Recent results on $\eta \to 3\pi$

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ArXiv:1610.03494 + in preparation

Outline :

1. Introduction and Motivation
2. Dispersive analysis
3. Results
4. Conclusion and outlook

1. Introduction and Motivation

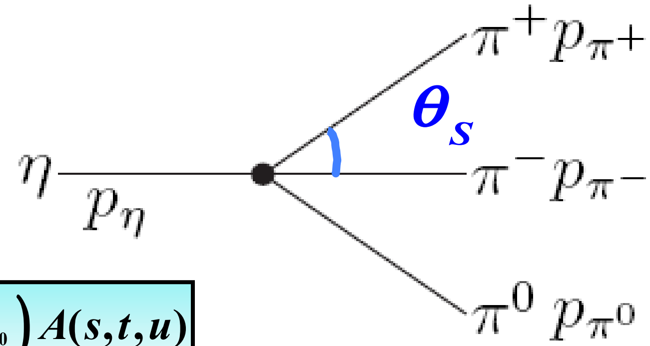
1.1 Decays of the η

- η decay from PDG:

η DECAY MODES

	Mode	Fraction (Γ_i/Γ)	Scale factor/ Confidence level
	Neutral modes		
Γ_1	neutral modes	(72.12 ± 0.34) %	S=1.2
Γ_2	2γ	(39.41 ± 0.20) %	S=1.1
Γ_3	$3\pi^0$	(32.68 ± 0.23) %	S=1.1
	Charged modes		
Γ_8	charged modes	(28.10 ± 0.34) %	S=1.2
Γ_9	$\pi^+\pi^-\pi^0$	(22.92 ± 0.28) %	S=1.2
Γ_{10}	$\pi^+\pi^-\gamma$	(4.22 ± 0.08) %	S=1.1

1.2 Definitions



- η decay: $\eta \rightarrow \pi^+ \pi^- \pi^0$

$$\boxed{\left\langle \pi^+\pi^-\pi^0{}_{out} \middle| \eta \right\rangle = i(2\pi)^4 \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)}$$

- Mandelstam variables $s=\left(p_{\pi^+}+p_{\pi^-}\right)^2,\ t=\left(p_{\pi^-}+p_{\pi^0}\right)^2,\ u=\left(p_{\pi^0}+p_{\pi^+}\right)^2$

  ➡ only two independent variables

$$s+t+u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2 \equiv 3s_0$$

- 3 body decay ➡ Dalitz plot

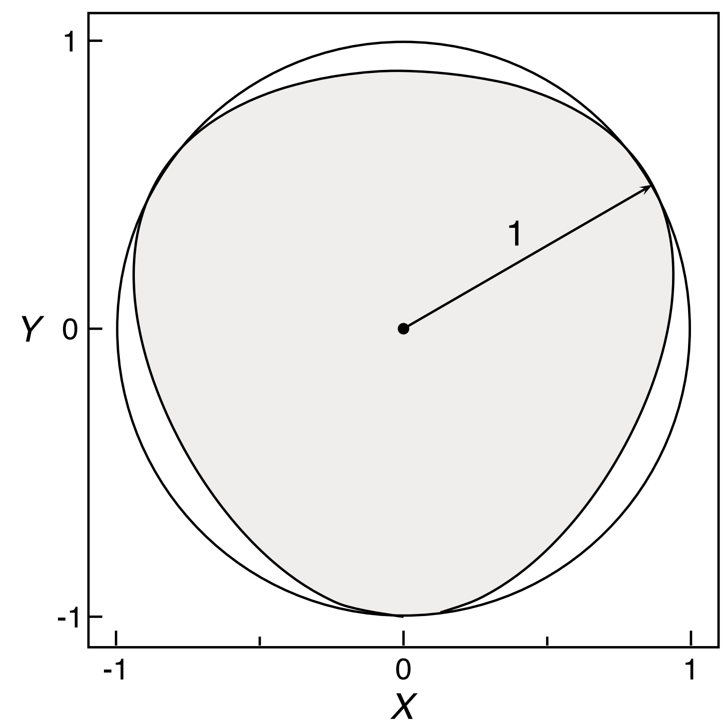
$$\boxed{\left|A(s,t,u)\right|^2 = N\left(1 + aY + bY^2 + dX^2 + fY^3 + \ldots\right)}$$

Expansion around X=Y=0

$$X = \sqrt{3}\,\frac{T_+ - T_-}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

$$Q_c \equiv M_\eta - 2M_{\pi^+} - M_{\pi^0}$$



1.3 Why is it interesting to study $\eta \to 3\pi$?

- Decay forbidden by isospin symmetry

⇨ $\boxed{A = (m_u - m_d) A_1 + \alpha_{em} A_2}$

- α_{em} effects are small

  Sutherland'66, Bell & Sutherland'68
  Baur, Kambor, Wyler'96, Ditsche, Kubis, Meissner'09

- Decay rate measures the size of isospin breaking ($m_u - m_d$) in the SM:

$$L_{QCD} \to \boxed{L_{IB} = -\frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right)}$$

⇨ Unique access to ($m_u - m_d$)

1.4 Quark mass ratios

- Instead of ($m_u - m_d$) of estimate quark mass ratio Q (or R):

$$\boxed{A(s,t,u) \propto B_0\left(m_u - m_d\right) = \frac{1}{Q^2}\frac{M_K^2\left(M_K^2 - M_\pi^2\right)}{M_\pi^2} + O(M^3)}$$

$$= -\frac{1}{R}\left(M_K^2 - M_\pi^2\right) + O(M^2)$$

- $\boxed{Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}}$ or $\boxed{R \equiv \frac{m_s - \hat{m}}{m_d - m_u}}$

$$\left[\hat{m} \equiv \frac{m_d + m_u}{2}\right]$$

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- Normalized Amplitude: $\boxed{A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)}$

1.4 Quark mass ratios

- In the following, extraction of Q from $\eta \rightarrow \pi^+ \pi^- \pi^0$

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{Q^4}\frac{M_K^4}{M_\pi^4}\frac{\left(M_K^2 - M_\pi^2\right)^2}{6912\pi^3 F_\pi^4 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \left|M(s,t,u)\right|^2$$

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Determined from experiment

Determined from:
- Dispersive calculation
- ChPT

Fit to Dalitz distr.

- Aim: Compute M(s,t,u) with the *best accuracy*

1.5 Computation of the amplitude

- What do we know?

- Compute the amplitude using ChPT :

$$\Gamma_{\eta\to 3\pi} = \left(\underset{\text{LO}}{66} + \underset{\text{NLO}}{94} + \underset{\text{NNLO}}{\ldots} + \ldots\right)\text{eV} = \left(300 \pm 12\right)\text{eV}$$

PDG'16

LO: *Osborn, Wallace'70*

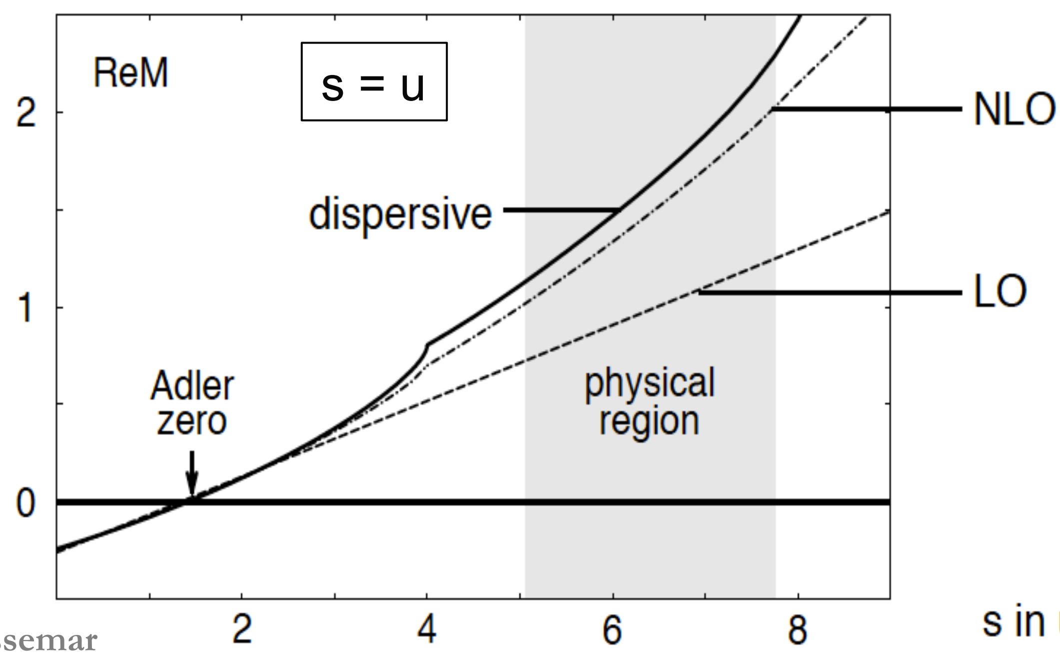
NLO: *Gasser & Leutwyler'85*

NNLO: *Bijnens & Ghorbani'07*

The Chiral series has convergence problems ⇒ Large *$\pi\pi$ final state interactions*

Roiesnel & Truong'81

Anisovich & Leutwyler'96



s in units of M_π

1.5 Computation of the amplitude

- What do we know?

- The amplitude has an Adler zero: soft pion theorem *Adler'85*

  ➡ Amplitude has a zero for :

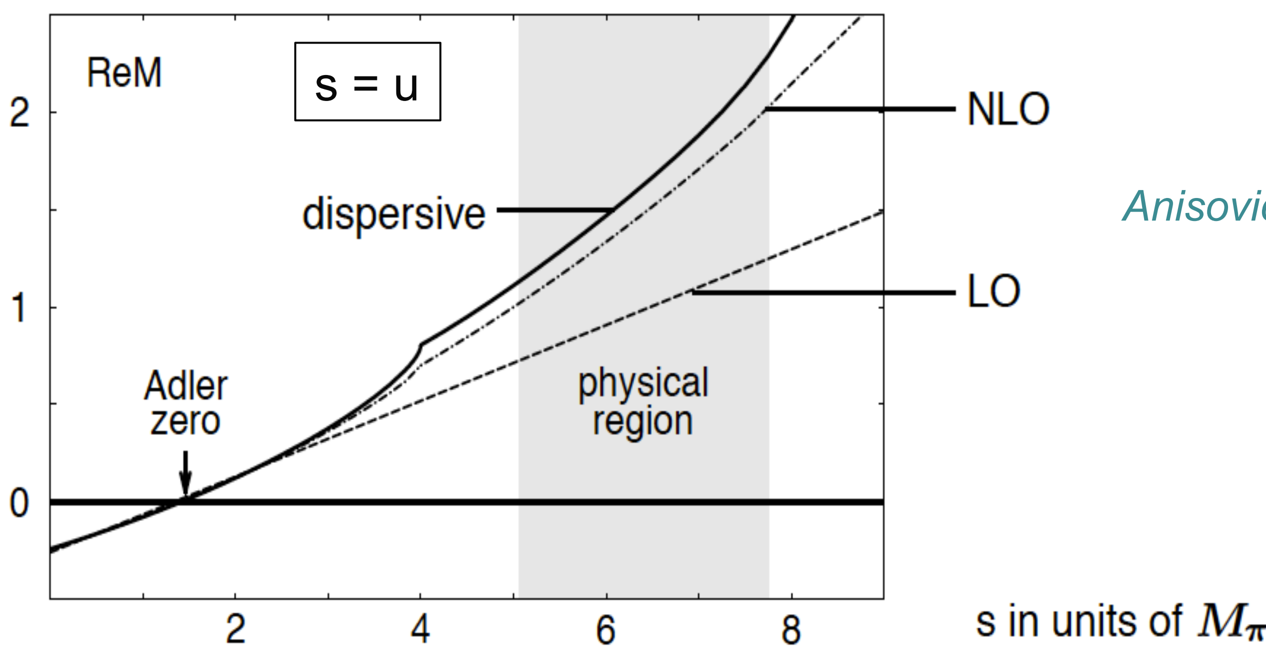
$$p_{\pi^+} \to 0 \;\Rightarrow\; s = u = 0,\; t = M_\eta^2$$

$$p_{\pi^-} \to 0 \;\Rightarrow\; s = t = 0,\; u = M_\eta^2$$

$$M_\pi \neq 0 \;\Rightarrow\; s = u = \frac{4}{3}M_\pi^2,\; t = M_\eta^2 + \frac{M_\pi^2}{3}$$

$$s = t = \frac{4}{3}M_\pi^2,\; u = M_\eta^2 + \frac{M_\pi^2}{3}$$

SU(2) corrections



Anisovich & Leutwyler'96

1.6 Neutral channel

- What do we know?

- We can relate charged and neutral channel

$$\boxed{\bar{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)}$$

⇨ *Consistency* check

- Ratio of decay width precisely measured

$$\boxed{r = \frac{\Gamma\left(\eta \to \pi^0\pi^0\pi^0\right)}{\Gamma\left(\eta \to \pi^+\pi^-\pi^0\right)}} = 1.426 \pm 0.026$$

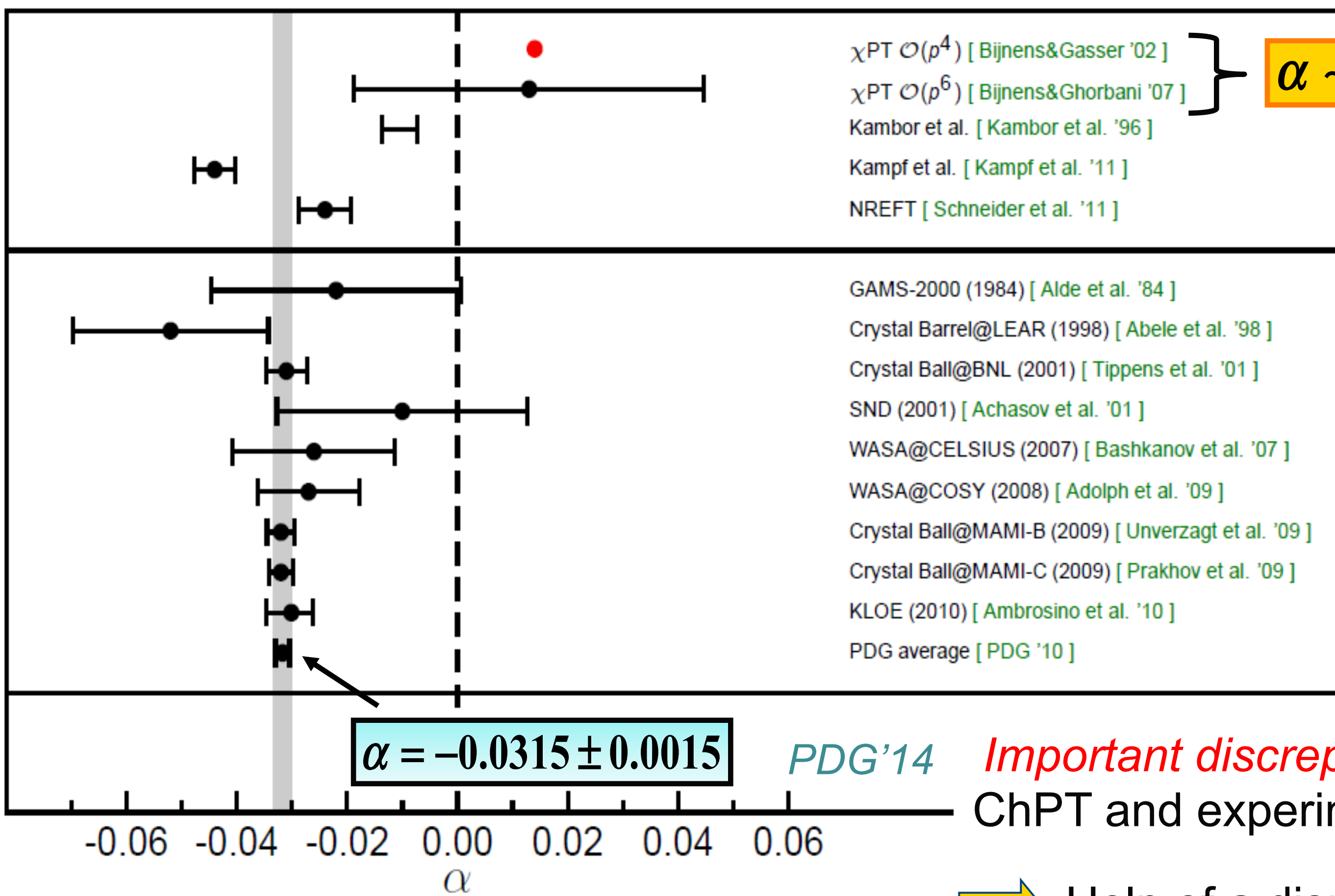
PDG'16

1.6 Neutral Channel : $\eta \to \pi^0 \pi^0 \pi^0$

- Decay amplitude $\boxed{\Gamma_{\eta\to 3\pi} \propto \left|\bar{A}\right|^2 \propto 1 + 2\alpha Z}$ with $Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2$

$Q_n \equiv M_\eta - 3M_{\pi^0}$

$\alpha \sim 0.015$



$\boxed{\alpha = -0.0315 \pm 0.0015}$ *PDG'14*

Important discrepancy between ChPT and experiment!

⇨ Help of a dispersive treatment?

2. Dispersive Analysis of $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

2.1 Why a new dispersive analysis?

- Large $\pi\pi$ final state interactions

  ⇨ call for a dispersive treatment :
  - analyticity, unitarity and crossing symmetry
  - Take into account all the rescattering effects

- Several new ingredients:
  - New inputs available: extraction $\pi\pi$ phase shifts has improved

    Ananthanarayan et al'01, Colangelo et al'01
    Descotes-Genon et al'01
    Kaminsky et al'01, Garcia-Martin et al'09

  - New experimental programs, precise Dalitz plot measurements

    TAPS/CBall-MAMI (Mainz), WASA-Celsius (Uppsala), WASA-Cosy (Juelich)
    CBall-Brookhaven, CLAS, GlueX (JLab), KLOE I-II (Frascati)
    BES III (Beijing) ⇨ see talks by *S. Giovannella*
    S. Fang

2.1 Why a new dispersive analysis?

- Large $\pi\pi$ final state interactions

  ⇒ call for a dispersive treatment :

  – analyticity, unitarity and crossing symmetry

  – Take into account all the rescattering effects

- Several new ingredients:

  – Possible improvements:

    – Electromagnetic effects, complete analysis of $\mathcal{O}(e^2 m)$ effects

      Ditsche, Kubis, Meissner'09

    – Isospin breaking effects

      Gullstrom, Kupsc, Rusetsky'09, Schneider, Kubis, Ditsche'11

    – Inelasticities *Albaladejo & Moussallam'15*

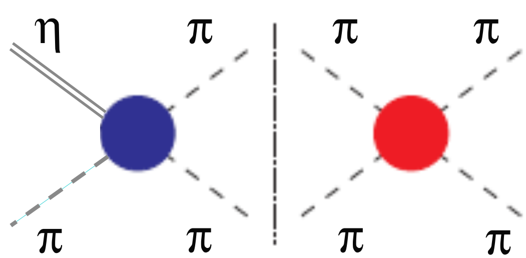
- Match to ChPT amplitude to obtain Q from rates ⇒ see later

2.2 Method: Representation of the amplitude

- Consider the s channel ⇨ Partial wave expansion of M(s,t,u) and isospin projection:

$$\boxed{M(s,t,u) = f_0^0(s) + f_1^1(s)\cos\theta + \ldots}$$

- Elastic unitarity:



$$disc\left[f_\ell^I(s)\right] \propto t_\ell^{I*}(s) f_\ell^I(s)$$

with $t_\ell^I(s)$ partial wave of elastic ππ scattering

Watson's theorem

- M(s,t,u) right-hand branch cut in the complex s-plane starting at the ππ threshold

- Left-hand cut present due to crossing

- Same situation in the t- and u-channel

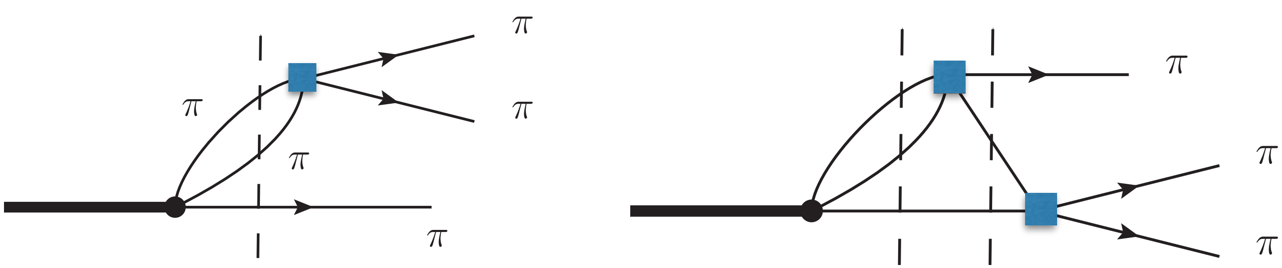
2.2 Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$\boxed{M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)}$$

Fuchs, Sazdjian & Stern'93
Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I



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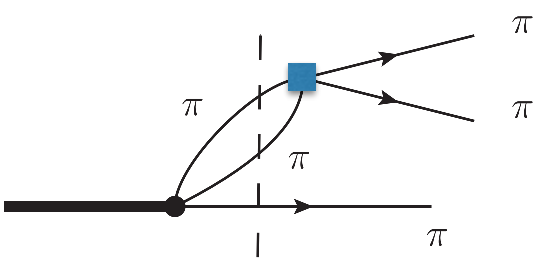
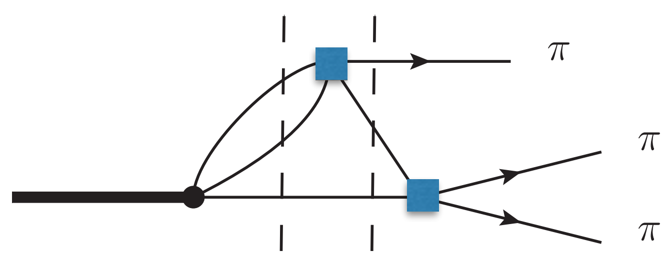




- Functions of only one variable with only right-hand cut of the partial wave

Elastic unitarity

$$disc\left[M_I(s)\right] \equiv disc\left[f_\ell^I(s)\right] \Rightarrow \boxed{disc\left[f_1^I(s)\right] \propto t_1^*(s)\, f_1^I(s)}$$

2.3 Unitarity relation for the $M_I(s)$

- Elastic Unitarity

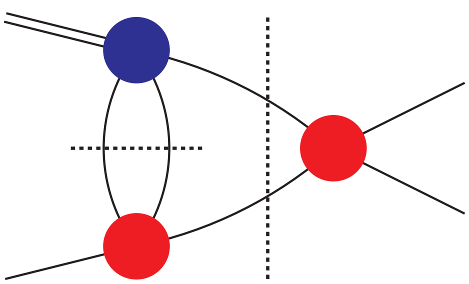
$$\left[\ell = 1 \text{ for } I = 1,\ \ell = 0 \text{ otherwise}\right]$$

$$\boxed{disc\left[M_I\right] = disc\left[f_\ell^I(s)\right] = \theta\left(s - 4M_\pi^2\right)\left[M_I(s) + \hat{M}_I(s)\right]\sin\delta_\ell^I(s)e^{-i\delta_\ell^I(s)}}$$

ππ phase shift

δ_1^I phase of the partial wave $f_1^I(s)$

Watson theorem: elastic ππ scattering phase shifts

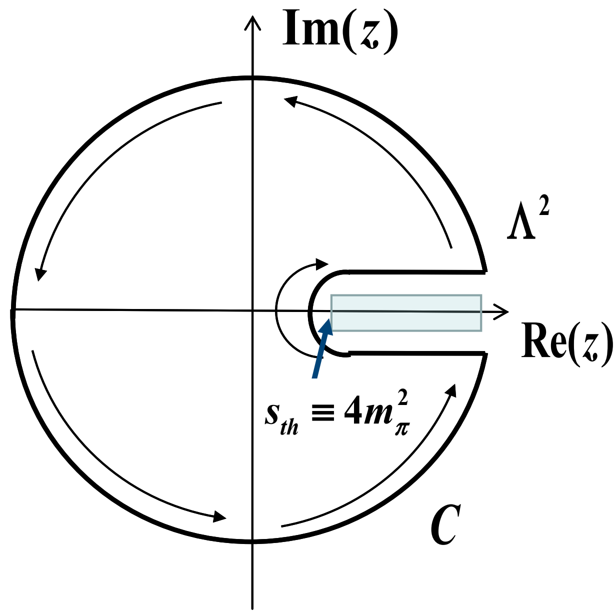


2.4 Dispersion relations for $M_I(s)$

- Knowing the discontinuity of M_I ⇨ write a dispersion relation for it
- Cauchy Theorem and Schwarz reflection principle

$$M_I(s) = \frac{1}{\pi} \oint_C dz \frac{M_I(z)}{z-s}$$

$$= \frac{1}{2i\pi} \int_{s_{th}=4M_\pi^2}^{\Lambda^2} dz \frac{disc\left[M_I(z)\right]}{z-s-i\varepsilon} + \frac{1}{2i\pi} \int_{|z|=\Lambda^2} dz \frac{M_I(z)}{z-s}$$



⇨ $$\boxed{M_I(s) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{disc\left[M_I(z)\right]}{z-s-i\varepsilon} dz}$$

M_I can be reconstructed everywhere from the knowledge of $disc\left[M_I(s)\right]$

- If M_I doesn't converge fast enough for $|s| \to \infty$ ⇨ subtract the dispersion relation

$$\boxed{M_I(s) = P_{n-1}(s) + \frac{s^n}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n} \frac{disc\left[M_I(s')\right]}{\left(s'-s-i\varepsilon\right)}}$$

$P_{n-1}(s)$ polynomial

2.4 Dispersion relations for $M_I(s)$

- $$\boxed{disc\left[M_I\right] = disc\left[f_\ell^I(s)\right] = \theta\left(s - 4M_\pi^2\right)\left[M_I(s) + \hat{M}_I(s)\right]\sin\delta_\ell^I(s)e^{-i\delta_\ell^I(s)}}$$

- Write a dispersion relation for $\dfrac{M_I(s)}{\Omega_I(s)}$:

$$\boxed{M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{\left|\Omega_0(s')\right|\left(s' - s - i\varepsilon\right)}\right)}$$

Omnès function

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Similarly for M_1 and M_2

2.4 Dispersion relations for $M_I(s)$

- $$\boxed{disc\left[M_I\right] = disc\left[f_\ell^I(s)\right] = \theta\left(s-4M_\pi^2\right)\left[M_I(s)+\hat{M}_I(s)\right]\sin\delta_\ell^I(s)e^{-i\delta_\ell^I(s)}}$$

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- Inputs needed : S and P-wave phase shifts of ππ scattering

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Similarly for M_1 and M_2

- Inputs needed : S and P-wave phase shifts of ππ scattering

- $\hat{M}_I(s)$: singularities in the t and u channels, depend on the other $M_I(s)$
  subtract $M_I(s)$ from the partial wave projection of $M(s,t,u)$
  ⇨ Angular averages of the other functions ⇨ Coupled equations

Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$\boxed{M(s,t,u) = M_0(s) + \left(s-u\right) M_1(t) + \left(s-t\right) M_1(u) + M_2(t) + M_2(u) - \frac{2}{3} M_2(s)}$$

Fuchs, Sazdjian & Stern'93

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- Dispersion relation for the M_I's

$$\boxed{M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n} \frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)} \right)}$$

Omnès function

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

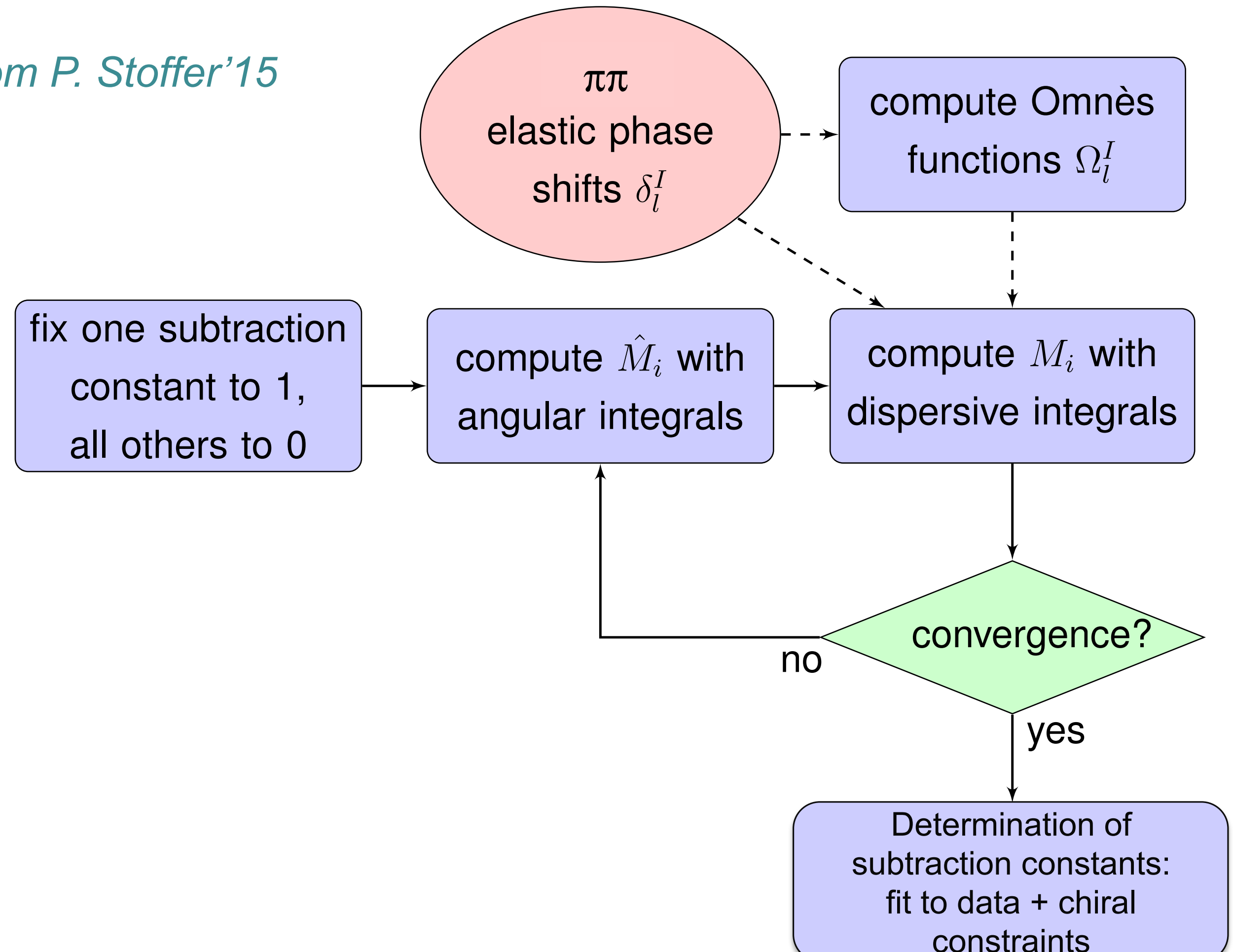
- Solution depends on *subtraction constants* only ⇨ solve by iterative procedure

2.5 Iterative Procedure

- Solution *linear* in the *subtraction constants* *Anisovich & Leutwyler'96*

$$M(s,t,u) = \alpha_0 M_{\alpha_0}(s,t,u) + \beta_0 M_{\beta_0}(s,t,u) + \dots$$ ⇨ makes the fit much easier

Adapted from P. Stoffer'15



2.6 Subtraction constants

- Extension of the numbers of parameters compared to *Anisovich & Leutwyler'96*

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$
$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$
$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

- In the work of *Anisovich & Leutwyler'96* matching to one loop ChPT
  Use of the SU(2) x SU(2) chiral theorem
  ⇨ The amplitude has an *Adler zero* along the line s=u

- Now data on the Dalitz plot exist from KLOE, WASA, MAMI and BES III

  ⇨ Use the data to directly fit the subtraction constants

- However normalization to be fixed to ChPT!

2.7 Subtraction constants

- The subtraction constants are

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$

$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$

$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 + \delta_0 s^3$$

Only *6 coefficients* are of physical relevance

- They are determined from combining ChPT with a fit to KLOE Dalitz plot

- Taylor expand the dispersive M_I
Subtraction constants ⟺ Taylor coefficients

$$M_0(s) = A_0 + B_0 s + C_0 s^2 + D_0 s^3 + \ldots$$

$$M_1(s) = A_1 + B_1 s + C_1 s^2 + \ldots$$

$$M_2(s) = A_2 + B_2 s + C_2 s^2 + D_2 s^3 +$$

- Gauge freedom in the decomposition of M(s,t,u)

2.7 Subtraction constants

- Build some gauge independent combinations of Taylor coefficients

$$H_0 = A_0 + \frac{4}{3}A_2 + s_0\left(B_0 + \frac{4}{3}B_2\right)$$

$$H_1 = A_1 + \frac{1}{9}(3B_0 - 5B_2) - 3C_2 s_0$$

$$H_2 = C_0 + \frac{4}{3}C_2, \qquad H_3 = B_1 + C_2$$

$$H_4 = D_0 + \frac{4}{3}D_2, \qquad H_5 = C_1 - 3D_2$$

$\Rightarrow$

$$H_0^{ChPT} = 1 + 0.176 + O\left(p^4\right)$$

$$h_1^{ChPT} = \frac{1}{\Delta_{\eta\pi}}\left(1 - 0.21 + O\left(p^4\right)\right)$$

$$h_2^{ChPT} = \frac{1}{\Delta_{\eta\pi}^2}\left(4.9 + O\left(p^4\right)\right)$$

$$h_3^{ChPT} = \frac{1}{\Delta_{\eta\pi}^2}\left(1.3 + O\left(p^4\right)\right)$$

$$\left[h_i \equiv \frac{H_i}{H_0}\right]$$

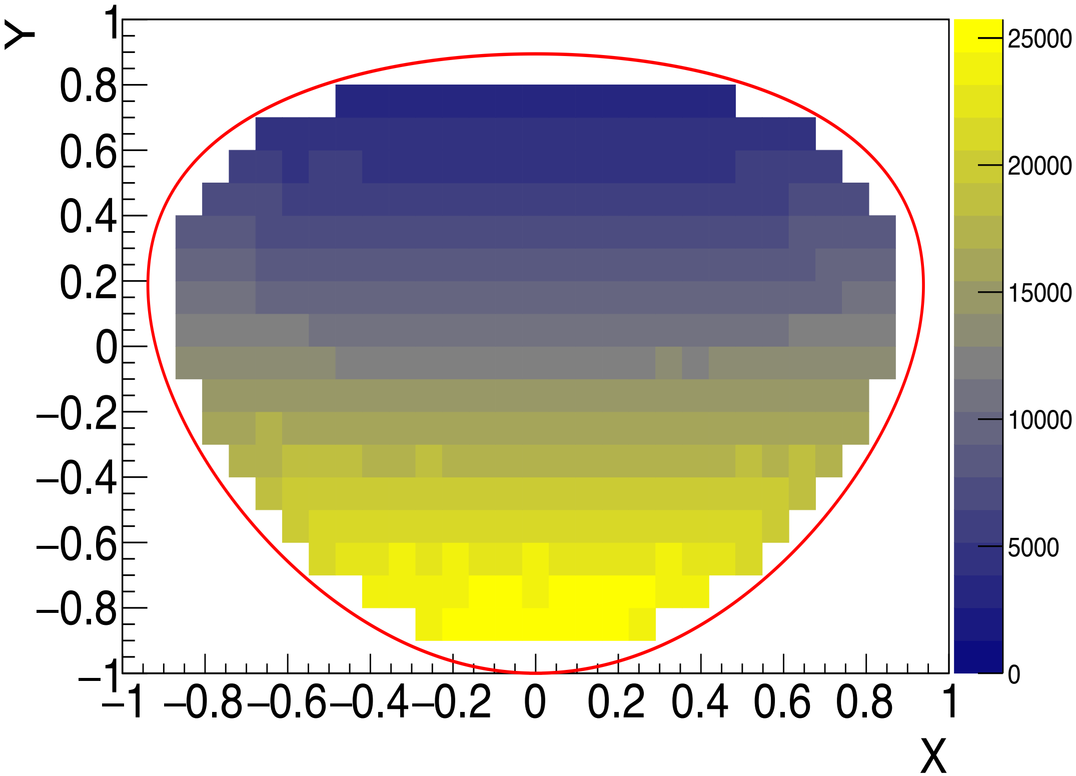
$\Rightarrow$

$$\boxed{\chi^2_{theo} = \sum_{i=1}^{3}\left(\frac{h_i - h_i^{ChPT}}{\sigma_{h_i^{ChPT}}}\right)^2}$$

$$\boxed{\sigma_{h_i^{ChPT}} = 0.3\left|h_i^{NLO} - h_i^{LO}\right|}$$

2.8 Fitting procedure

- Fit of $h_{1,2,3,4,5}$ using KLOE data and NLO ChPT



$$\chi^2_{\mathbf{exp}} = \sum_{bins=(376)} \left(\frac{N_{events} - N_{theory}}{\sigma_{N_{events}}} \right)^2$$

$$\boxed{\chi^2 = \chi^2_{theo} + \chi^2_{\mathbf{exp}}}$$

⇨ Very good fit $\chi^2_{\mathbf{exp}} = \mathbf{380.2}$ for 371 data points

3. Results

3.1 Isospin breaking corrections

- Dispersive calculations in the isospin limit ⇨ to fit to data one has to include isospin breaking corrections

- $$M_{c/n}(s,t,u) = M_{disp}(s,t,u)\,\frac{M_{DKM}(s,t,u)}{\tilde{M}_{GL}(s,t,u)}$$

with M_{DKM} : amplitude at one loop with $\mathcal{O}(e^2 m)$ effects

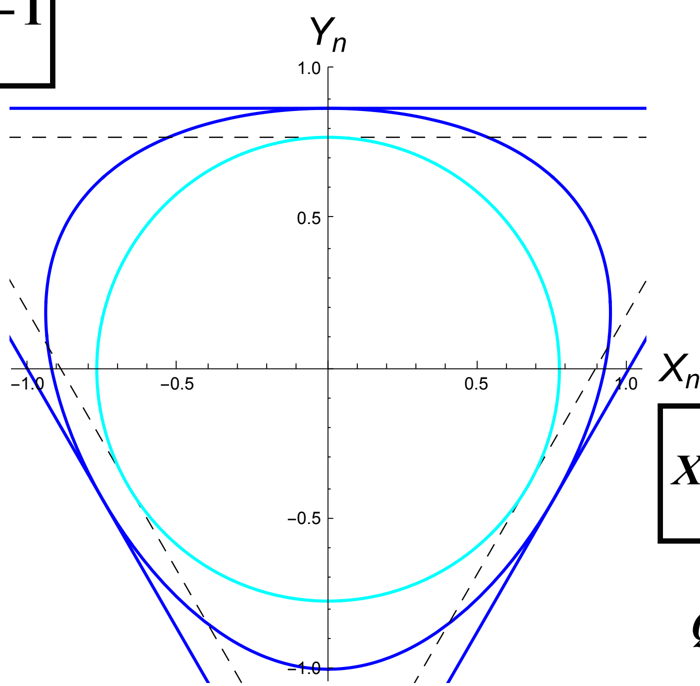
Ditsche, Kubis, Meissner'09

M_{GL}: amplitude at one loop in the isospin limit

Gasser & Leutwyler'85

$$Y_n = \frac{3T_3}{Q_n} - 1$$

Neutral channel



$$X_n = \sqrt{3}\,\frac{T_2 - T_1}{Q_n}$$

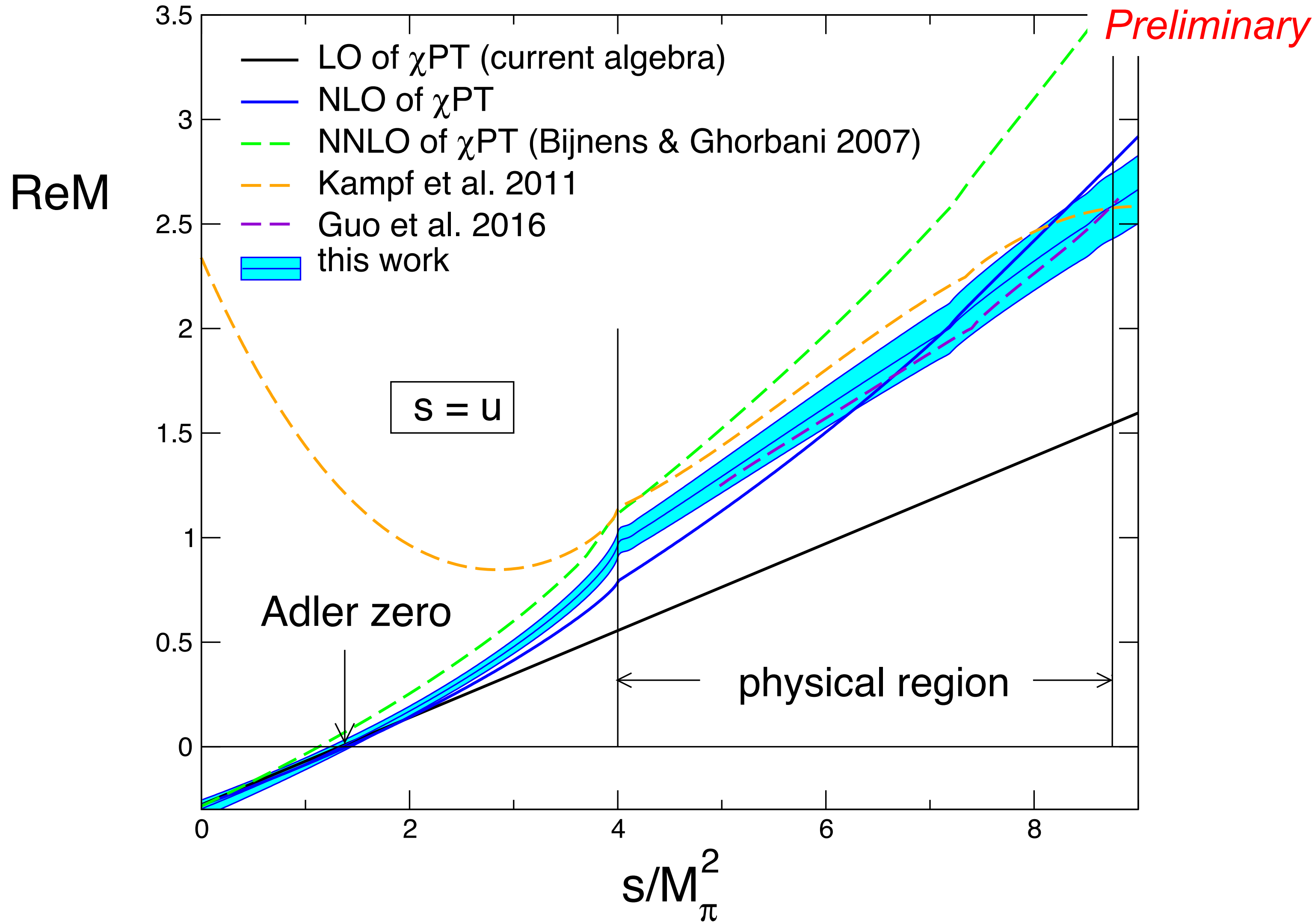
$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

Kinematic map:
isospin symmetric boundaries ⇨ physical boundaries

$$M_{GL} \rightarrow \tilde{M}_{GL}$$

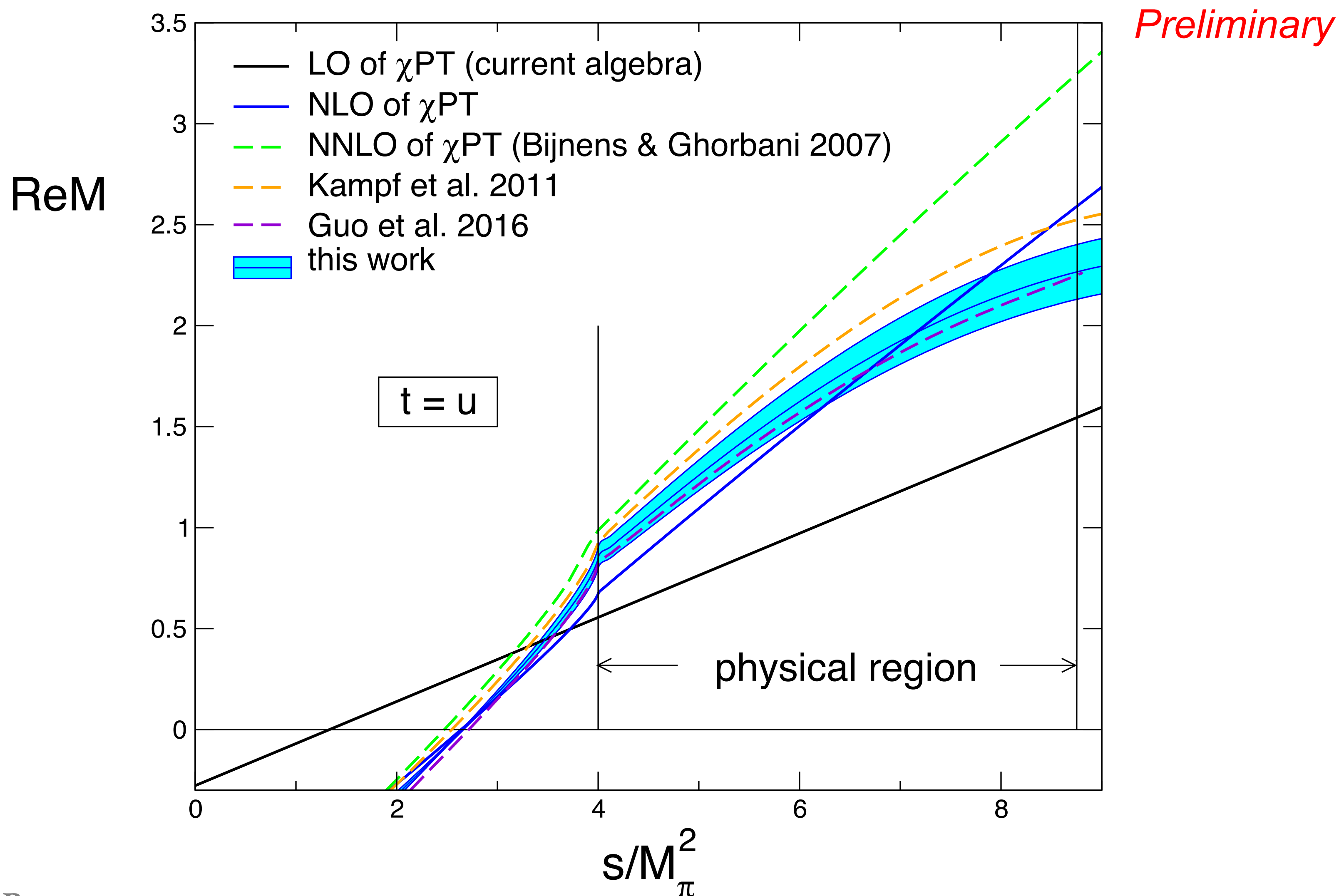
3.2 Amplitude for $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

- The amplitude along the line s = u :



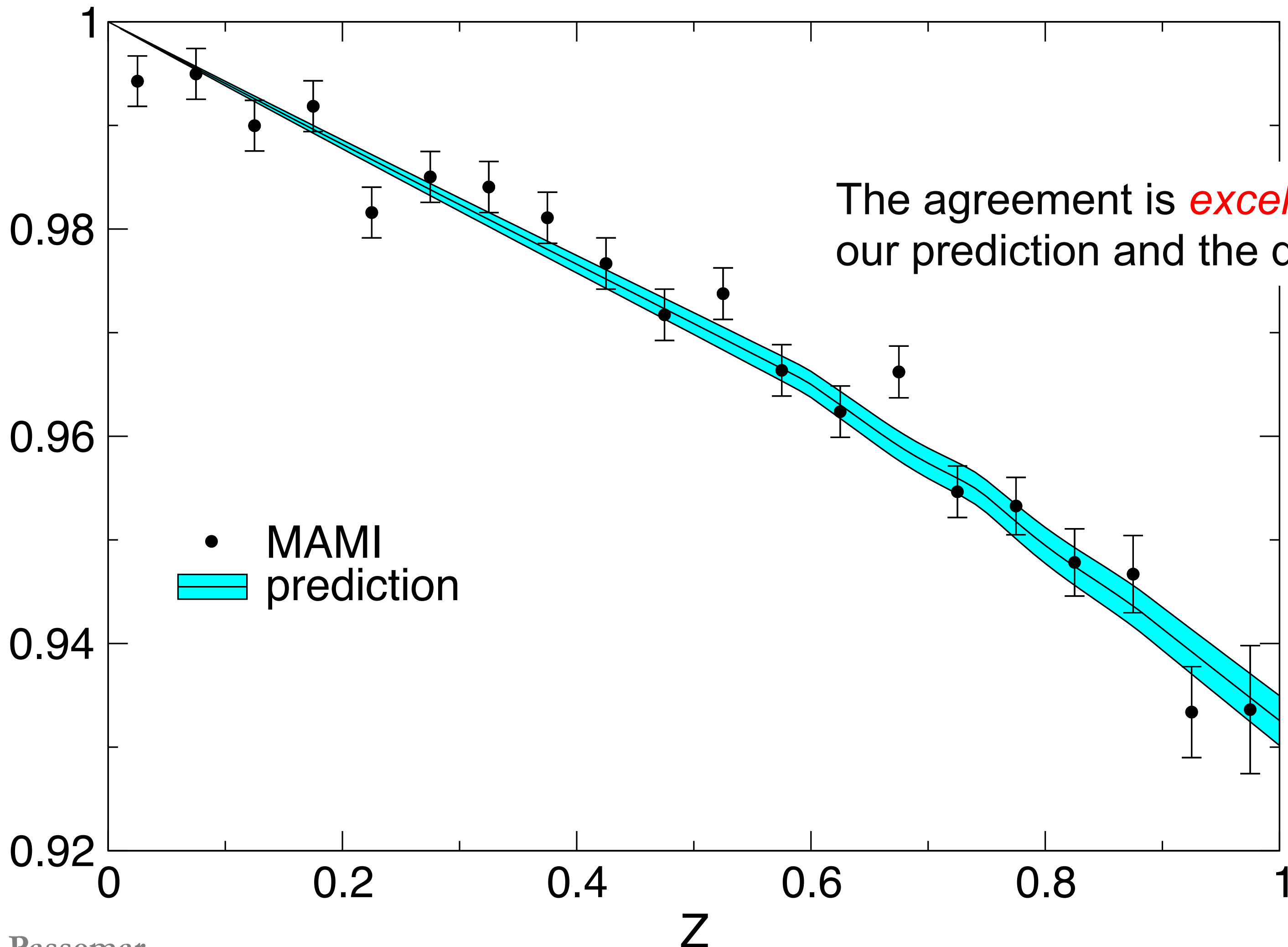
3.2 Amplitude for $\eta \to \pi^+ \pi^- \pi^0$ decays

- The amplitude along the line t = u :



3.3 Z distribution for $\eta \rightarrow \pi^0 \pi^0 \pi^0$ decays

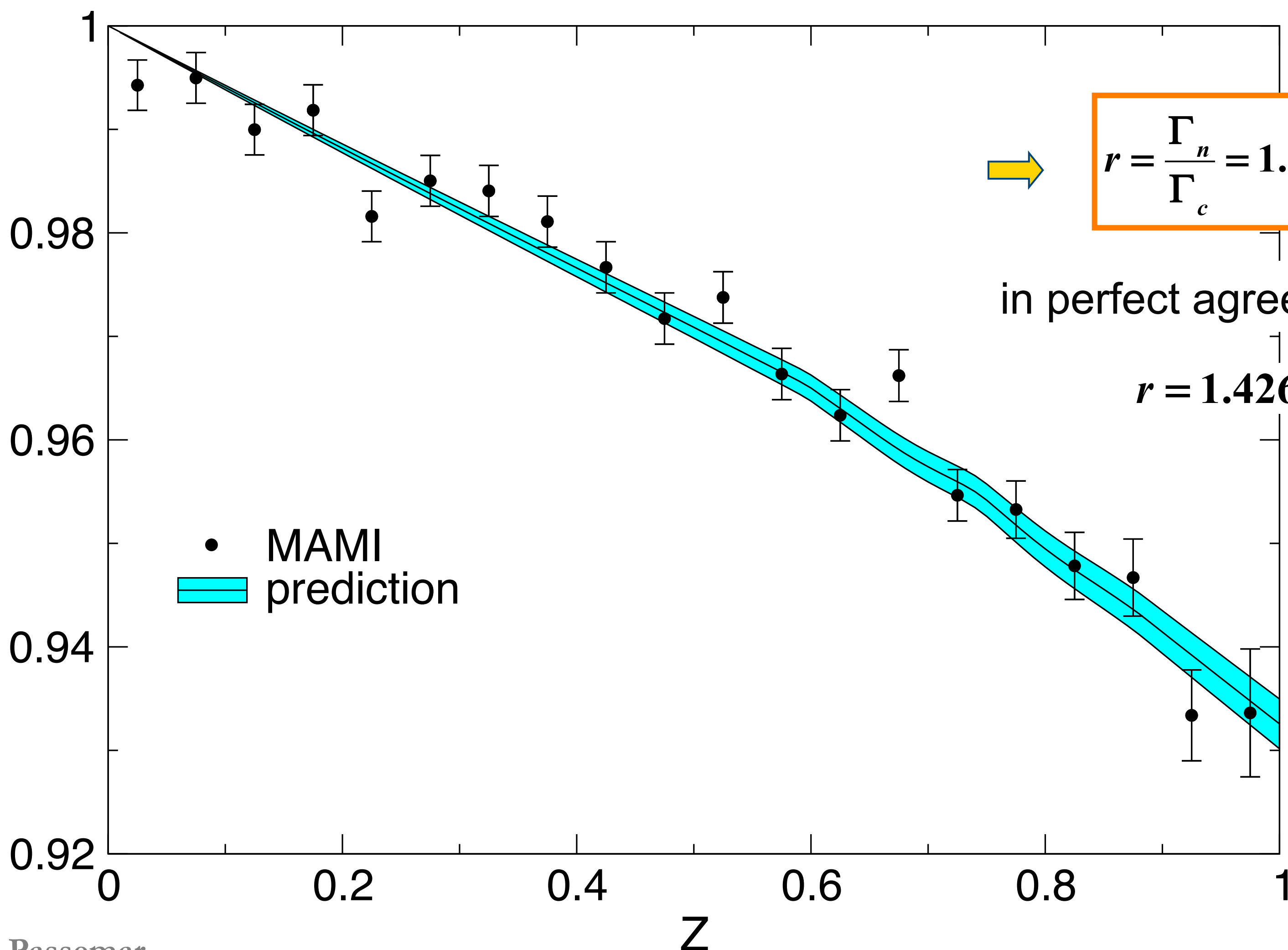
- The amplitude squared in the neutral channel is



The agreement is *excellent* between our prediction and the data!

3.3 Z distribution for $\eta \to \pi^0 \pi^0 \pi^0$ decays

- The amplitude squared in the neutral channel is



$$r = \frac{\Gamma_n}{\Gamma_c} = 1.4394(5)$$

in perfect agreement with PDG

$$r = 1.426 \pm 0.026$$

3.4 Extraction of Q and α

- Determination of Q from the dispersive approach :

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{Q^4}\frac{M_K^4}{M_\pi^4}\frac{\left(M_K^2-M_\pi^2\right)^2}{6912\pi^3 F_\pi^4 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \; \left|M(s,t,u)\right|^2$$

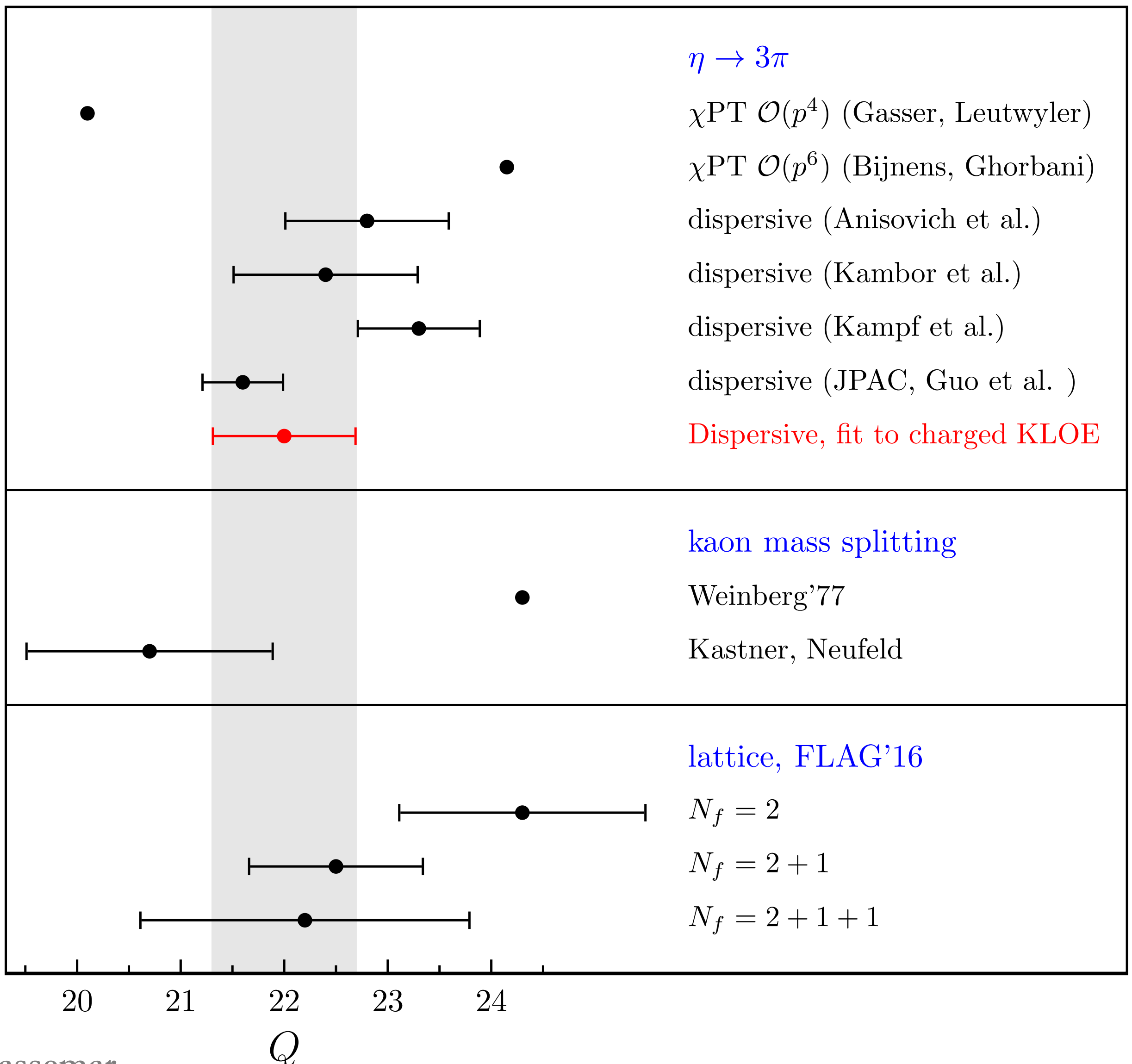
$$\Gamma_{\eta\to 3\pi} = 300 \pm 12 \text{ eV}$$ PDG'14

$$\left(Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \right)$$

- Determination of α

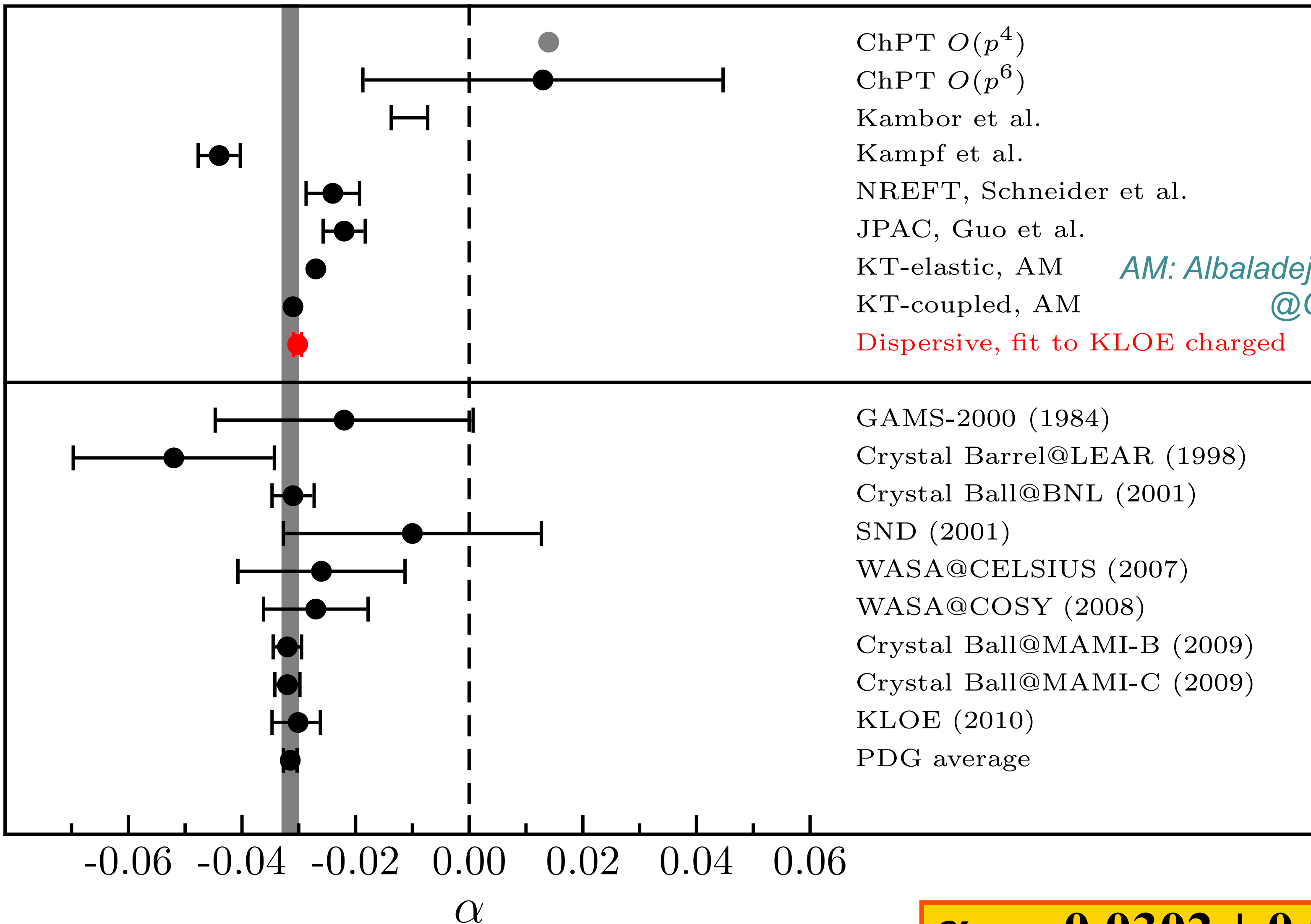
$$\left|A_n(s,t,u)\right|^2 = N\left(1+2\alpha Z\right)$$

3.5 Comparison of results for Q



$\boxed{Q = 22.0 \pm 0.7}$

3.6 Comparison of results for α

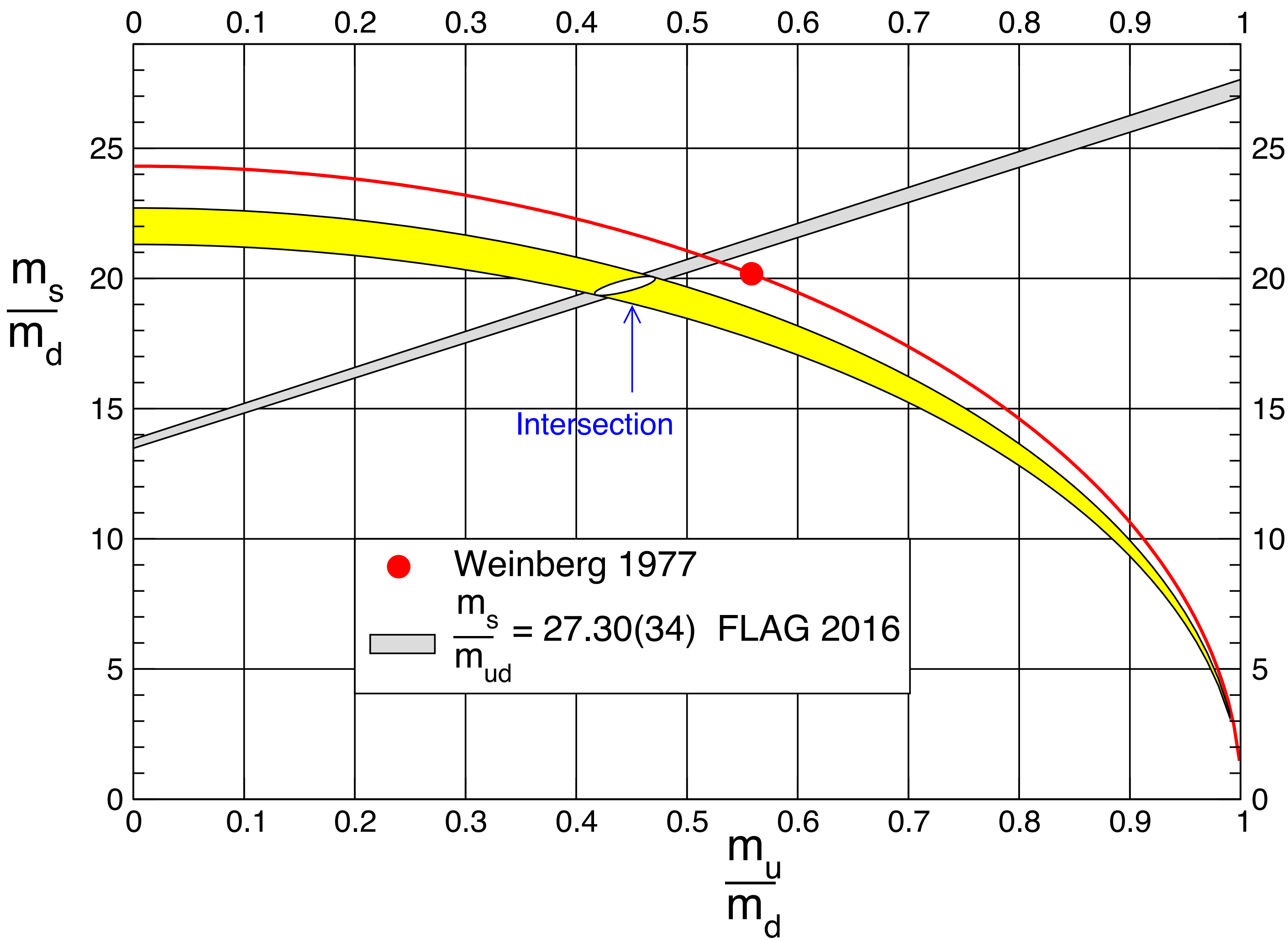


AM: Albaladejo & Moussallam
@CD15

$$\alpha = -0.0302 \pm 0.0011$$

3.7 Light quark masses

Courtesy of H.Leutwyler



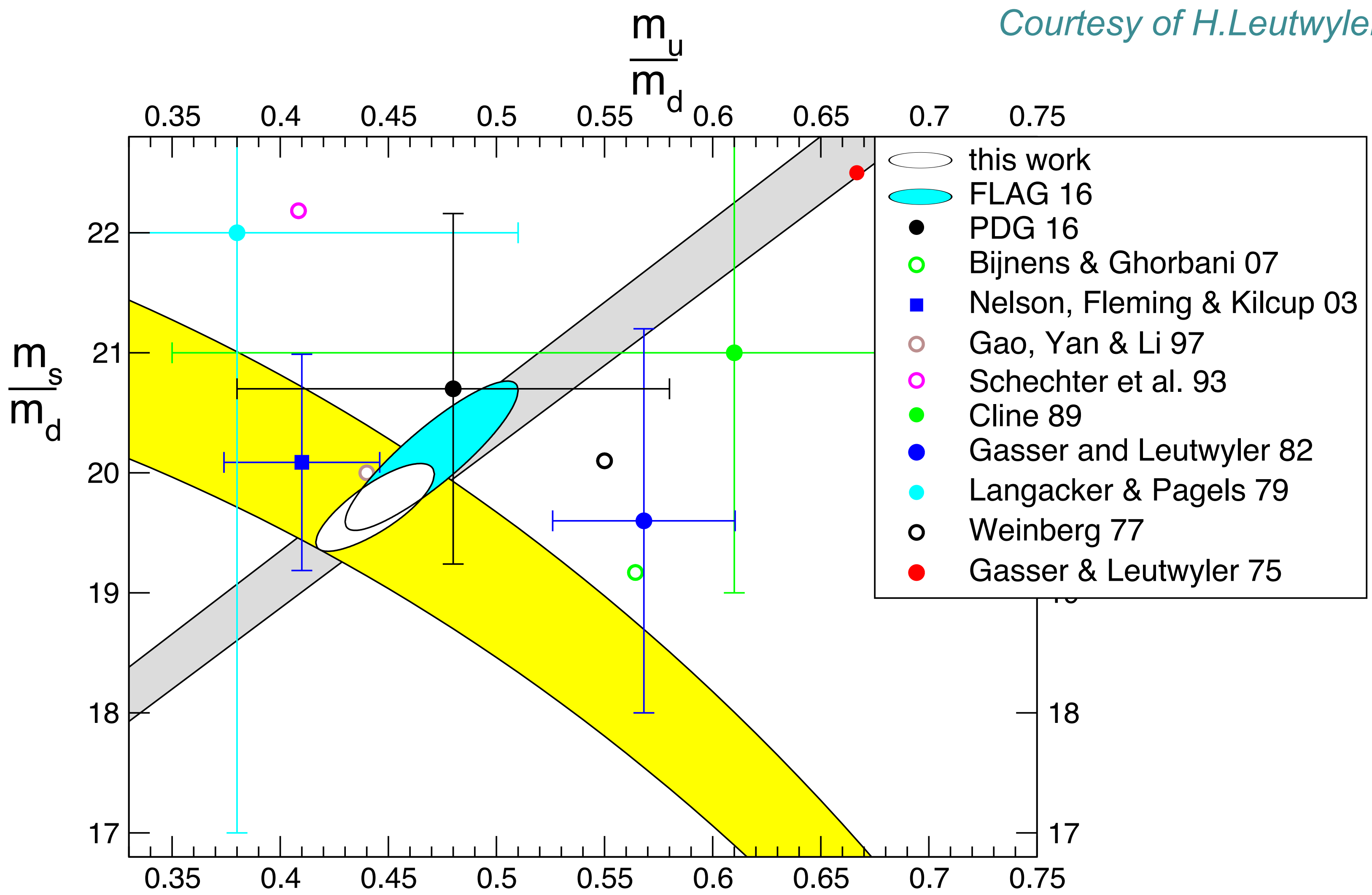
$$Q = 22.0 \pm 0.7$$

$$\frac{m_u}{m_d} = 0.44 \pm 0.03$$

- Smaller values for Q ⇨ smaller values for ms/md and mu/md than LO ChPT

3.7 Light quark masses

Courtesy of H.Leutwyler



4. Conclusion and outlook

4.1 Conclusion

- $\eta \rightarrow 3\pi$ decays represent a very clean source of information on the quark mass ratio *Q*

- A reliable extraction of *Q* requires having the strong rescattering effects in the final state under control

- This is possible thanks to dispersion relations

  ⇒ need to determine unknown subtraction constants

- This was done up to now relying exclusively on ChPT but precise measurements have become available

  - In the charged channel: *KLOE*, *WASA and BES III*
  - In the neutral channel: *MAMI-B*, *MAMI-C*, *WASA*
  - More results are expected: *CLAS and GlueX (Jlab)*

  ⇒ very interesting to get a reliable extraction of Q

- At the level of accuracy need to take the e.m. corrections into account

- Combine ChPT and the data in the optimal way

4.2 Outlook

- Our results give a consistent picture between
  - the KLOE'16 Dalitz plot charged measurement
  - the measurement in the neutral channel
  - theoretical requirements: e.g. Adler zero

- Extract very reliably Q with careful estimate of all uncertainties

- Outlook:
  - Estimate the experimental systematic on Q ⇒ collaboration with experimentalists
  - Perform our analysis with other experimental results: WASA, BES III etc..
  - Assume the subtraction constants real ⇒ investigate relaxing this condition
  - Matching to NNLO ChPT

    ⇒ Constraints from experiment: possible insights on C_i values

5. Back-up

4.3 Hat functions

- Discontinuity of M_I: by definition $disc\left[M_I(s)\right] \equiv disc\left[f_1^I(s)\right]$

  $\Rightarrow$ $\boxed{f_1^I(s) = M_I(s) + \hat{M}_I(s)}$

  with $\hat{M}_I(s)$ real on the right-hand cut

- The left-hand cut is contained in $\hat{M}_I(s)$

- Determination of $\hat{M}_I(s)$:
  subtract M_I from the partial wave projection of $M(s,t,u)$
  $$M(s,t,u) = M_0(s) + \left(s-u\right) M_1(t) + \ldots$$

- $\hat{M}_I(s)$ singularities in the t and u channels, depend on the other M_I
  Angular averages of the other functions $\Rightarrow$ Coupled equations

4.3 Hat functions

- Ex: $$\boxed{\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + 2(s-s_0)\langle M_1\rangle + \frac{20}{9}\langle M_2\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle}$$

where $$\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I\big(t(s,z)\big),$$

$$z = \cos\theta \quad \text{scattering angle}$$

Non trivial angular averages $\Rightarrow$ need to deform the integration path to avoid crossing cuts

Anisovich & Anselm'66

3.3 Different recent analyses

1. *Schneider, Kubis, Ditsche 2011*: 2-loop NREFT approach
   - allows investigation of isospin-violating corrections
   - relations between charged and neutral Dalitz plots

2. *Kampf, Knecht, Novotny, Zdrahal 2011*: Analytic dispersive approach
   - Amplitudes involve 6 parameters (subtraction constants)
   - Fit to Dalitz plot distribution (KLOE 2008: $\eta \to \pi^+\pi^-\pi^0$)
   - Predict Dalitz plot parameter α (neutral decay mode)
   - Match to absorptive part of NNLO chiral amplitude where differences between NLO and NNLO are small $\Rightarrow$ R (Q)

   Problem: do not reproduce the Adler's zero

3.3 Different recent analyses

3. *Guo et al. 2015: JPAC* analysis, Khuri Treiman equations solved numerically using Pasquier inversion techniques

- Madrid/Cracow ππ phase shifts, 3 subtraction constants
- Fit experimental Dalitz plot (*WASA/COSY* 2014: $\eta \to \pi^+\pi^-\pi^0$, *KLOE* 2016)
  ⇒ predict Dalitz plot parameter α
- Match to NLO ChPT near Adler zero ⇒ Q

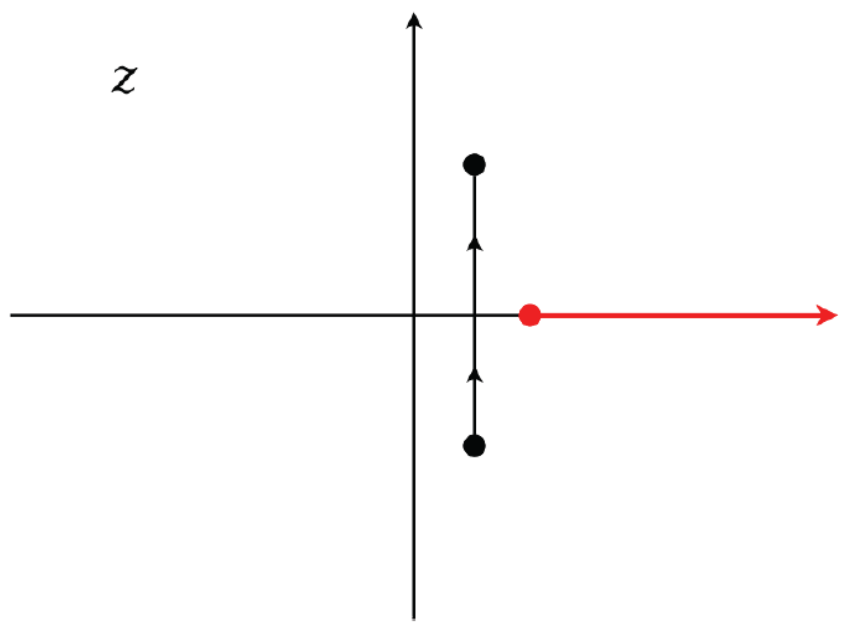
Discontinuities of the $M_I(s)$

- Ex: $\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + 2(s - s_0)\langle M_1\rangle + \frac{20}{9}\langle M_2\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle$

  where $\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),\ z = \cos\theta$ scattering angle

  Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts

  Anisovich & Anselm'66



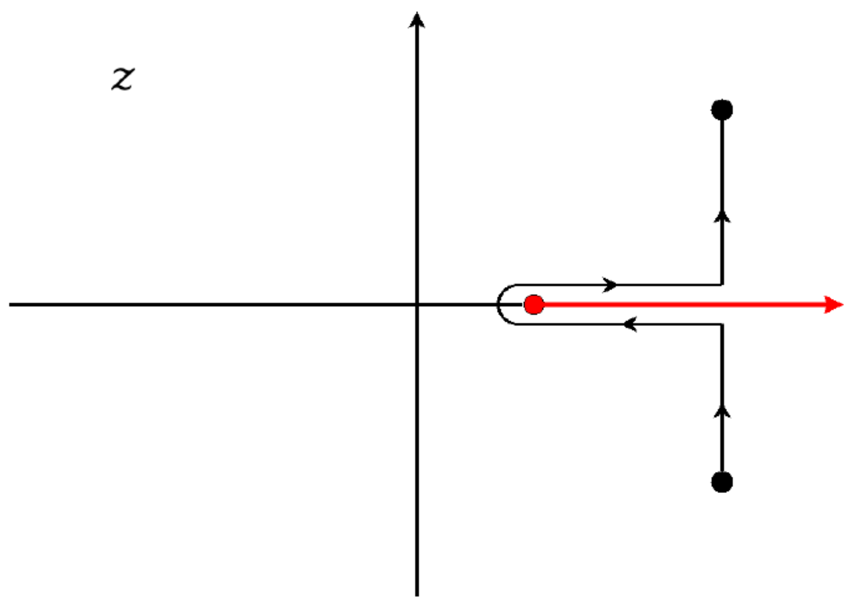
Discontinuities of the $M_I(s)$

- Ex: $\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + 2(s - s_0)\langle M_1 \rangle + \frac{20}{9}\langle M_2 \rangle + \frac{2}{3}\kappa(s)\langle zM_1 \rangle$

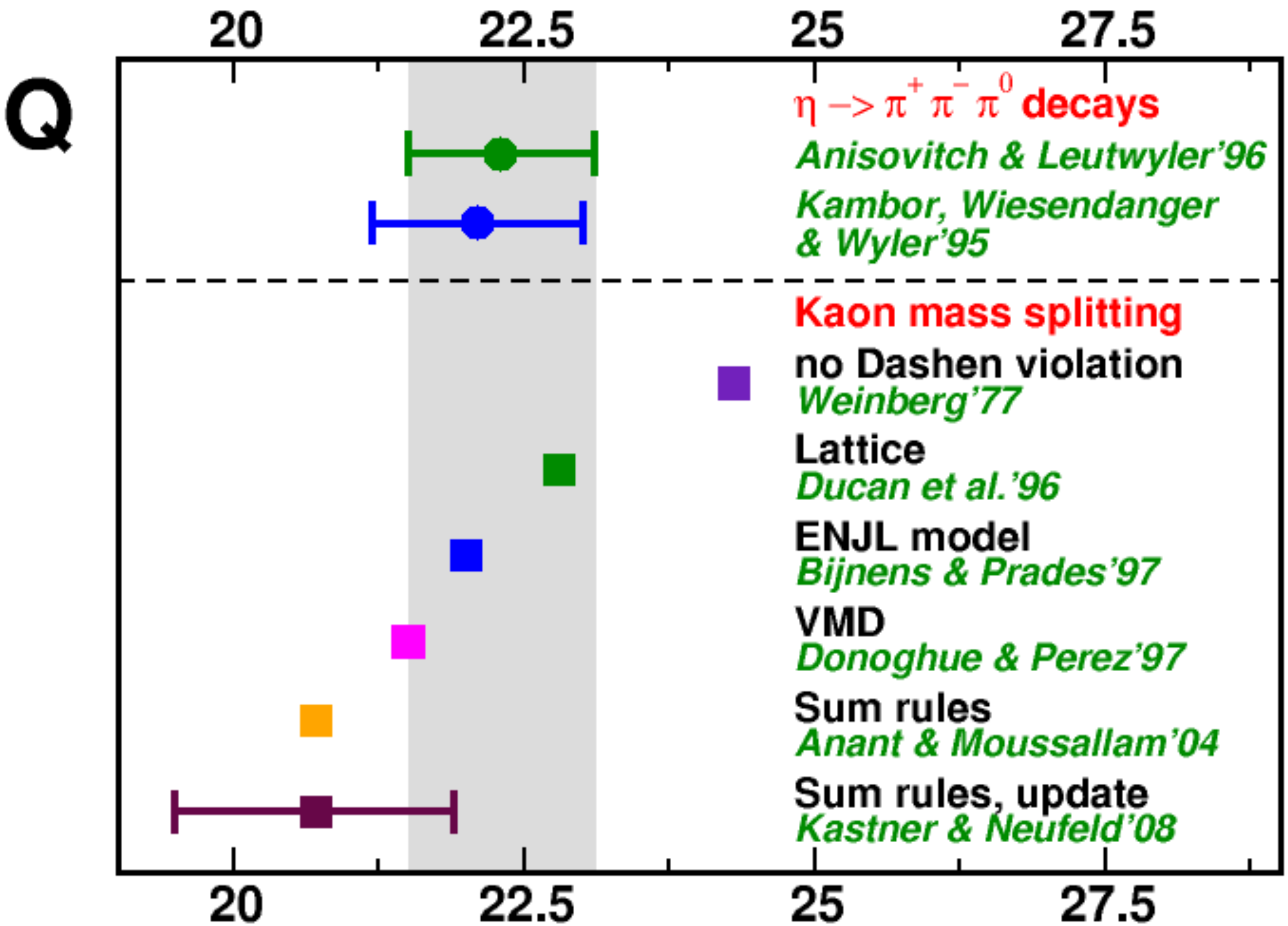
  where $\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),\ z = \cos\theta$ scattering angle

  Non trivial angular averages ⇒ need to deform the integration path to avoid crossing cuts

  Anisovich & Anselm'66



3.7 Comparison of values of Q



$Q = 22.3 \pm 0.9$

$Q = 22.1 \pm 0.9$

$Q = 24.2$

$Q = 22.8$

$Q = 22$

$Q = 21.5$

$Q = 20.7$

$Q = 20.7 \pm 1.2$

➡ Fair agreement with the determination from meson masses

Comparison with Q from meson mass splitting

- $Q^2 = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2}\left[1 + O(m_q^2)\right]$ is only valid for e=0

- Including the electromagnetic corrections, one has

$$\boxed{Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)}}$$

$\Rightarrow$ $Q_D = 24.2$

- Corrections to the Dashen's theorem

$\Rightarrow$ The corrections can be large due to $e^2 m_s$ corrections:

$$\boxed{\left(M_{K^+}^2 - M_{K^0}^2\right)_{\text{em}} - \left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\text{em}} = e^2 M_K^2\left(A_1 + A_2 + A_3\right) + O\left(e^2 M_\pi^2\right)}$$

Urech'98,
Ananthanarayan & Moussallam'04

3.6 Corrections to Dashen's theorem

- Dashen's Theorem

$$\left(M^2_{K^+} - M^2_{K^0}\right)_{\mathrm{em}} = \left(M^2_{\pi^+} - M^2_{\pi^0}\right)_{\mathrm{em}} \Rightarrow \boxed{\left(M_{K^+} - M_{K^0}\right)_{\mathrm{em}} = 1.3 \text{ MeV}}$$

- With higher order corrections
  - Lattice : $\left(M_{K^+} - M_{K^0}\right)_{\mathrm{em}} = 1.9 \text{ MeV}, \; Q = 22.8$ *Ducan et al.'96*
  - ENJL model: $\left(M_{K^+} - M_{K^0}\right)_{\mathrm{em}} = 2.3 \text{ MeV}, \; Q = 22$ *Bijnens & Prades'97*
  - VMD: $\left(M_{K^+} - M_{K^0}\right)_{\mathrm{em}} = 2.6 \text{ MeV}, \; Q = 21.5$ *Donoghue & Perez'97*
  - Sum Rules: $\left(M_{K^+} - M_{K^0}\right)_{\mathrm{em}} = 3.2 \text{ MeV}, \; Q = 20.7$ *Anant & Moussallam'04*

  Update $\Rightarrow$ $Q = 20.7 \pm 1.2$ *Kastner & Neufeld'07*

4.2 Method: Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$\boxed{M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)}$$

Fuchs, Sazdjian & Stern'93

Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

- Functions of only one variable with only right-hand cut of the partial wave ⇨ $disc\left[M_I(s)\right] \equiv disc\left[f_1^I(s)\right]$

- Elastic unitarity *Watson's theorem*

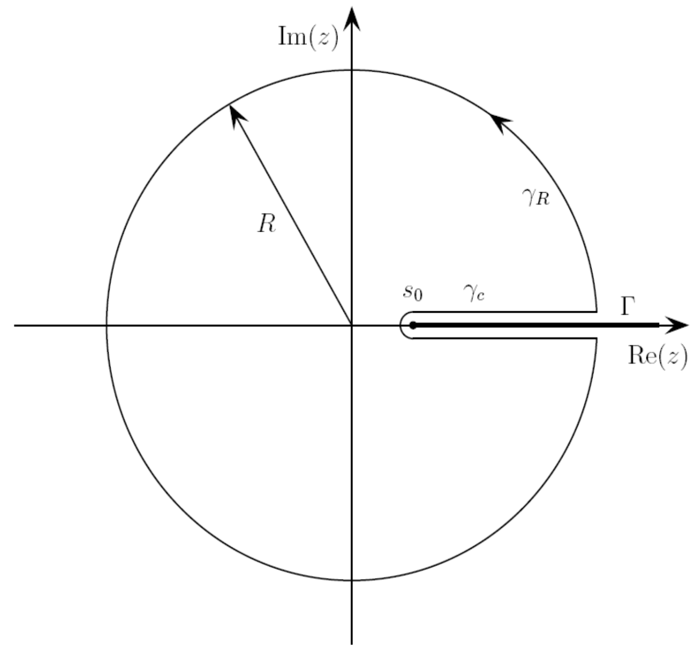
$$\boxed{disc\left[f_1^I(s)\right] \propto t_1^*(s) f_1^I(s)}$$

with $t_1(s)$ partial wave of elastic $\pi\pi$ scattering

4.2 Method: Representation of the amplitude

- Knowing the discontinuity of M_I ⇨ write a dispersion relation for it

- Cauchy Theorem and Schwarz reflection principle



⇨ $$\boxed{M_I(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{disc\left[M_I(s')\right]}{s'-s-i\varepsilon} ds'}$$

M_I can be reconstructed everywhere from the knowledge of $disc\left[M_I(s)\right]$

- If M_I doesn't converge fast enought for $|s| \to \infty$ ⇨ subtract the dispersion relation

$$\boxed{M_I(s) = P_{n-1}(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n} \frac{disc\left[M_I(s')\right]}{(s'-s-i\varepsilon)}}$$

$P_{n-1}(s)$ polynomial

4.4 Dispersion Relations for the $M_I(s)$

- $$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|\left(s'-s-i\varepsilon\right)}\right)$$

  Omnès function

  Similarly for M_1 and M_2

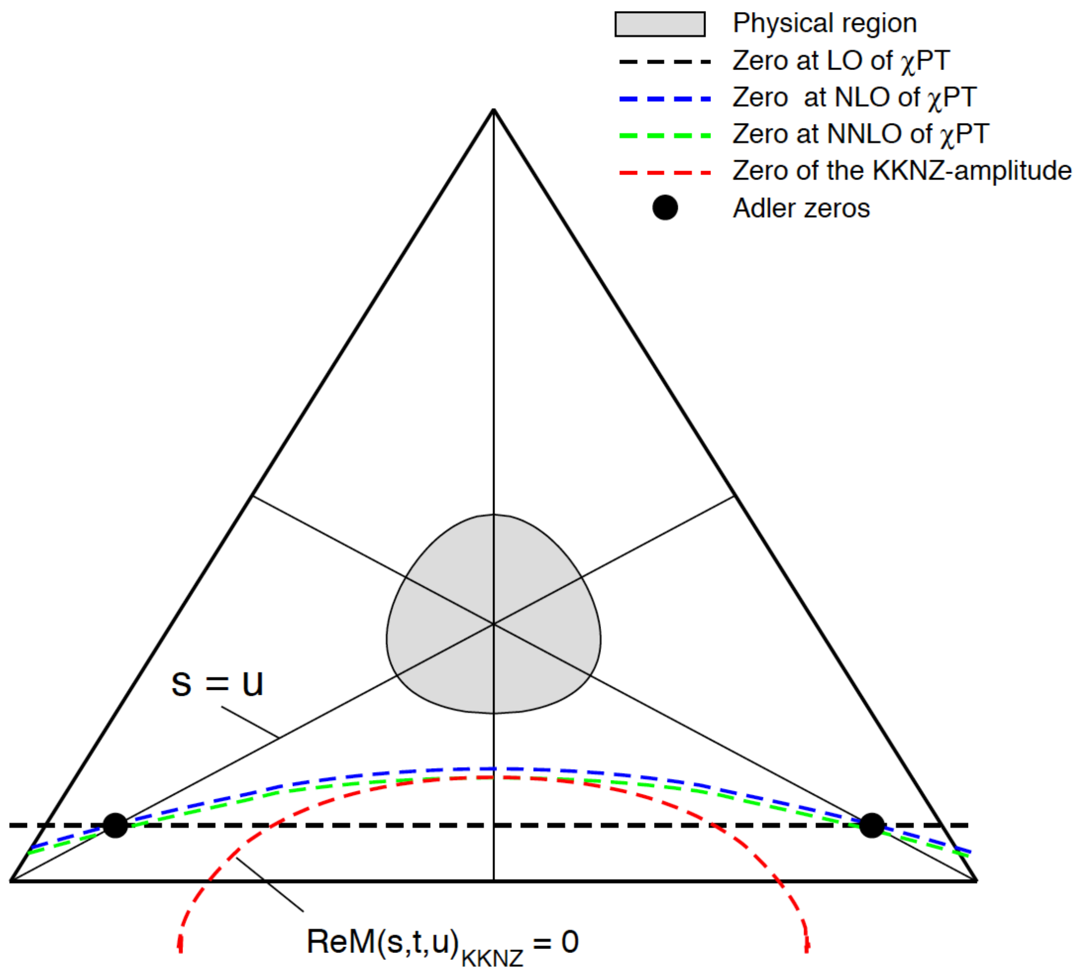
- Four subtraction constants to be determined: α_0, β_0, γ_0 and one more in M_1 (β_1)

- Inputs needed for these and for the $\pi\pi$ phase shifts δ_l^I
  - M_0: $\pi\pi$ scattering, ℓ=0, I=0
  - M_1: $\pi\pi$ scattering, ℓ=1, I=1
  - M_2: $\pi\pi$ scattering, ℓ=0, I=2

- Solve dispersion relations numerically by an iterative procedure

5.4 Comparison with KKNZ

- Adler zero not reproduced!



1.5 Quark mass ratios

- Mass formulae to second chiral order *Gasser & Leutwyler'85*

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

with $\Delta_M = \dfrac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{-logs}$

$$\left[\hat{m} \equiv \frac{m_d + m_u}{2}\right]$$

- The same O(m) correction appears in both ratios

⇨ Take the double ratio

$$\boxed{Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}}\left[1 + O(m_q^2, e^2)\right]$$

Very Interesting quantity to determine since Q^2 does not receive any correction at NLO!