Dispersion relations for hadronic light-by-light scattering and the muon g-2

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CERN

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Outline

- Introduction: the anomalous magnetic moment of the muon and its hadronic contributions. Dispersive approach to hadronic light-by-light (HLbL) scattering

- Lorentz structure of HLbL tensor: gauge invariance and crossing symmetry

- Master formula for the HLbL contribution to $(g-2)_\mu$

- Focus on pion-pole, pion-box and $\pi\pi$ rescattering contributions

- Summary and outlook

Colangelo, Hoferichter, Procura, Stoffer, JHEP 1505 (2015) + work in progress

Colangelo, Hoferichter, Procura, Stoffer, JHEP 1409 (2014)

Colangelo, Hoferichter, Kubis, Procura, Stoffer, PLB 738 (2014)

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QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
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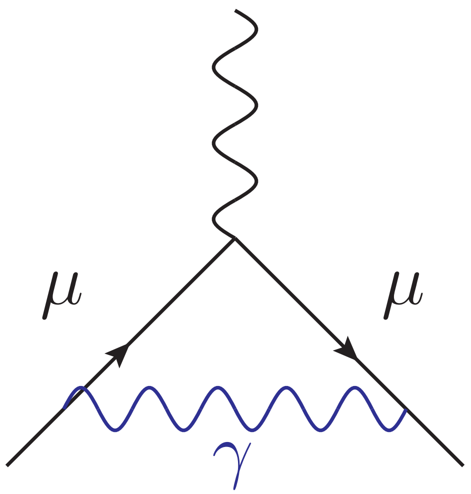

Schwinger 1948

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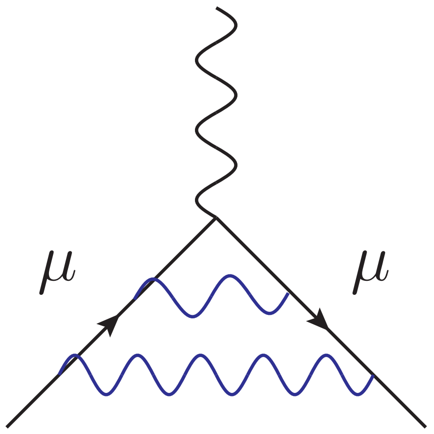

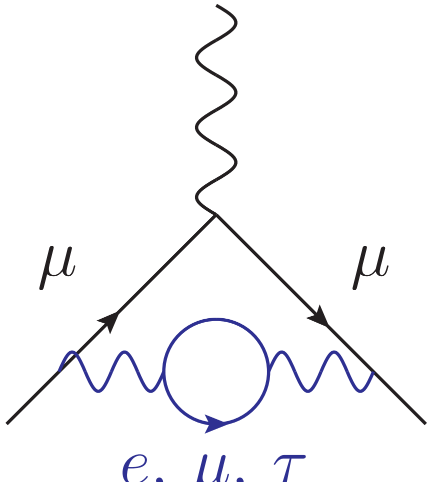

Petermann 1957

Sommerfield 1957

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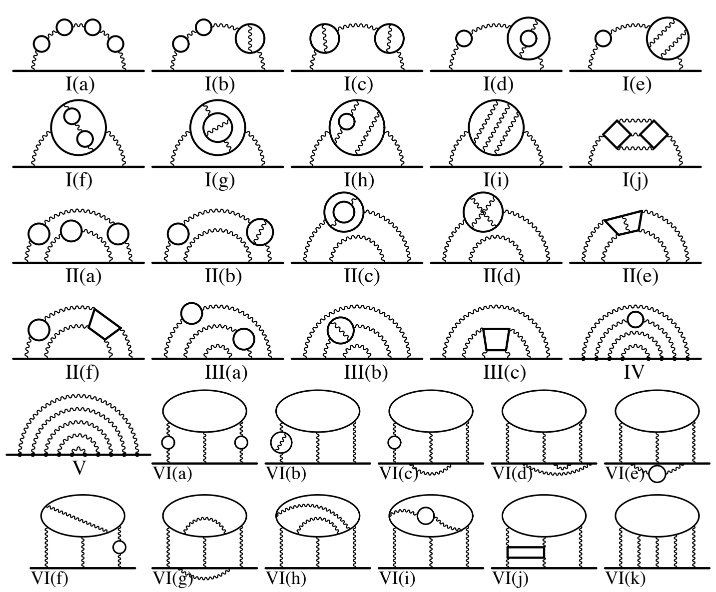

Kinoshita et al. 2012

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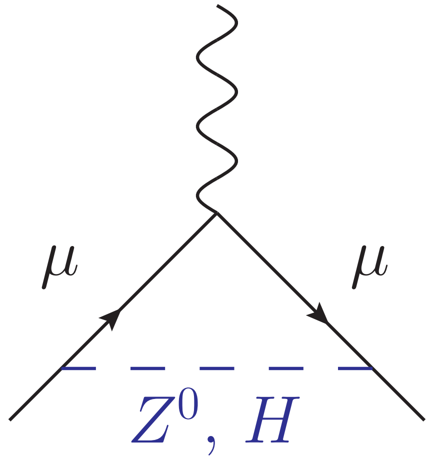

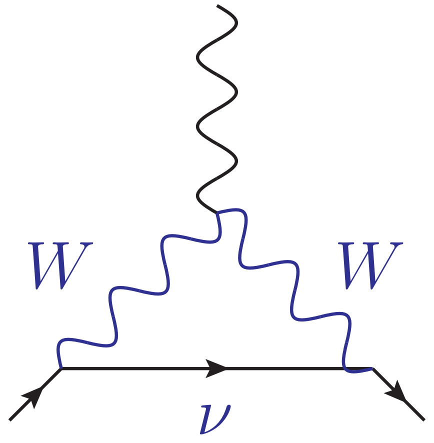

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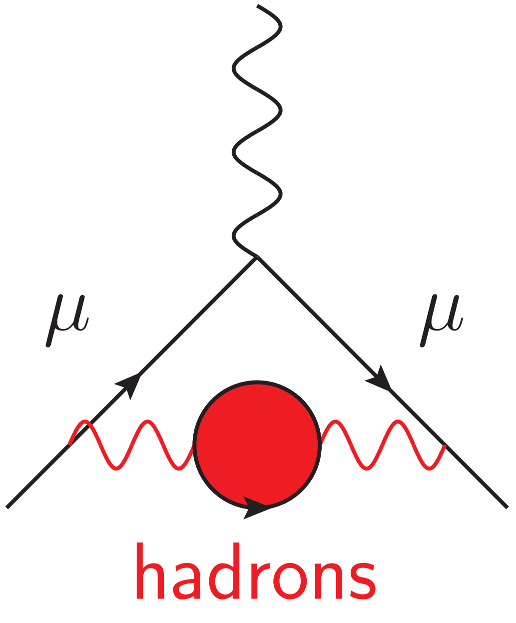

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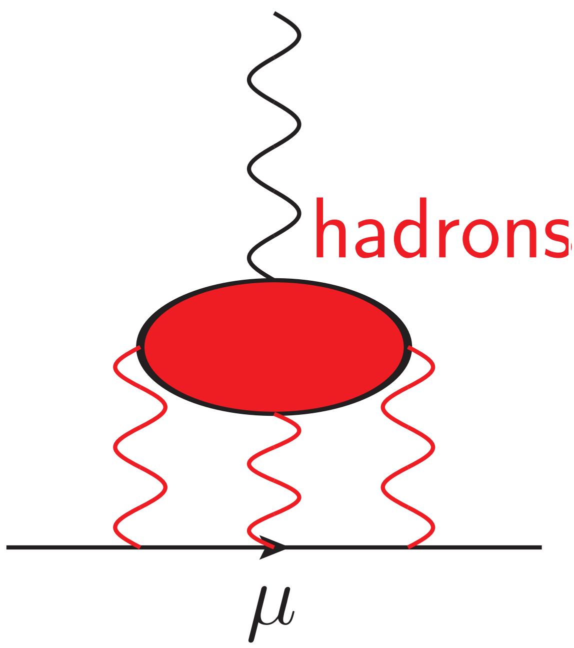

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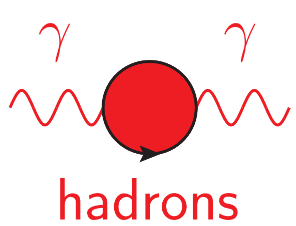
- New experiments at FNAL and J-PARC aim at improving the experimental precision

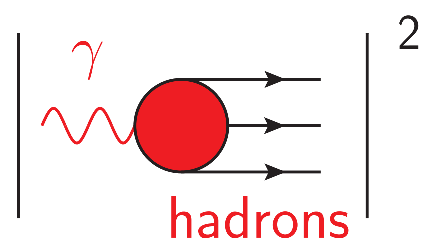
  ▶ important to scrutinize theory predictions and get reliable uncertainties

Introduction: hadronic vacuum polarization

- Limiting factor in the accuracy of SM predictions for $a_\mu = (g-2)_\mu$ is control over hadronic contributions, responsible for most of the theory uncertainty

- HVP is directly related via the optical theorem to $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$

Im 

⇔ 

$\propto \sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$

Obtained by integrating the R-ratio weighted with a perturbative QED kernel :

$$a_\ell^{\text{HVP-LO}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4M_\pi^2}^{\infty} \frac{dt}{t} K(t) R^{\text{had}}(t)$$

dominated by the low-energy region

- dedicated e^+e^- program (BaBar, BESIII, KLOE2 ...) to improve accuracy

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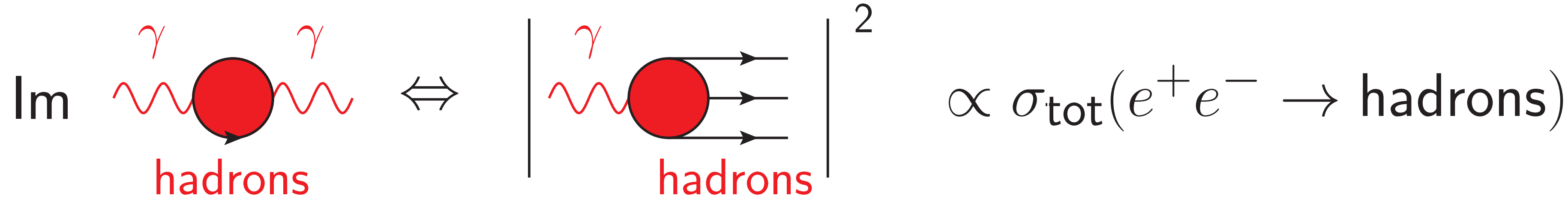

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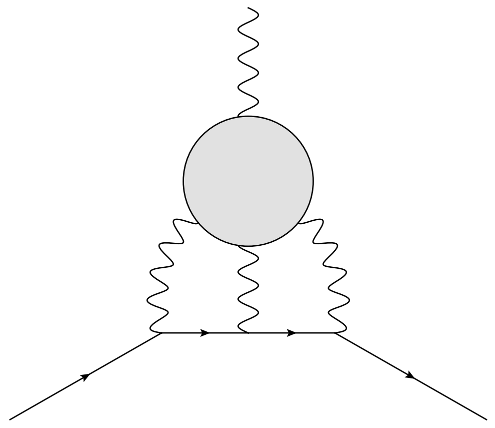
- Lattice QCD determination of the HVP-LO : recent progress

Blum et al., Burger et al., Chakraborty et al., ...

Introduction: hadronic light-by-light

- Hadronic light-by-light (HLbL) is more problematic: model calculations and some high-energy and low-energy constraints. Uncontrolled uncertainties



a_μ^{HLbL} in 10^{-11} units

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops + other subleading in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

The two global evaluations: Bijnens, Pallante, Prades (1995, 1996) and Hayakawa, Kinoshita, Sanda (1995, 1996)

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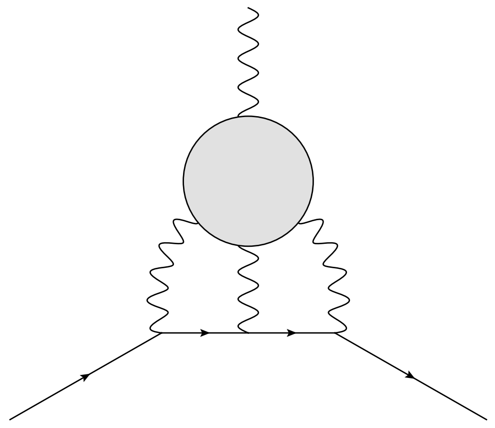

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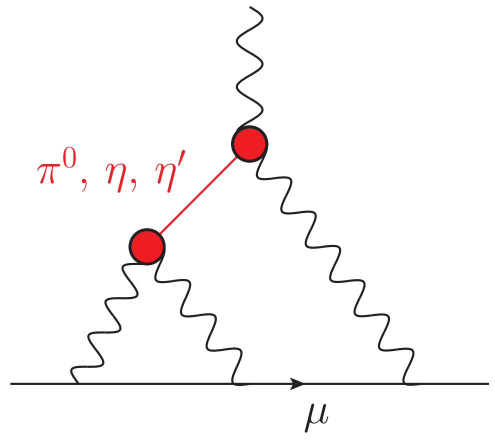

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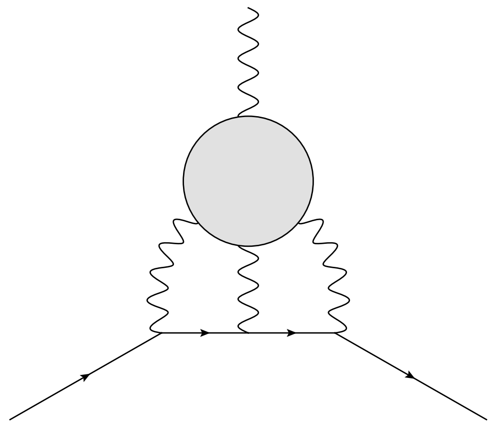

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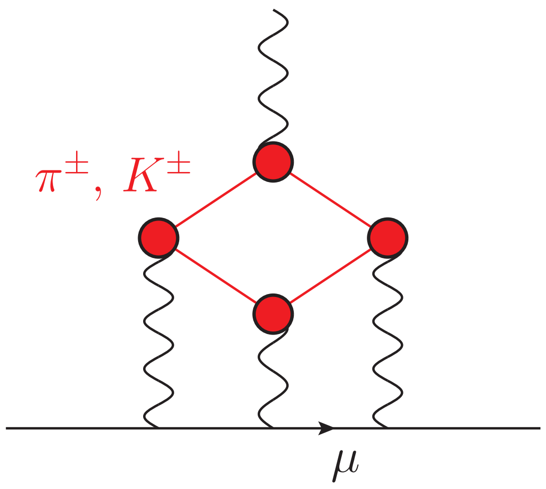

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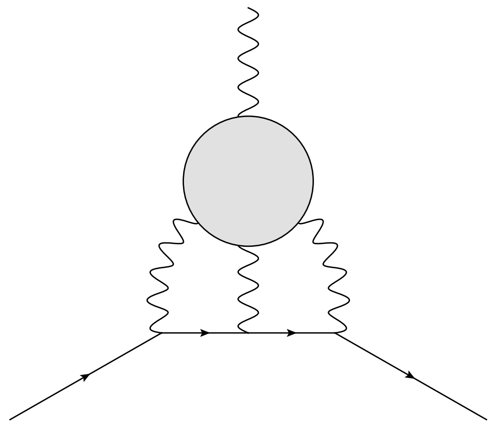

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Jegerlehner (2015)

≈ 8±3

102±39

The two most often quoted estimates: Prades, de Rafael, Vainshtein (2009) and Jegerlehner, Nyffeler (2009)

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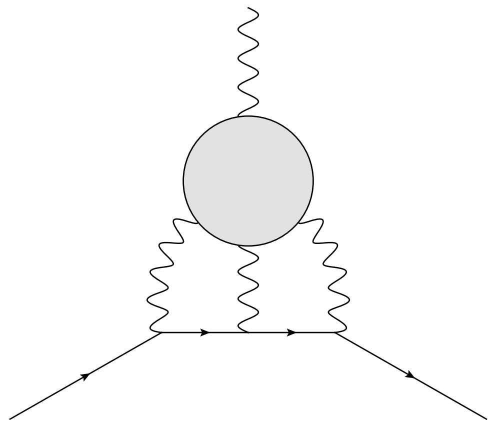


- ▶ a reliable uncertainty estimate is still an open issue

- How to reduce model dependence ? Recent strategies for an improved calculation :

  - ▶ lattice QCD: first computations at physical pion masses with leading disconnected contributions performed — Blum et al. (2015, 2016)

  - ▶ dispersion theory to make the evaluation as data driven as possible

Our strategy for HLbL

- Exploits fundamental principles :
  - gauge invariance and crossing symmetry
  - unitarity and analyticity

  to relate HLbL to experimentally accessible quantities

- Much more challenging task than for the hadronic vacuum polarization due to the complexity of the HLbL tensor, which is the key object of our analysis

- Defines and relates single contributions to HLbL to form factors and cross sections

Alternative: dispersive treatment of the HLbL contribution to Pauli form factor by Pauk and Vanderhaeghen (2014) (so far only single-meson pole contributions)

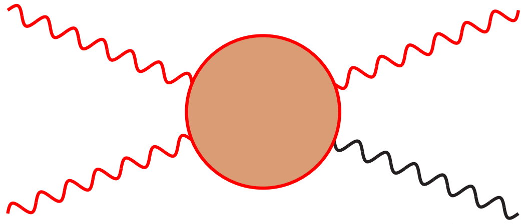
- The HLbL tensor: gauge invariance and crossing symmetry

- Master formula for the HLbL contribution to $(g-2)_\mu$

- Dispersive representation of scalar functions at fixed photon virtualities

The HLbL tensor

- The fully off-shell HLbL tensor :



$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -i\int d^4x\, d^4y\, d^4z\, e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\langle 0|T\{j^\mu_{\rm em}(x)j^\nu_{\rm em}(y)j^\lambda_{\rm em}(z)j^\sigma_{\rm em}(0)\}|0\rangle$$

- Mandelstam variables:

$$s=(q_1+q_2)^2,\ t=(q_1+q_3)^2,\ u=(q_2+q_3)^2$$

- Anomalous magnetic moment: Pauli form factor at zero momentum transfer

Lorentz structure of HLbL tensor

- Based on Lorentz covariance the HLbL tensor can be decomposed in 138 structures

$$
\begin{aligned}
\Pi^{\mu\nu\lambda\sigma} &= g^{\mu\nu} g^{\lambda\sigma}\, \Pi^1 + g^{\mu\lambda} g^{\nu\sigma}\, \Pi^2 + g^{\mu\sigma} g^{\nu\lambda}\, \Pi^3 \\
&+ \sum_{\substack{i=2,3,4\\ j=1,3,4}} \sum_{\substack{k=1,2,4\\ l=1,2,3}} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma\, \Pi^4_{ijkl} \\
&+ \sum_{\substack{i=2,3,4\\ j=1,3,4}} g^{\lambda\sigma} q_i^\mu q_j^\nu\, \Pi^5_{ij} + \sum_{\substack{i=2,3,4\\ k=1,2,4}} g^{\nu\sigma} q_i^\mu q_k^\lambda\, \Pi^6_{ik} + \sum_{\substack{i=2,3,4\\ l=1,2,3}} g^{\nu\lambda} q_i^\mu q_l^\sigma\, \Pi^7_{il} \\
&+ \sum_{\substack{j=1,3,4\\ k=1,2,4}} g^{\mu\sigma} q_j^\nu q_k^\lambda\, \Pi^8_{jk} + \sum_{\substack{j=1,3,4\\ l=1,2,3}} g^{\mu\lambda} q_j^\nu q_l^\sigma\, \Pi^9_{jl} + \sum_{\substack{k=1,2,4\\ l=1,2,3}} g^{\mu\nu} q_k^\lambda q_l^\sigma\, \Pi^{10}_{kl}
\end{aligned}
$$

- In 4 space-time dimensions there are 2 linear relations among these 138 structures

Eichmann, Fischer, Heupel, Williams (2014)

- Scalar functions encode the hadronic dynamics and depend on 6 kinematic variables

- This set of functions is hugely redundant: Ward identities imply 95 linear relations between these scalar functions (kinematic zeros)

Lorentz structure of HLbL tensor

- Following Bardeen and Tung (1968) - "BT"- we contracted the HLBL tensor with

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

  - 95 structures project to zero

- $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles eliminated by taking linear combinations of structures

- This procedure introduces kinematic singularities in the scalar functions : degeneracies in these BT Lorentz structures, e.g. as $q_1 \cdot q_2 \to 0,\ q_3 \cdot q_4 \to 0$

$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

Lorentz structure of HLbL tensor

- Following Tarrach (1975) we extended BT set to incorporate $X_i^{\mu\nu\lambda\sigma}$, $Y_i^{\mu\nu\lambda\sigma}$ to obtain a ("BTT") generating set of structures even for $q_1 \cdot q_2 \to 0,\, q_3 \cdot q_4 \to 0$

$$\boxed{\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)}$$

- Lorentz structures are manifestly gauge invariant

- crossing symmetry is manifest (only 7 genuinely different structures, the remaining ones being obtained by crossing)

- the BTT scalar functions are free of kinematic singularities and zeros : their analytic structure is dictated by dynamics only. This makes them suitable for a dispersive treatment

- The HLbL tensor: gauge invariance and crossing symmetry
- Master formula for the HLbL contribution to $(g-2)_\mu$
- Dispersive representation of scalar functions at fixed photon virtualities

Master formula for a_μ^{HLbL}

- Differentiating the Ward identity with respect to q_4 ,

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_4 - q_1 - q_2) = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2)$$

one obtains the relation

$$a_\mu^{\mathrm{HLbL}} = -\frac{1}{48 m_\mu} \mathrm{Tr}\left((\not{p} + m_\mu)[\gamma^\rho, \gamma^\sigma](\not{p} + m_\mu) \Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p) \right)$$

where $p^2 = m_\mu^2$ and

$$\Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p) = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \gamma^\mu \frac{(\not{p} + \not{q}_1 + m_\mu)}{(p + q_1)^2 - m_\mu^2} \gamma^\lambda \frac{(\not{p} - \not{q}_2 + m_\mu)}{(p - q_2)^2 - m_\mu^2} \gamma^\nu$$
$$\times \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2) \Big|_{q_4 = 0}$$

Master formula for a_μ^{HLbL}

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- Since there are no kinematic singularities in the BTT scalar functions,

$$\begin{aligned} a_\mu^{\mathrm{HLbL}} = &-\frac{e^6}{48 m_\mu} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{1}{(p + q_1)^2 - m_\mu^2} \frac{1}{(p - q_2)^2 - m_\mu^2} \\ &\times \mathrm{Tr}\left((\not{p} + m_\mu)[\gamma^\rho, \gamma^\sigma](\not{p} + m_\mu)\gamma^\mu(\not{p} + \not{q}_1 + m_\mu)\gamma^\lambda(\not{p} - \not{q}_2 + m_\mu)\gamma^\nu \right) \\ &\times \sum_{i=1}^{54} \left(\frac{\partial}{\partial q_4^\rho} T^i_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2) \right) \Bigg|_{q_4 = 0} \Pi_i(q_1, q_2, -q_1 - q_2) \end{aligned}$$

Master formula for a_μ^{HLbL}

- Only 12 linear combinations of the scalar functions contribute to a_μ^{HLbL} :

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \bar{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 ((p+q_1)^2 - m_\mu^2)((p-q_2)^2 - m_\mu^2)}$$

- the functions $\hat{T}_i$ contain trace and derivative (calculated)

- Wick rotation of q_1, q_2 and p (allowed even in the presence of anomalous cuts)

- 5 out of 8 integrals can be done analytically, without knowing the scalar functions

Master formula for a_μ^{HLbL}

- We obtained a general master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau)\bar{\Pi}_i(Q_1, Q_2, \tau)$$

- $Q_i^2 = -q_i^2$ are Euclidean momenta and $Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$: space-like kinematics

- We determined the integration kernels T_i .
  The scalar functions $\bar{\Pi}_i$ are linear combinations of the BTT Π_i

- Generalization of the formula for the pion pole in Knecht and Nyffeler (2002)

- Our goal: dispersive representation of $\bar{\Pi}_i$ at fixed photon virtualities

- The HLbL tensor: gauge invariance and crossing symmetry
- Master formula for the HLbL contribution to $(g-2)_\mu$
- Dispersive representation of scalar functions at fixed photon virtualities

Mandelstam representation

- Analytic properties of scalar functions relevant for the evaluation of a_μ^{HLbL} : right- and left-hand cuts, double spectral regions (box topologies)...

- Very complex analytic structure: approximations are required. We order the contributions according to the mass of intermediate states: the lightest states are expected to be the most important (in agreement with model calculations)

- Here we consider the 2 lowest-lying contributions: one- and two-pion intermediate states in all channels

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

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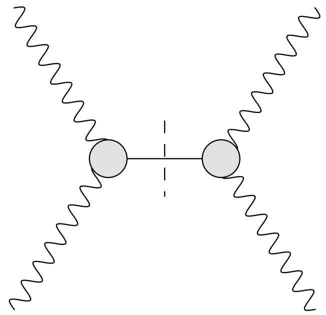
one-pion intermediate state :



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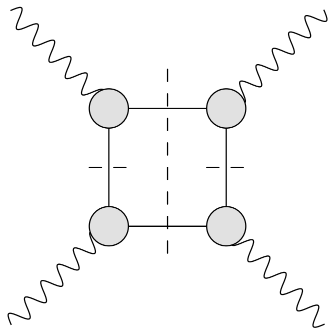
two-pion intermediate state in both channels :



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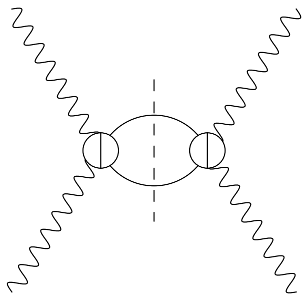
two-pion intermediate state in the direct channel:



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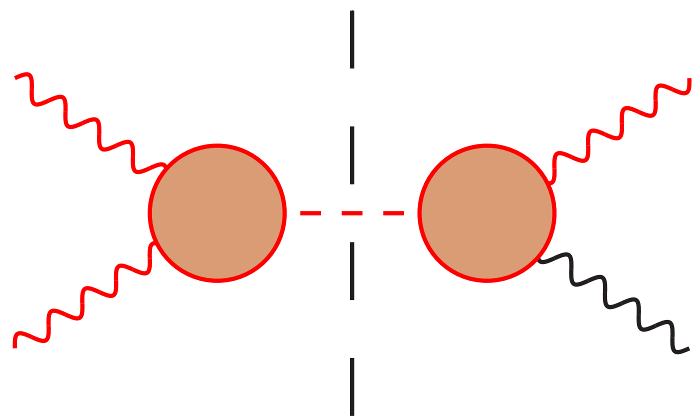
higher intermediate states: neglected here

The pion-pole contribution

- From the unitarity relation with only π^0 intermediate state, the pole residues in each channel are given by products of doubly-virtual and singly-virtual pion transition form factors ($\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$, input for our analysis)



$$a_\mu^{\pi^0\text{-pole}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \left(T_1(Q_1,Q_2,\tau)\bar{\Pi}_1^{\pi^0\text{-pole}}(Q_1,Q_2,\tau) + T_2(Q_1,Q_2,\tau)\bar{\Pi}_2^{\pi^0\text{-pole}}(Q_1,Q_2,\tau)\right)$$

with

$$\bar{\Pi}_1^{\pi^0\text{-pole}} = -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_3^2,0)}{Q_3^2+M_\pi^2} \qquad \bar{\Pi}_2^{\pi^0\text{-pole}} = -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)}{Q_2^2+M_\pi^2}$$

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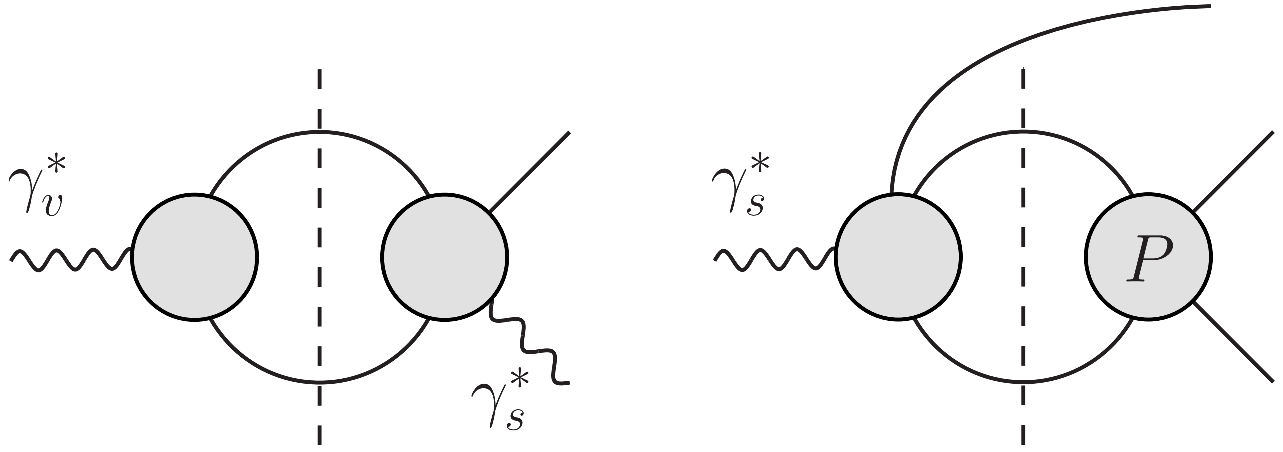
- Data on doubly-virtual pion-photon interaction not available. However, these form factors can be reconstructed dispersively. This requires as input :

  - pion vector form factor
  - $\gamma^* \to 3\pi$ amplitude
  - $\pi\pi$ scattering amplitude



Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

- Pseudoscalar poles with higher masses can be treated analogously

Pion-box contribution

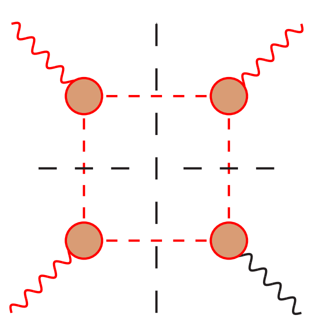
- Defined by simultaneous two-pion cuts in two channels

- Contribution to scalar functions as dispersive integral of double spectral functions

$$\Pi_i = \frac{1}{\pi^2}\int ds'dt' \frac{\boxed{\rho_i^{st}(s',t')}}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

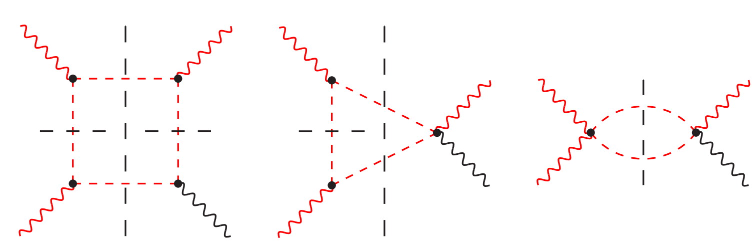
- Dependence on q_i^2 carried by the pion vector FFs for each off-shell photon

- sQED loop projected onto the BTT structures fulfills the same Mandelstam representation of the pion box, the only difference being the pion vector FFs :



$$\equiv F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)$$

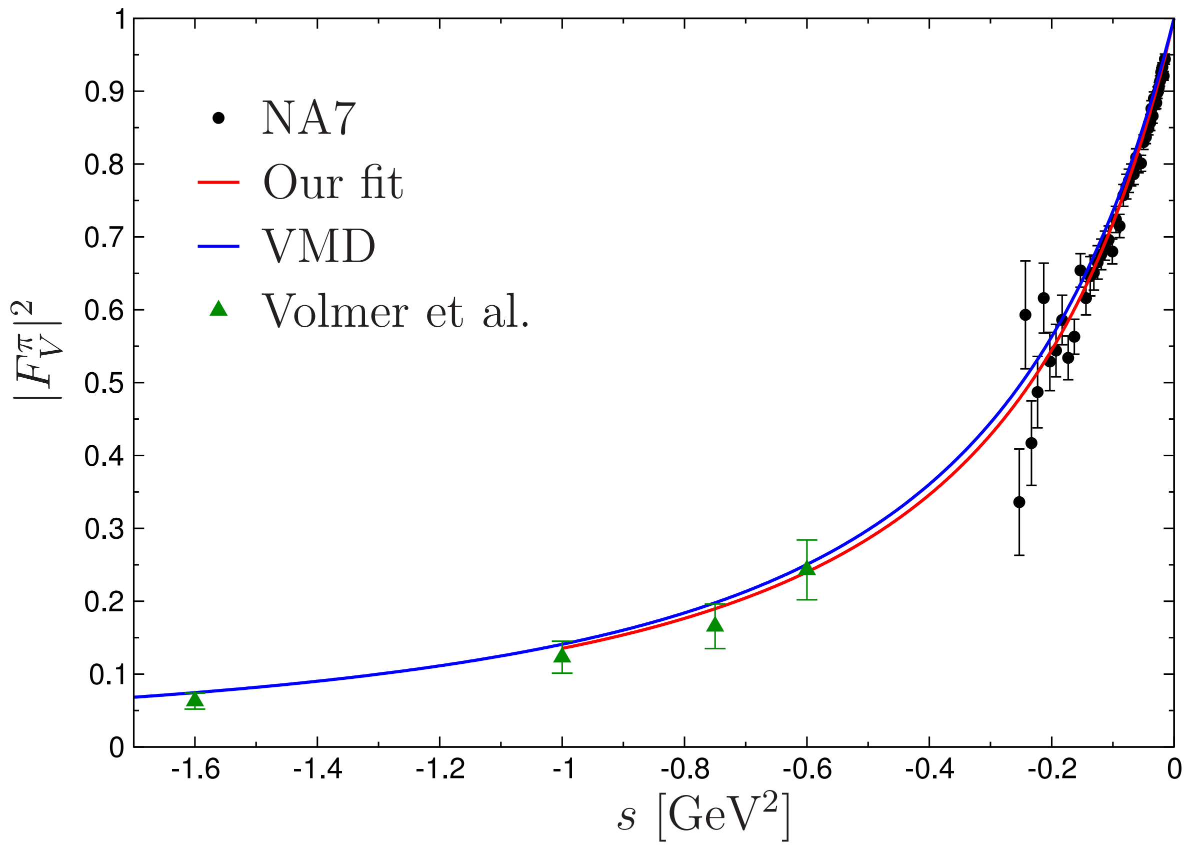
$\times \Big[$



$\Big]$

Numerics for the pion-box contribution

- The only input: pion vector form factor in the space-like region

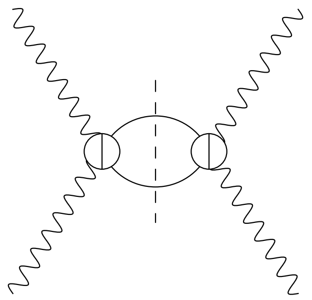


- Preliminary results: $a_\mu^{\pi\text{-box}} = -15.9 \times 10^{-11}$, $a_\mu^{\pi\text{-box,VMD}} = -16.4 \times 10^{-11}$
  vs $a_\mu^{K\text{-box,VMD}} = -0.5 \times 10^{-11}$

- Rapid convergence: $Q_{\text{max}} = \{1, 1.5\}$ GeV $\Rightarrow$ $a_\mu^{\pi\text{-box}} = \{95, 99\}\%$ of full result

The remaining ππ contribution

- Two-pion cut only in the direct channel:
  LH cut due to multi-particle intermediate states
  in the crossed channel neglected

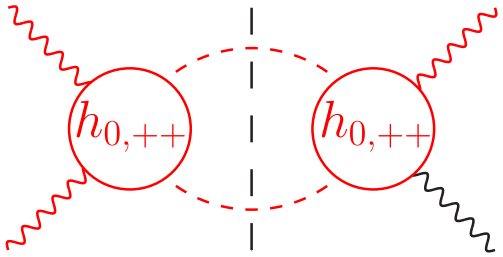


- Unitarity relates this contribution to the subprocess $\gamma^*\gamma^{(*)} \to \pi\pi$

- Our goal is a dispersive reconstruction of helicity partial waves for $\gamma^*\gamma^* \to \pi\pi$

Colangelo, Hoferichter, MP, Stoffer (2014)

$$\mathrm{Im}\, h^J_{++,++}(s;q_1^2,q_2^2;q_3^2,0) = \frac{\sigma(s)}{16\pi} h^*_{J,++}(s;q_1^2,q_2^2)\, h_{J,++}(s;q_3^2,0)$$



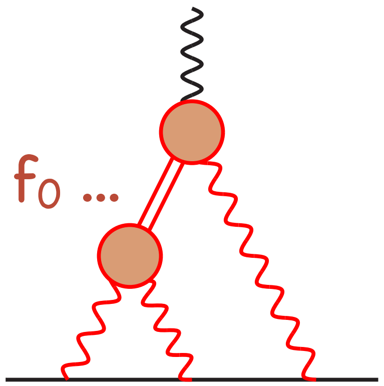
then project onto BTT basis and use our master formula.
We have recently extended our formalism to arbitrary partial waves.

- We checked that the PW expansion converges for FsQED (pion box)

ππ rescattering : preliminary results

- The framework for a dispersive reconstruction of $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves : Roy-Steiner equations, respecting analyticity, unitarity and crossing

- Omnès-type solutions allow for the summation of ππ rescattering effects in the direct channel (effects of resonances coupling to ππ)



- We solved dispersion relations for $\gamma^*\gamma^* \to \pi\pi$ S-waves taking :
  - pion pole as only LH singularity (pion VFF accounts for the off-shell behavior)
  - ππ phase shifts from SU(2) inverse amplitude method (reproduce $f_0(500)$)

a_μ^{HLbL} in 10^{-11} units

Λ	1 GeV	1.5 GeV	2 GeV	∞
$I = 0$	-9.2	-9.5	-9.3	-8.8
$I = 2$	2.0	1.3	1.1	0.9

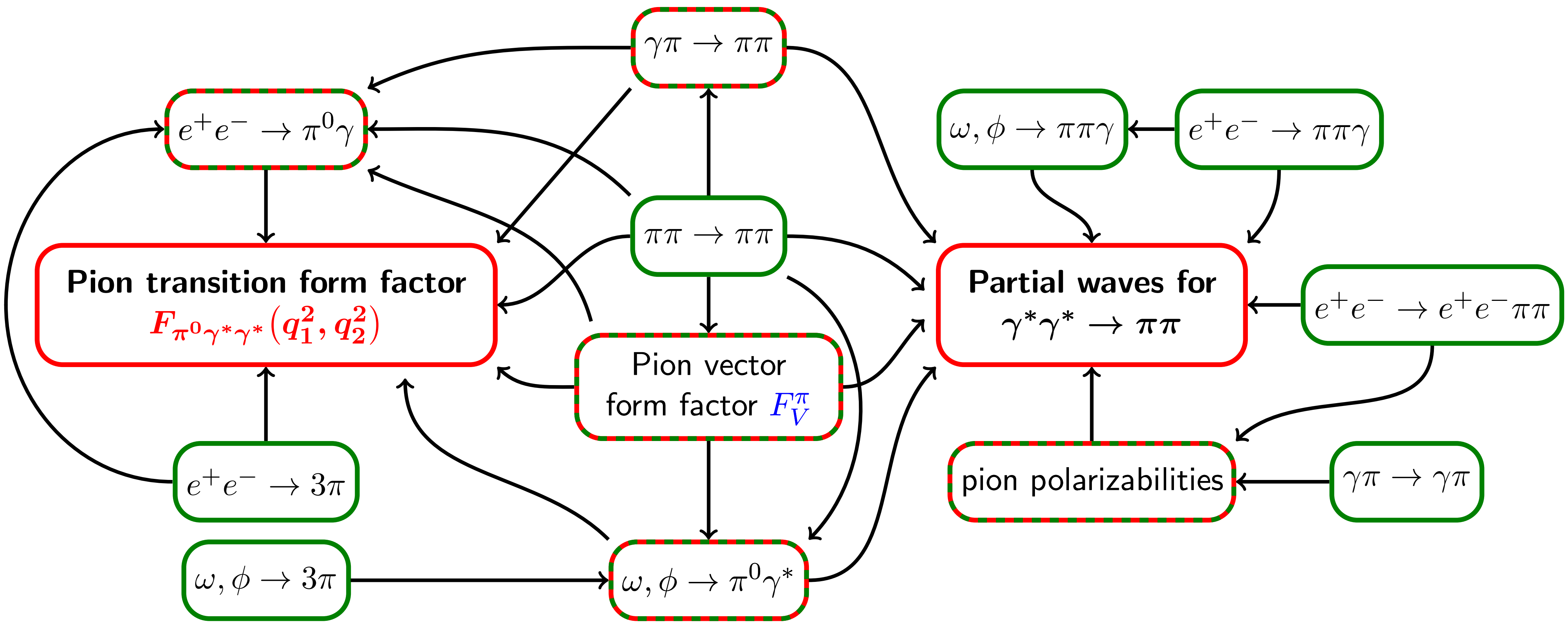
Summary and Outlook

- Dispersive approach to HLbL scattering based on general principles: gauge invariance and crossing symmetry, unitarity and analyticity
- Derivation of a set of structures according to Bardeen-Tung-Tarrach (BTT) such that the scalar functions are free of kinematic singularities and zeros
- Derivation of a general master formula for a_μ^{HLbL} in terms of BTT functions
- Single- and double-pion intermediate states are taken into account.
  Results can be extended to other pseudoscalar poles and two-meson states
- Preliminary numerical results for pion box and $\pi\pi$ rescattering
- Future work: refined analysis of $\pi\pi$ rescattering, reliable uncertainty estimates, higher intermediate states. Investigate and incorporate pQCD constraints
- First step towards a reduction of model dependence of HLbL: within a dispersive framework, relations with experimentally accessible (or dispersively reconstructed) quantities (form factors, scattering amplitudes)

Additional slides

A roadmap for HLbL

Colangelo, Hoferichter, Kubis, MP, Stoffer (2014)



Artwork by M. Hoferichter