
PETRA
SPS
Tevatron
$\alpha_s(M_Z) = 0.1171 \pm^{0.0075}_{0.0050}$ (3-jet mass)
$\alpha_s(M_Z) = 0.1185 \pm 0.0006$ (World average)
10
100
1000
Q [GeV]

Hadronic cross sections characterized by large energy scales (e.g. the production of heavy objects) can be reliably computed using perturbative QCD + precisely measured parton densities.

Typical examples:

- **Drell-Yan pairs: virtual photons with large virtuality, or weak vector bosons;**
- **heavy quarks and heavy quark pairs;**
- **Higgs boson;**
- **jets at large transverse momentum**

We do not know how to prove color confinement by a first-principle calculation: the hadron-formation scale is well outside the perturbative domain. However:

- **The fact that $\alpha_S(\mu^2)$ grows very large at energy scales of the order of the typical hadrom mass is consistent with confinement;**
- **lattice calculations show that for example the strong quark-antiquark potential is Coulomb-like at short distances, but grows linearly ar large distances:**

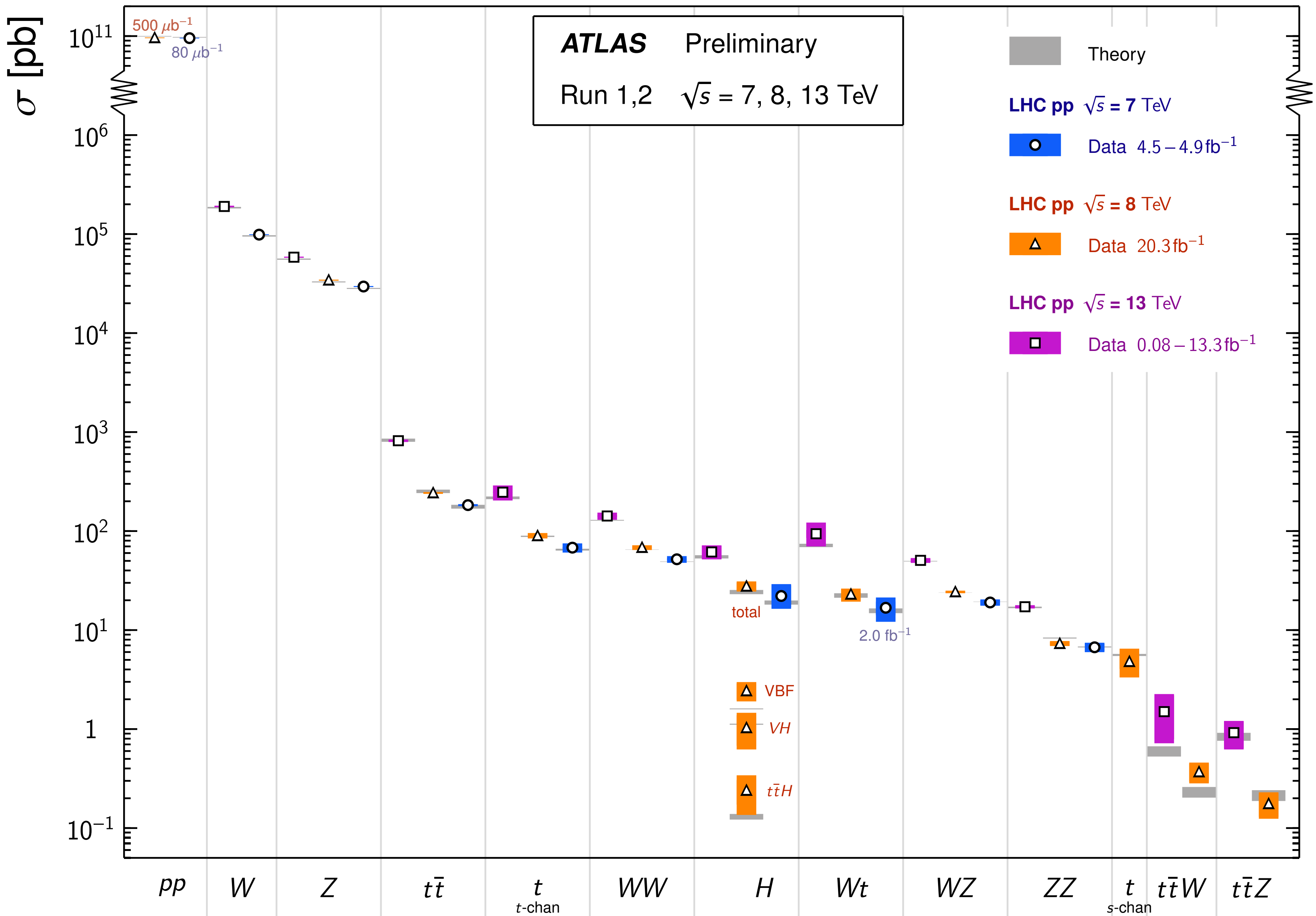
$$V_{q\bar{q}}(r) = C_F \left[\frac{\alpha_S(1/r^2)}{r} + \ldots + \sigma r \right]$$

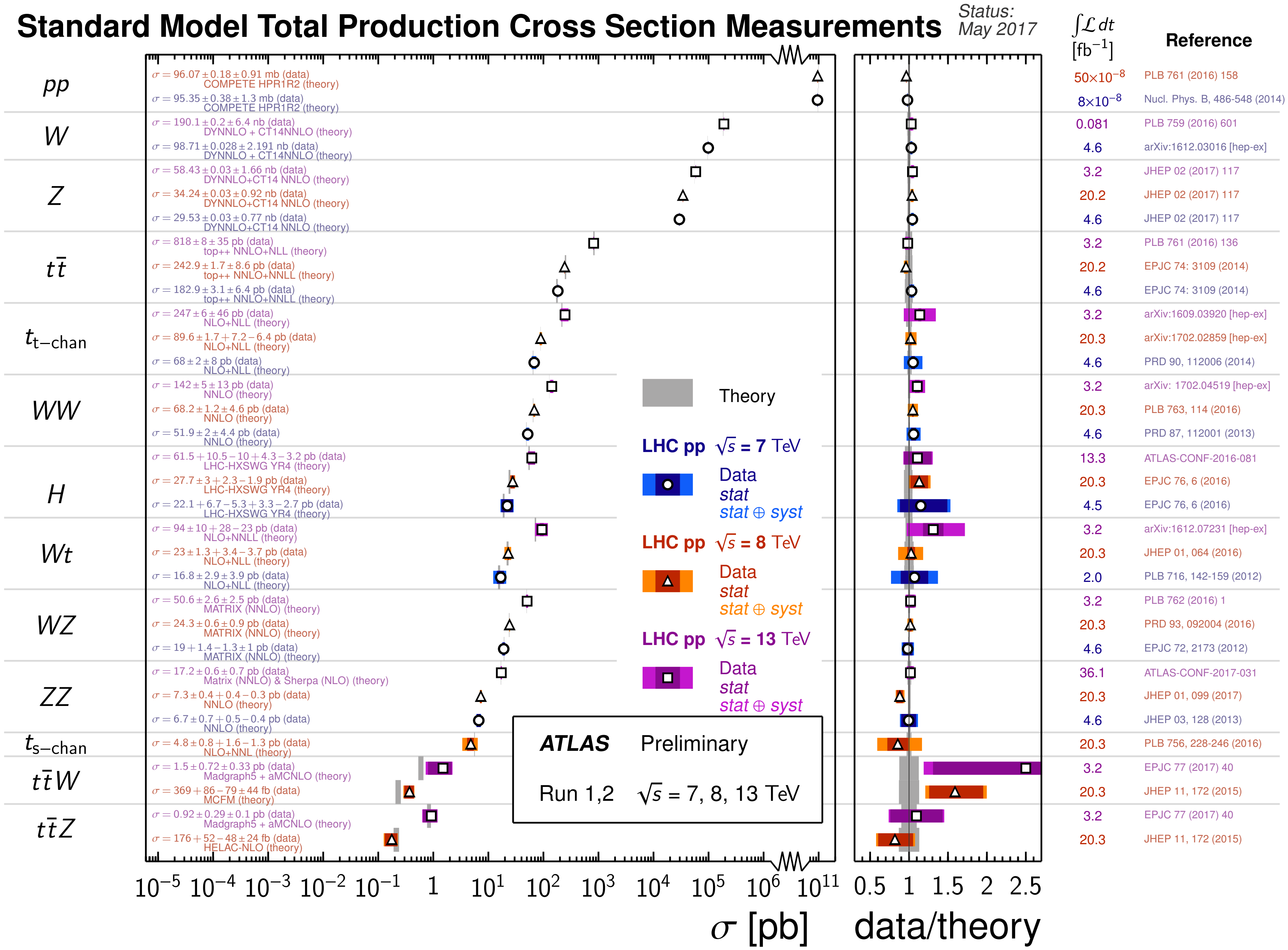
The linearly increasing term in the potential makes it energetically impossible to separate a $q\bar{q}$ pair. If the pair is created at one spacetime point, for example in e^+e^- annihilation, and then the quark and the antiquark start moving away from each other in the center-of-mass frame. It is energetically favourable to create additional pairs.

Confinement provide an explanation of the short range of nuclear forces: massless gluon exchange would be long range, but nucleons are color singlets, and cannot exchange colour octet gluons. The lightest colour singlet hadronic particles are pions, and the range of nuclear forces is fixed by the pion mass $r \sim \frac{1}{m_\pi} \sim 10^{-13}$ cm, since $V(r) = \frac{e^{-m_\pi r}}{r}$.

Standard Model Total Production Cross Section Measurements

Status: May 2017





Standard Model Total Production Cross Section Measurements
Status: May 2017
∫ℒdt [fb⁻¹]
Reference
pp
W
Z
tt̄
t_t−chan
WW
H
Wt
WZ
ZZ
t_s−chan
tt̄W
tt̄Z
σ = 96.07 ± 0.18 ± 0.91 mb (data)
COMPETE HPR1R2 (theory)
σ = 95.35 ± 0.38 ± 1.3 mb (data)
COMPETE HPR1R2 (theory)
σ = 190.1 ± 0.2 ± 6.4 nb (data)
DYNNLO + CT14NNLO (theory)
σ = 98.71 ± 0.028 ± 2.191 nb (data)
DYNNLO + CT14NNLO (theory)
σ = 58.43 ± 0.03 ± 1.66 nb (data)
DYNNLO+CT14 NNLO (theory)
σ = 34.24 ± 0.03 ± 0.92 nb (data)
DYNNLO+CT14 NNLO (theory)
σ = 29.53 ± 0.03 ± 0.77 nb (data)
DYNNLO+CT14 NNLO (theory)
σ = 818 ± 8 ± 35 pb (data)
top++ NNLO+NLL (theory)
σ = 242.9 ± 1.7 ± 8.6 pb (data)
top++ NNLO+NNLL (theory)
σ = 182.9 ± 3.1 ± 6.4 pb (data)
top++ NNLO+NNLL (theory)
σ = 247 ± 6 ± 46 pb (data)
NLO+NLL (theory)
σ = 89.6 ± 1.7 + 7.2 − 6.4 pb (data)
NLO+NLL (theory)
σ = 68 ± 2 ± 8 pb (data)
NLO+NLL (theory)
σ = 142 ± 5 ± 13 pb (data)
NNLO (theory)
σ = 68.2 ± 1.2 ± 4.6 pb (data)
NNLO (theory)
σ = 51.9 ± 2 ± 4.4 pb (data)
NNLO (theory)
σ = 61.5 + 10.5 − 10 + 4.3 − 3.2 pb (data)
LHC-HXSWG YR4 (theory)
σ = 27.7 ± 3 + 2.3 − 1.9 pb (data)
LHC-HXSWG YR4 (theory)
σ = 22.1 + 6.7 − 5.3 + 3.3 − 2.7 pb (data)
LHC-HXSWG YR4 (theory)
σ = 94 ± 10 + 28 − 23 pb (data)
NLO+NNLL (theory)
σ = 23 ± 1.3 + 3.4 − 3.7 pb (data)
NLO+NLL (theory)
σ = 16.8 ± 2.9 ± 3.9 pb (data)
NLO+NLL (theory)
σ = 50.6 ± 2.6 ± 2.5 pb (data)
MATRIX (NNLO) (theory)
σ = 24.3 ± 0.6 ± 0.9 pb (data)
MATRIX (NNLO) (theory)
σ = 19 + 1.4 − 1.3 ± 1 pb (data)
MATRIX (NNLO) (theory)
σ = 17.2 ± 0.6 ± 0.7 pb (data)
Matrix (NNLO) & Sherpa (NLO) (theory)
σ = 7.3 ± 0.4 + 0.4 − 0.3 pb (data)
NNLO (theory)
σ = 6.7 ± 0.7 + 0.5 − 0.4 pb (data)
NNLO (theory)
σ = 4.8 ± 0.8 + 1.6 − 1.3 pb (data)
NLO+NNL (theory)
σ = 1.5 ± 0.72 ± 0.33 pb (data)
Madgraph5 + aMCNLO (theory)
σ = 369 + 86 − 79 ± 44 fb (data)
MCFM (theory)
σ = 0.92 ± 0.29 ± 0.1 pb (data)
Madgraph5 + aMCNLO (theory)
σ = 176 + 52 − 48 ± 24 fb (data)
HELAC-NLO (theory)
Theory
LHC pp √s = 7 TeV
Data
stat
stat ⊕ syst
LHC pp √s = 8 TeV
Data
stat
stat ⊕ syst
LHC pp √s = 13 TeV
Data
stat
stat ⊕ syst
ATLAS Preliminary
Run 1,2 √s = 7, 8, 13 TeV
10⁻⁵ 10⁻⁴ 10⁻³ 10⁻² 10⁻¹ 1 10¹ 10² 10³ 10⁴ 10⁵ 10⁶ 10¹¹
σ [pb]
0.5 1 1.5 2 2.5
data/theory
50×10⁻⁸ PLB 761 (2016) 158
8×10⁻⁸ Nucl. Phys. B, 486-548 (2014)
0.081 PLB 759 (2016) 601
4.6 arXiv:1612.03016 [hep-ex]
3.2 JHEP 02 (2017) 117
20.2 JHEP 02 (2017) 117
4.6 JHEP 02 (2017) 117
3.2 PLB 761 (2016) 136
20.2 EPJC 74: 3109 (2014)
4.6 EPJC 74: 3109 (2014)
3.2 arXiv:1609.03920 [hep-ex]
20.3 arXiv:1702.02859 [hep-ex]
4.6 PRD 90, 112006 (2014)
3.2 arXiv: 1702.04519 [hep-ex]
20.3 PLB 763, 114 (2016)
4.6 PRD 87, 112001 (2013)
13.3 ATLAS-CONF-2016-081
20.3 EPJC 76, 6 (2016)
4.5 EPJC 76, 6 (2016)
3.2 arXiv:1612.07231 [hep-ex]
20.3 JHEP 01, 064 (2016)
2.0 PLB 716, 142-159 (2012)
3.2 PLB 762 (2016) 1
20.3 PRD 93, 092004 (2016)
4.6 EPJC 72, 2173 (2012)
36.1 ATLAS-CONF-2017-031
20.3 JHEP 01, 099 (2017)
4.6 JHEP 03, 128 (2013)
20.3 PLB 756, 228-246 (2016)
3.2 EPJC 77 (2017) 40
20.3 JHEP 11, 172 (2015)
3.2 EPJC 77 (2017) 40
20.3 JHEP 11, 172 (2015)

6. Open questions

1. **Why so many (19) free parameters?**
   - **Is there a grand unification?**
   - **What is the origin of flavour mixing?**
   - **Neutrino physics**
2. **Why is the Higgs boson so light?**
   - **Hierarchy and Naturalness**
3. **Cosmology-related questions:**
   - **the value of the cosmological constant**
   - **baryogenesis**
   - **dark matter**
4. **Gravity**

Cross sections and decay width

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \overline{\sum} |\mathcal{M}|^2 \; d\Phi(p_1, p_2; k_1, \dots k_n)$$

$$d\Gamma = \frac{1}{2M} \overline{\sum} |\mathcal{M}|^2 \; d\Phi(p_1, p_2; k_1, \dots k_n)$$

$$d\Phi(p_1, p_2; k_1, \dots k_n) = \frac{d^3 k_1}{(2\pi)^3 2E_{\vec{k}_1}} \cdots \frac{d^3 k_n}{(2\pi)^3 2E_{\vec{k}_n}} (2\pi)^4 \delta(p_1 + p_2 - k_1 - \dots - k_n)$$

External lines factors:

- **A factor 1 for each external scalar line;**
- **A factor $\bar{u}_s(p)$ for each external outgoing fermion line associated to a final-state fermion;**
- **A factor $\bar{v}_s(p)$ for each external outgoing fermion line associated to an initial-state antifermion;**
- **A factor $u_s(p)$ for each external incoming fermion line associated to an initial-state fermion;**
- **A factor $v_s(p)$ for each external incoming fermion line associated to a final-state antifermion;**
- **A factor $\epsilon_\mu^{(\lambda)}(p)$ for each external vector line associated to an initial-state vector;**
- **A factor $\epsilon_\mu^{(\lambda)*}(p)$ for each external vector line associated to a final-state vector.**

Furthermore,

$$(\not{p} - m)u_s(p) = 0$$
$$(\not{p} + m)v_s(p) = 0$$
$$p \cdot \epsilon^{(\lambda)}(p) = 0$$

and

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m$$
$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m$$
$$\sum_{\lambda=1}^{2} \epsilon_\mu^{(\lambda)}(p)\epsilon_\nu^{*(\lambda)}(p) \to -g_{\mu\nu} \qquad (\gamma)$$
$$\sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)}(p)\epsilon_\nu^{*(\lambda)}(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_V^2} \qquad (V = W, Z).$$