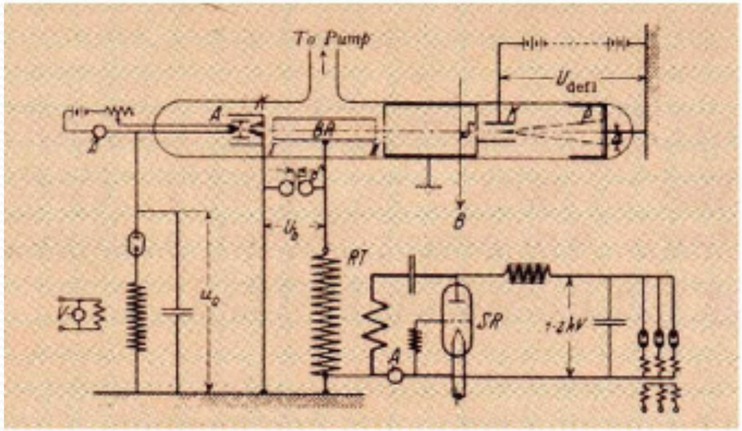
Circular Accelerators basics

M.E. Biagini, INFN-LNF
Corso di “Teoria degli acceleratori”
per dottorato Universita’ di Roma 3
LNF, 13 Giugno 2016

Why we need accelerated particles beams?

- To collide with other beams and understand the fundamental constituents of matter (fundamental physics)
- To modify matter (tumors therapy, ion implantation,...)
- To produce photoni beams (*synchrotron light*) to be used for different applications (ex. biology, material science, nano-technologies,....)

Short accelerators hystory

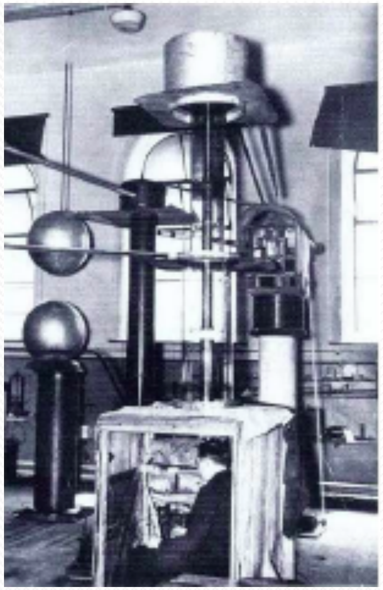


1928: Wideroe, Linac

1929: Van de Graaff Generator

1930: Lawrence, Ciclotron

1932: Cockcroft-Walton, Electrostatic Accelerator



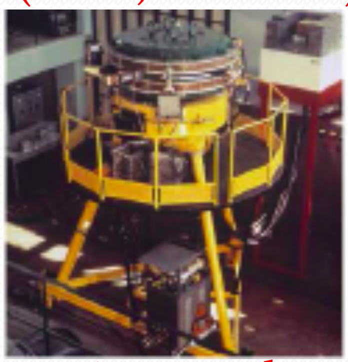
Short accelerators hystory (2)

1943: Synchrotron

1944: Phase stability Principle

1952: Strong Focusing

1961: First Collider e^+e^- (AdA, Frascati)

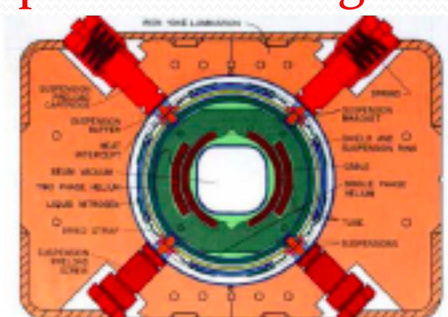


1966: Electron Cooling

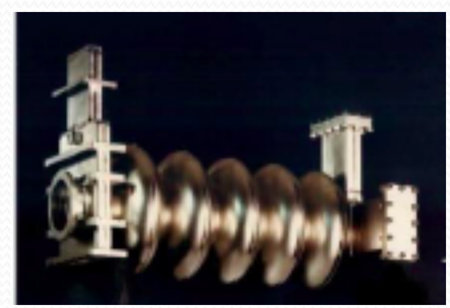
1968: Stochastic Cooling

1969: First Hadron collider

1983: SuperConducting Magnets

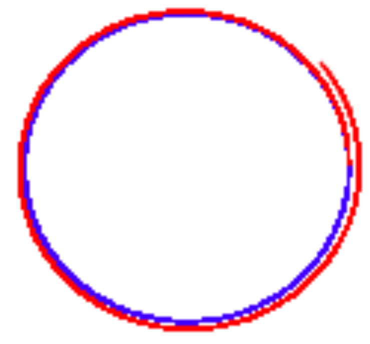


1994: SuperConducting RF



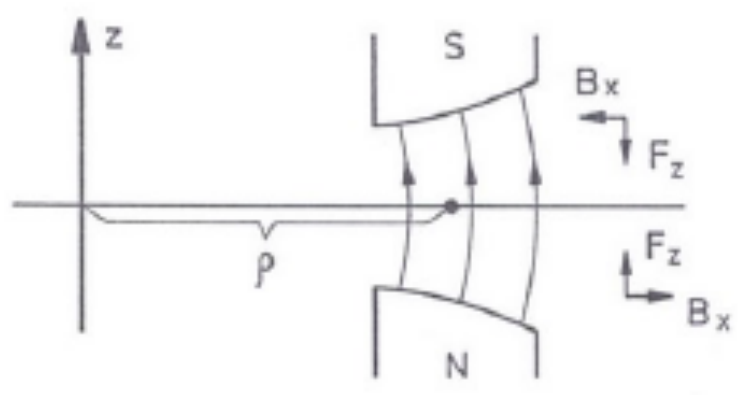
Focusing in a circular accelerator

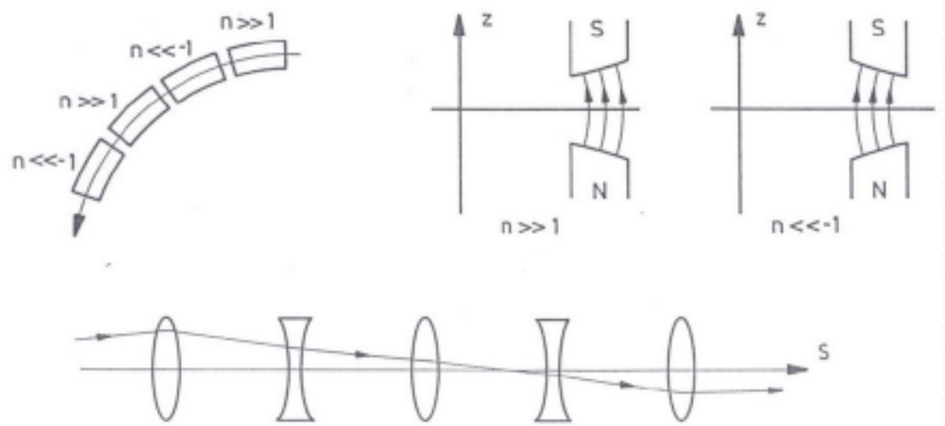
- To maintain particles inside the vacuum pipe in circular accelerators a magnetic field is used
- In a uniform magnetic field particles of different energy have different trajectories → instability → a "**transverse focusing**" in needed



Weak Focusing

Strong Focusing





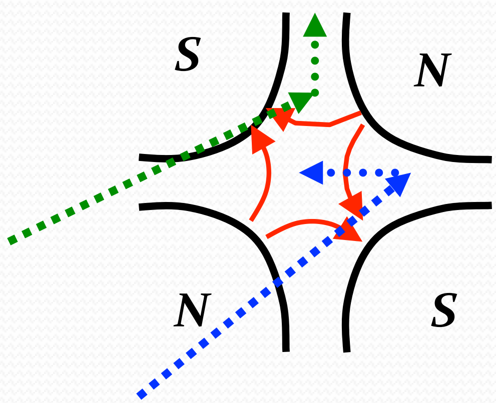
First solution: magnetic poles shaping a small gradient is introduced

Second solution: magnetic poles shaping + alternating poles

Strong focusing

- In weak focusing accelerators the beam is focused in one plane only
- *Alternating* a sequence of magnets of equal intensity focusing *in X e Y*, total effect is focusing in both planes "*strong focusing*" (o "*alternating gradient focusing*")
- *Quadrupole magnets* are used, *equivalent to optical lenses*, with high gradient and short focal length
- Two opposite sign quadrupoles together have a total focusing effect in both planes

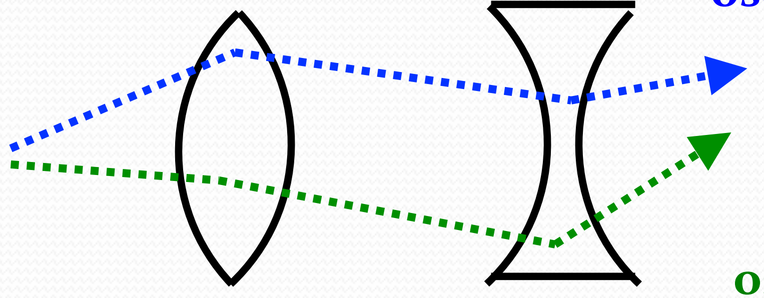
Defocusing in vertical plane



Focusing in horizontal plane

"QF" **"QD"**

Horizontal oscillation

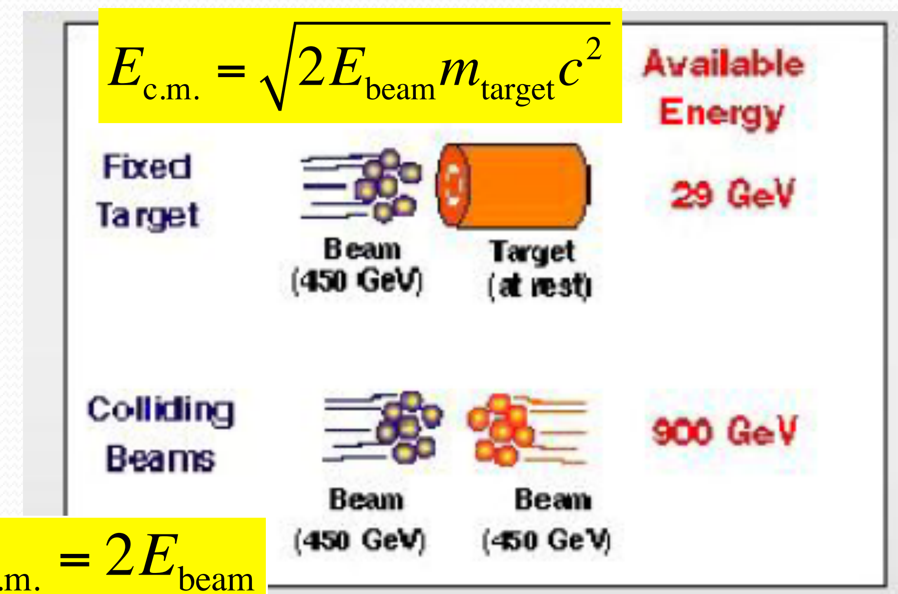


Vertical oscillation

Bruno Touschek (Frascati) great idea: e^+e^- colliders

- Collisions of one electron beam and one positron beam circulating in opposite direction
- In the annihilation process the colliding e^+e^- release all their energy in the center of mass, which is available for the production of other particles
- This allowed for a big increase of the energy available in the center of mass with respect to fixed target collisions

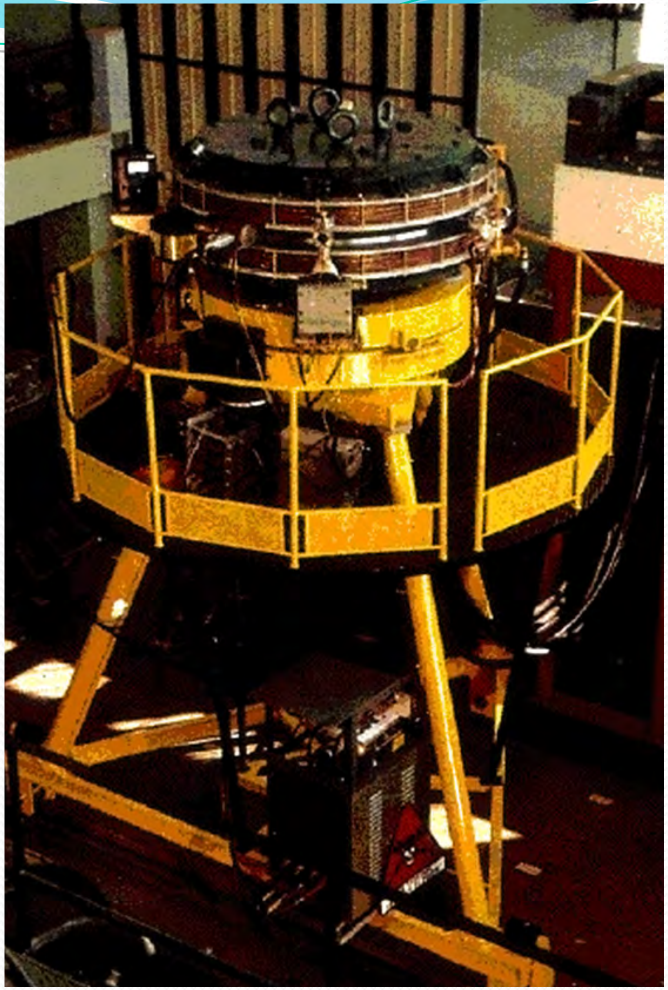
$$E_{\text{c.m.}} = \sqrt{2E_{\text{beam}} m_{\text{target}} c^2}$$

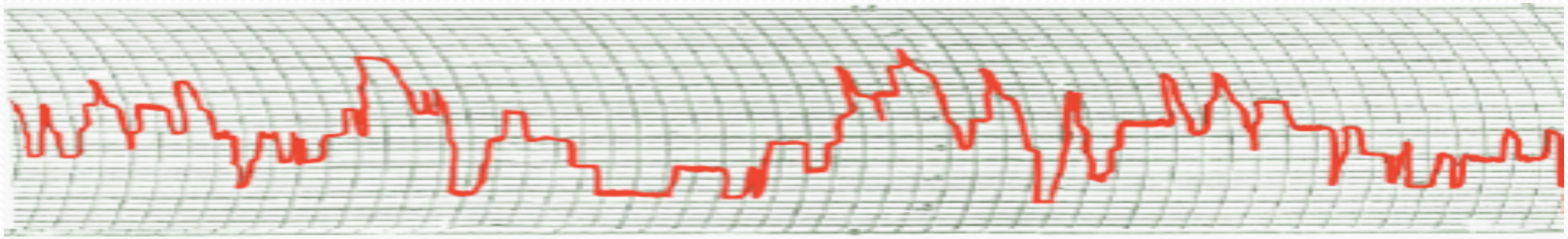


$$E_{\text{c.m.}} = 2E_{\text{beam}}$$

AdA: birth of a collider

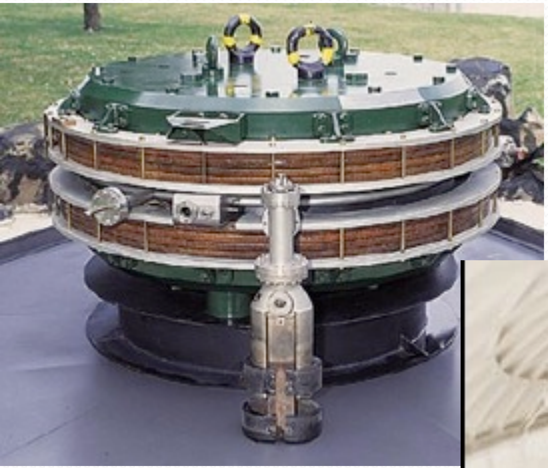
- First collider was AdA (Anello di Accumulazione, i.e. storage ring) 250 MeV (1965-69) built at LNF by Bruno Touschek and collaborators
- Test performed at LAL-Orsay (France) where a Linac for injecting electrons and positrons was available
- Ada success → ACO (France), SPEAR (SLAC, US) and Adone (Frascati) → start of colliders physics



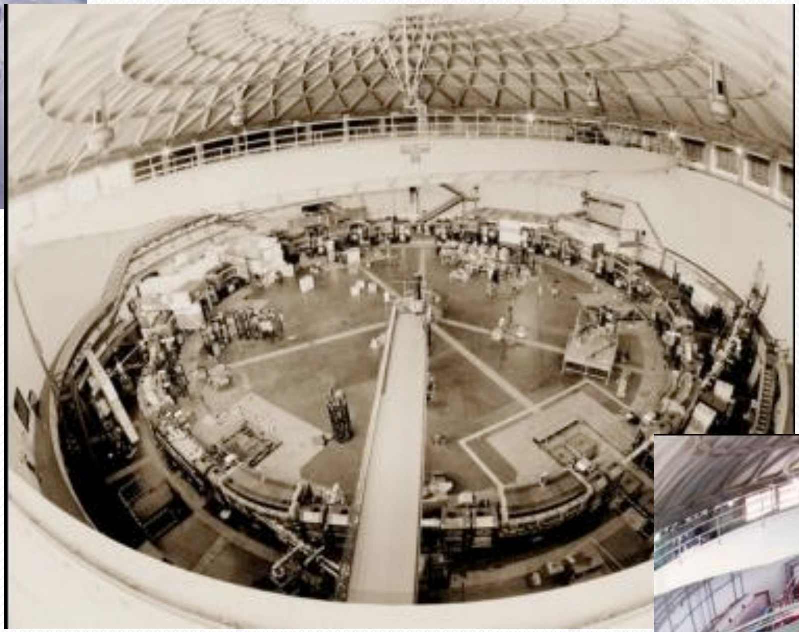


Track of first electron beam accumulated in AdA, lifetime 21 sec, N_e ~2.3

Colliders at Frascati…



AdA 0.25 GeV/beam

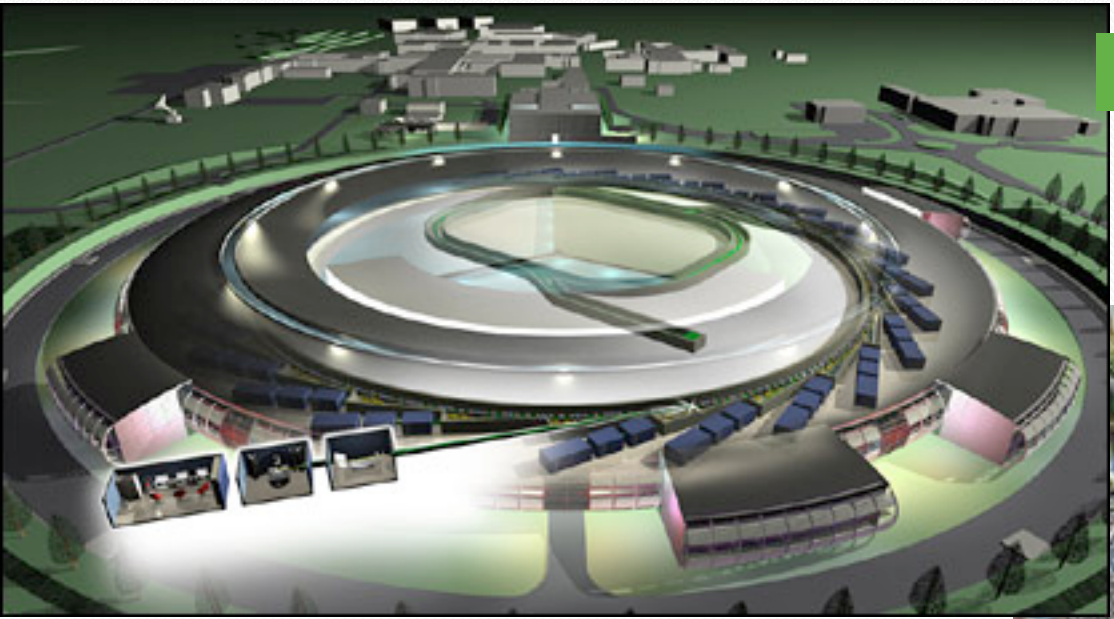


ADONE
1.5 GeV/beam

DAΦNE 0.51 GeV/beam

Two examples of other applications

- Synchrotron Light Sources: biology, materials science, chemistry, lithography, test of detectors, art, etc....
- Medical accelerators: electron or positron linac, synchrotrons for hadrotherapy with protons or ions

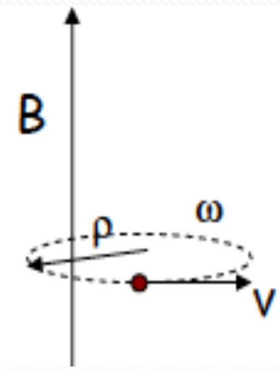


Diamond Light Source, Oxford, UK

CNAO, Pavia, Italy

Electromagnetic fields in a circular accelerator

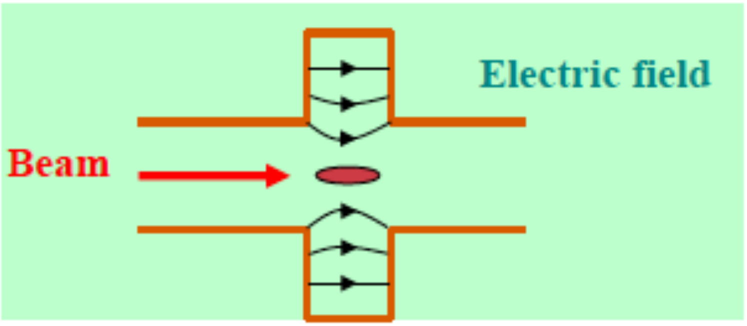
- Electric field E used for acceleration
- Magnetic field B used for bending and focusing
- A charge in a constant magnetic field makes a circular orbit with radius ρ and angular frequency ω



NB: **For relativistic particles the strength of a 1 Tesla bending magnet is equivalent to an electric field of $3\text{x}10^8$ V/m (far beyond technical limits)**

Acceleration **B=0**

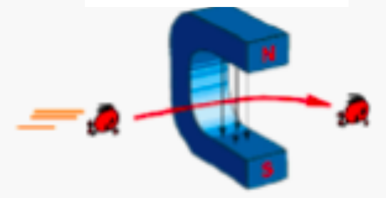
$\vec{E}$ = electric field



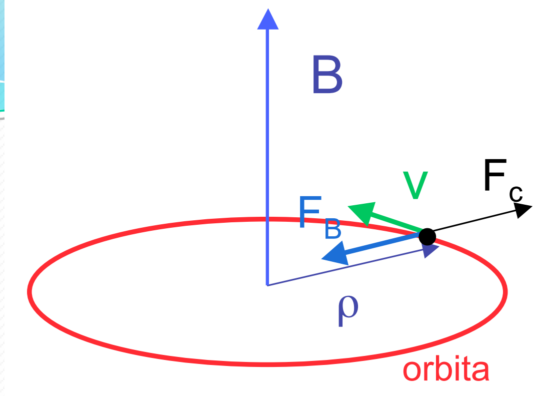
E=0 **Bending and focusing**

$\vec{B}$ = magnetic field

Deflection by magnetic field



Lorentz force



$$\boxed{\vec{F}_L = e\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

Force from E field is parallel to E field

Force from B field is perpendicular to velocity and B field

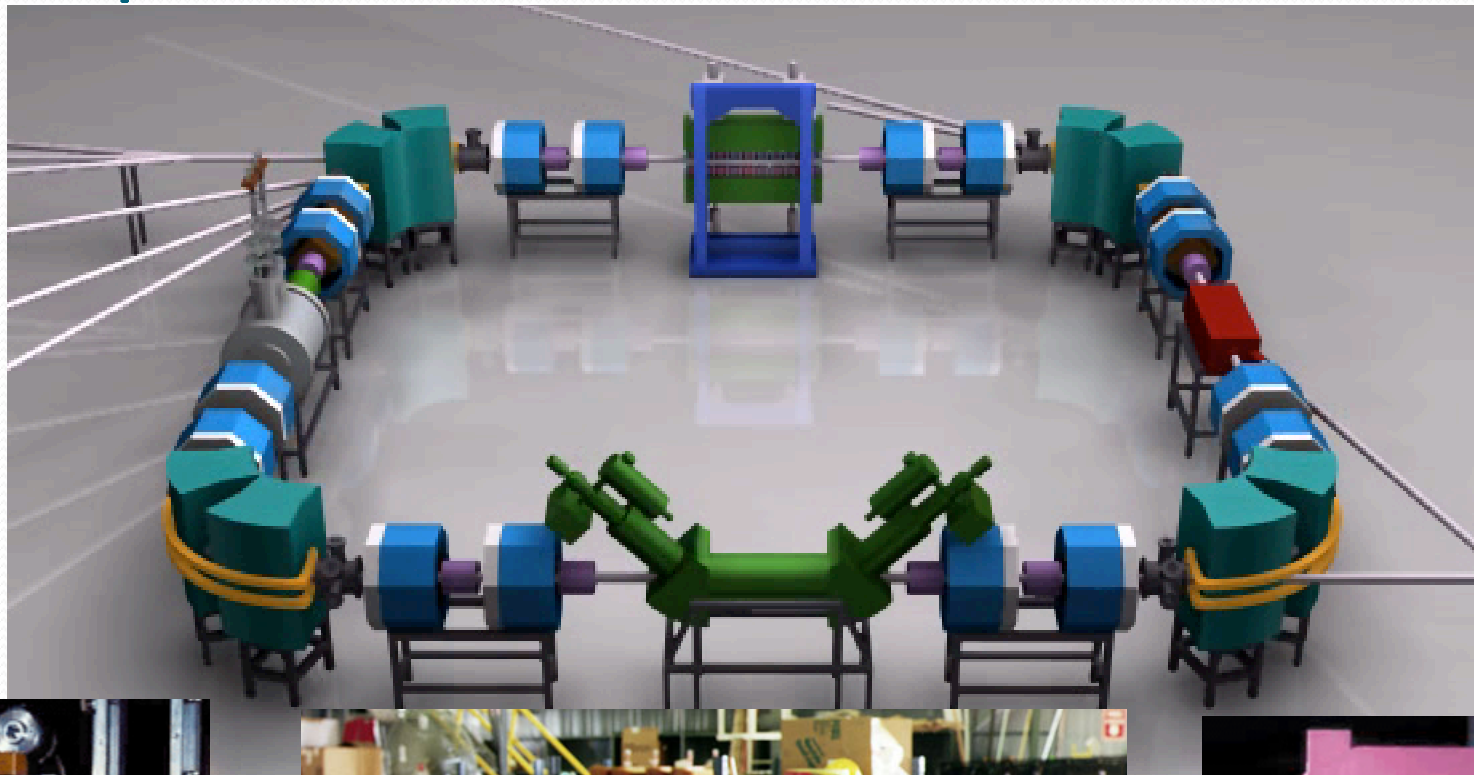
- The magnetic force $F_B = qvB$ is perpendicular to the particle velocity and bends the trajectory with a radius of curvature ρ
- The centrifugal force $F_c = \frac{mv^2}{\rho}$ balances the magnetic force:

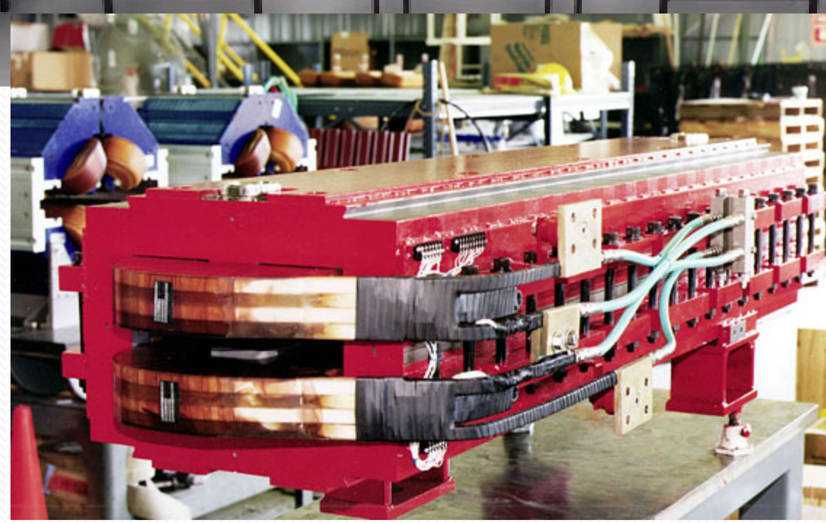
$$\boxed{qvB = \frac{mv^2}{\rho} \Rightarrow \rho = \frac{mv}{qB} = \frac{p}{qB}}$$

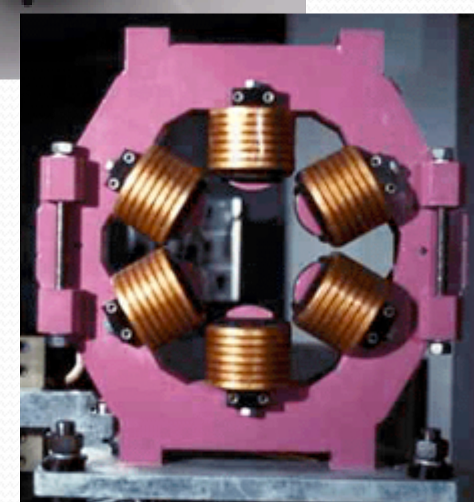
- Beam "rigidity"

$$\boxed{B\rho = \frac{p}{e}}$$

Components of a circular accelerator







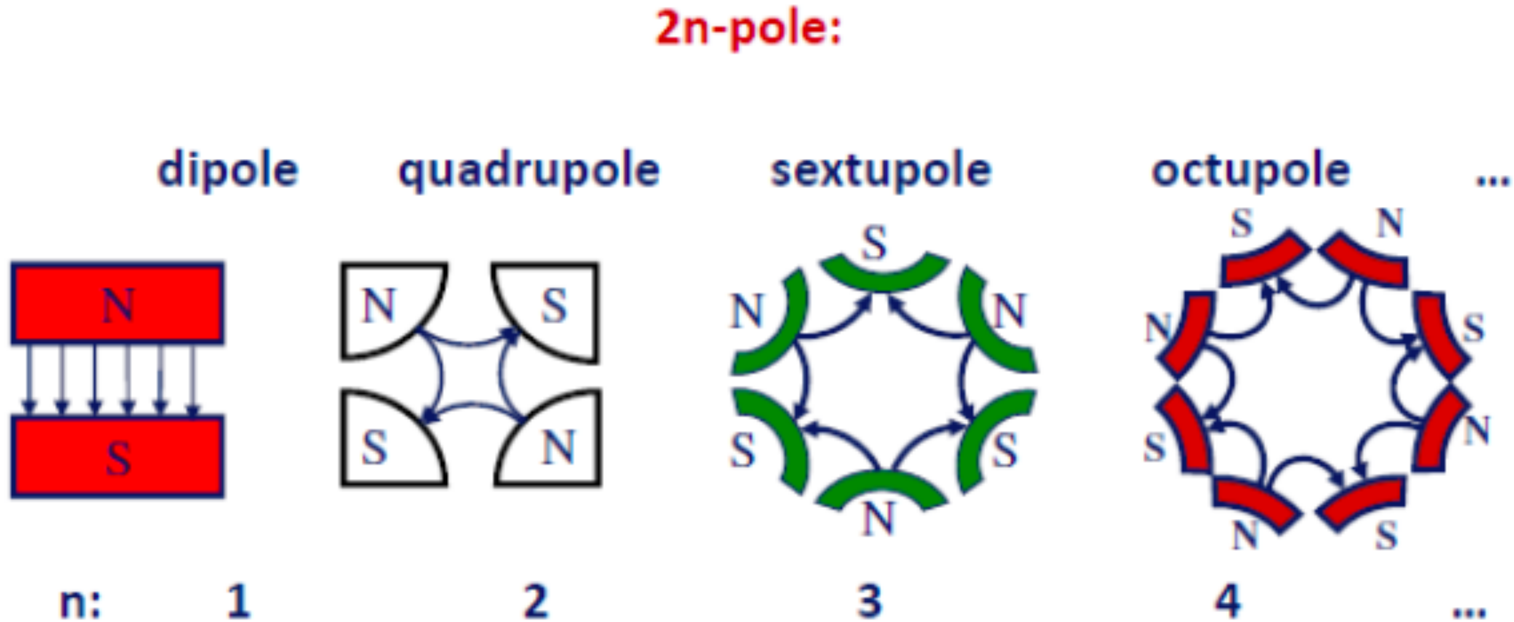
Main components of a storage ring

- Magnets with different characteristics are used to keep confined a beam of charged particles in a storage ring:
  - Dipoles: to guide the beam along a circular trajectory and to correct deviations from the ideal orbit
  - Quadrupoles: to focus the beam around the reference orbit and achieve small beam sizes at some positions
  - Sextupoles, octupoles, etc: magnets with non linear fields used to correct unwanted effects (chromaticity, etc…)
  - Wigglers and undulators: magnets with many poles with alternating polarity used to achieve synchrotron light beams with various wavelengths in the synchrotron light sources storage rings
- Charged particles bent on a circular trajectory in dipoles lose energy for synchrotron radiation. A Radio Frequency (RF) cavity, with a longitudinal electromagnetic field varying at high frequency, is used to restore the particle's energy

Main components of a storage ring (2)

- The beam travels in a vacuum chamber where a very low pressure is achieved by means of different pumping systems in order to minimize the interactions with the particles of the residual gas
- A cooling system is necessary for the magnets and RF
- A series of diagnostic systems is used to monitor the beam characteristics (current, beam position monitors, beam size monitor, luminosity monitor,…) and the accelerator performances
- An injector system is used to produce, accelerate and transport the beam inside the accelerator
- To inject the beam special pulsed magnets are used
- Collimators and masks are used to intercept the large amplitude particles and avoid damage of the accelerator systems and of the detector's components and performances
- A control system is managing the operation of the accelerator

Tipical magnetic fields



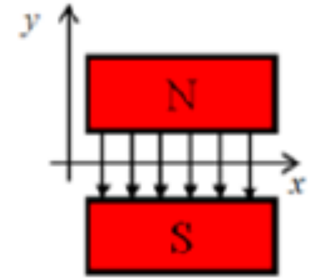
- Normal: gap appears in the horizontal plane
  - Skew: rotate around beam axis by $\pi/2n$

Properties and applications

Dipoles: used for guiding the particle trajectories

$B_x = 0$

$B_y = B_o = constant$

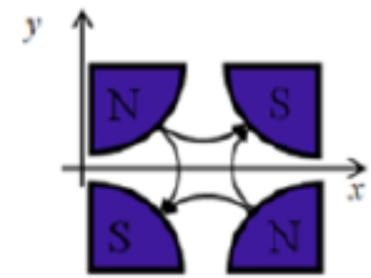


Quadrupoles: used to focus the particle trajectories

$B_x = G\,y$

$B_y = -G\,x$

$G = constant$

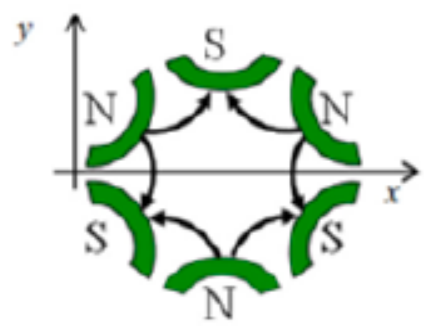


Sextupoles: used to correct chromatism and non linear terms

$B_x = 2\,S\,x\,y$

$B_y = S\,(x^2 - y^2)$

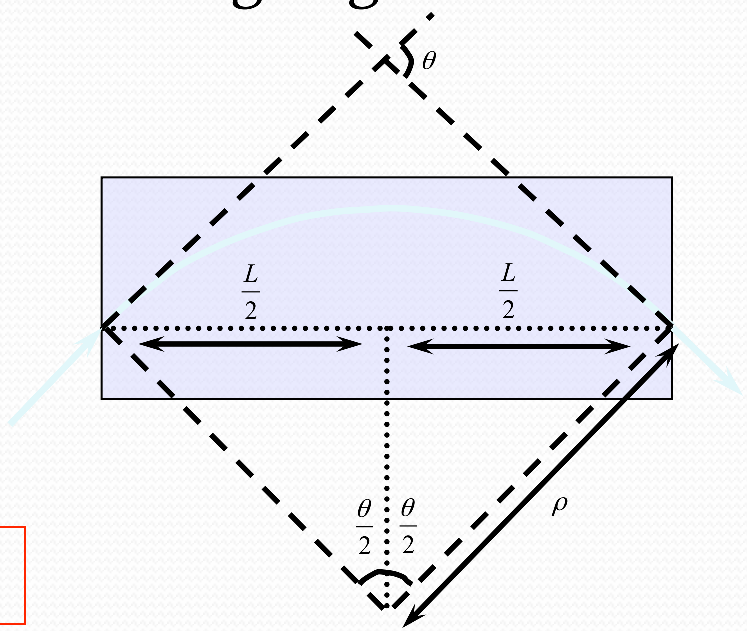
$S = constant$



Dipole magnet

- Magnet with 2 poles separated by a gap
- The dipole field is uniform and perpendicular to the orbit plane
- The particle is bent by an angle θ with a radius of curvature ρ
- Given the length L and the field B the bending angle is:

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \frac{L}{2\rho} = \frac{1}{2}\frac{LB}{(B\rho)}}$$

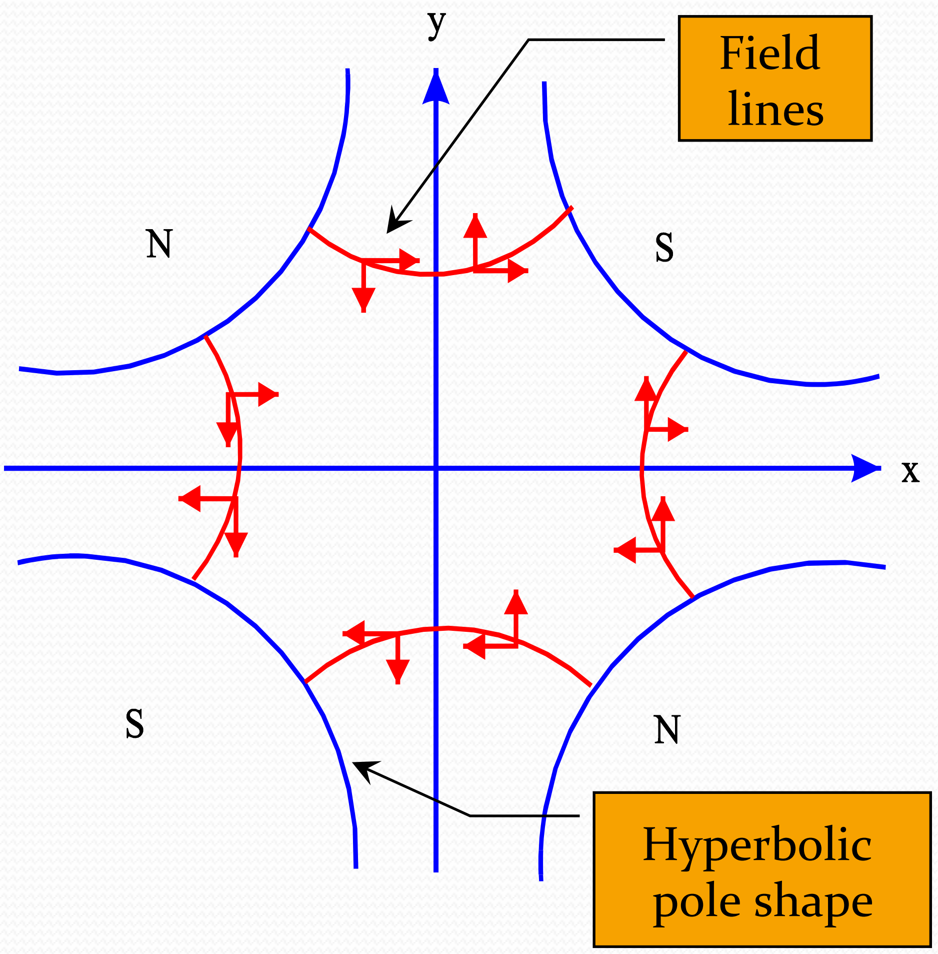


- For small θ : **θ=LB/(Bρ)**
- Bρ = p/e is the magnetic rigidity of a particle with charge *e*

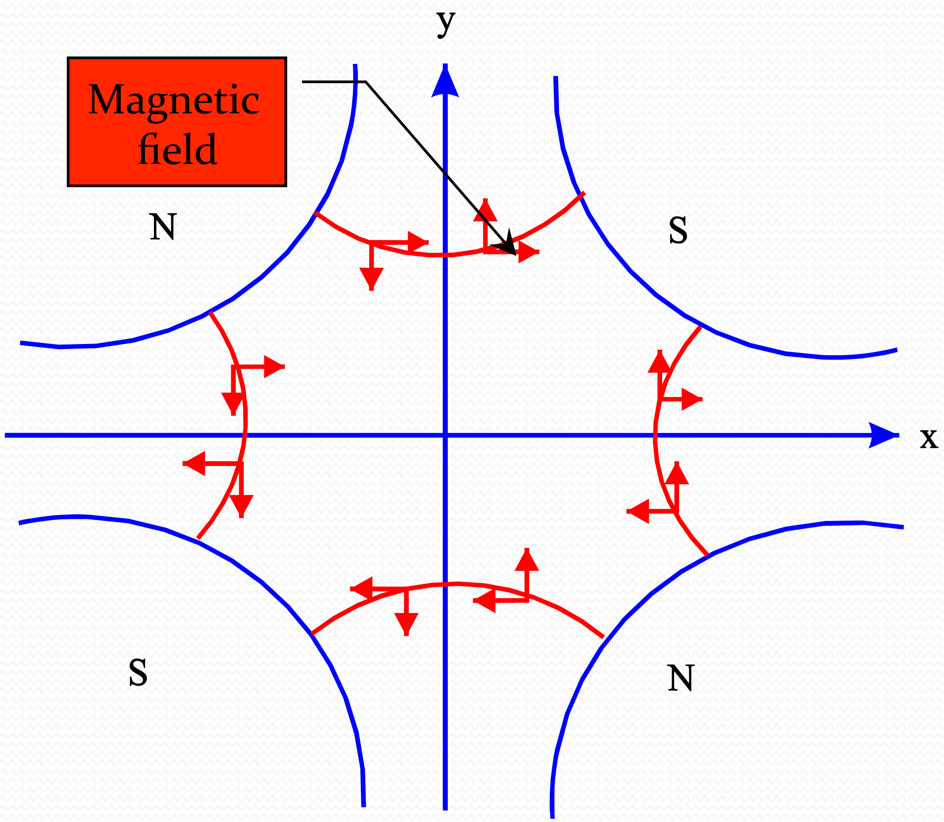
$$\boxed{\mathbf{B\rho\ [T{\cdot}m] = 3.3356 \cdot p\ [GeV/c]}}$$

Quadrupole magnet

- Magnet with 4 poles with hyperbolic contour
- Poles are symmetric with respect to x and y axes
- The field is zero in the center and varies linearly both in horizontal and vertical direction
- Depending on the field sign, a quadrupole is **focusing** in the horizontal plane (**QF**) or in the vertical one (**QD**), and **defocusing** in the other plane



Quadrupole field



QF: Focusing in x, defocusing in y

- On the X (horizontal) axis the field is vertical: $\boxed{B_y = G\,x}$
- On the Y (vertical) axis the field is horizontal: $\boxed{B_x = G\,y}$
- The gradient G is defined as:

$$\frac{dB_y}{dx} \quad [Tm^{-1}]$$

- The 'normalized' gradient K is:

$$K = \frac{G}{(B\rho)} \quad [m^{-2}]$$

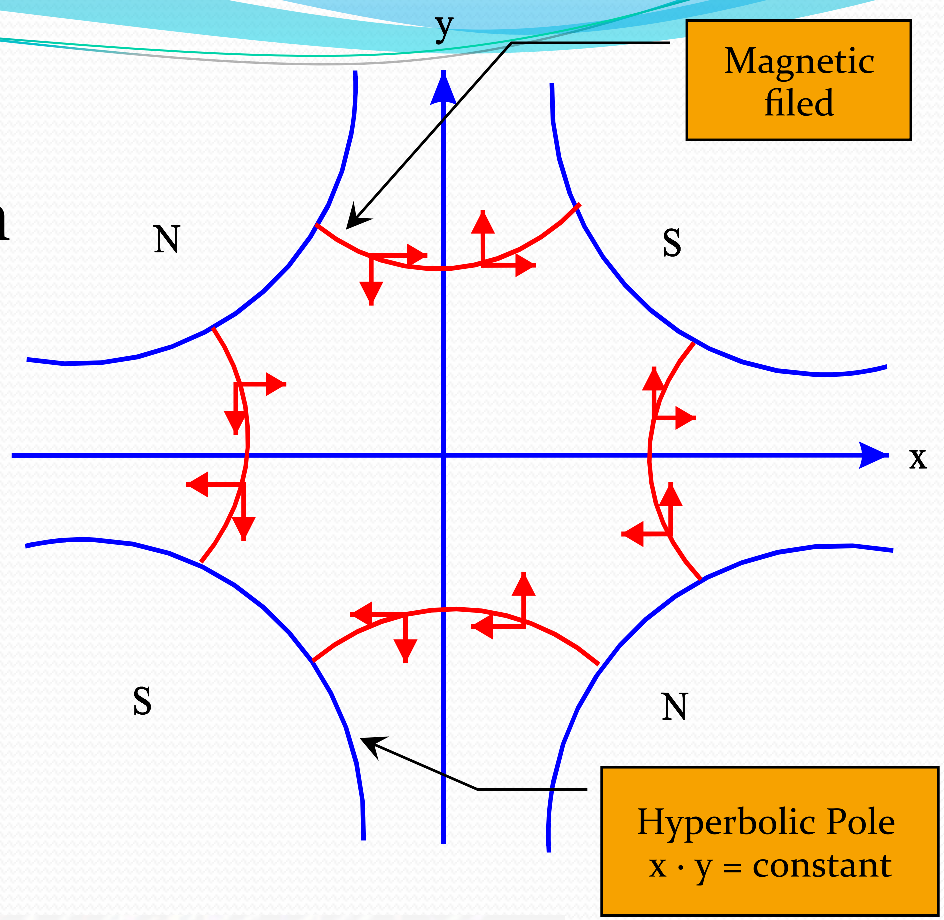
- The focal length is:

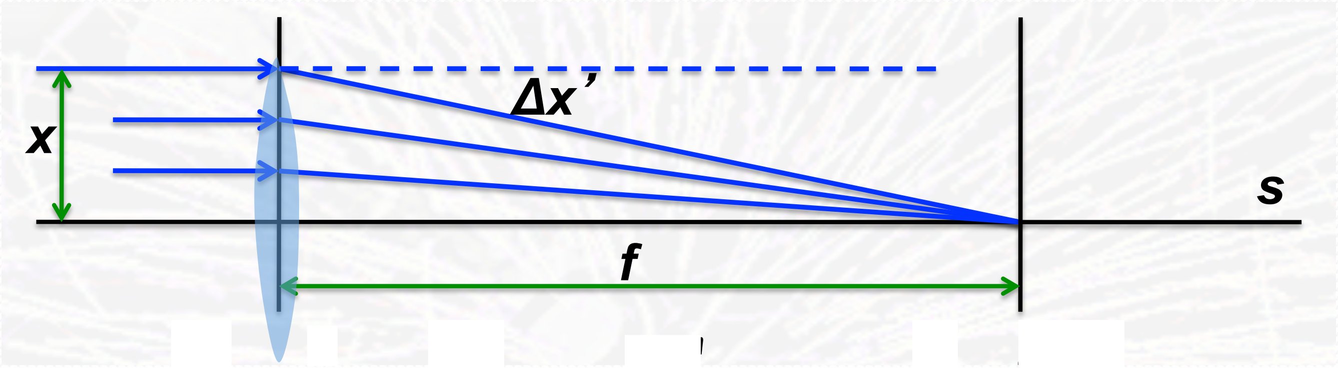
$$f = \frac{1}{KL_q} \quad [m]$$

Quadrupole kick

- The angular “kick” given to the particles is linear

$$\boxed{\begin{aligned}\Delta x' &= -Klx \\ \Delta y' &= Kly\end{aligned}}$$



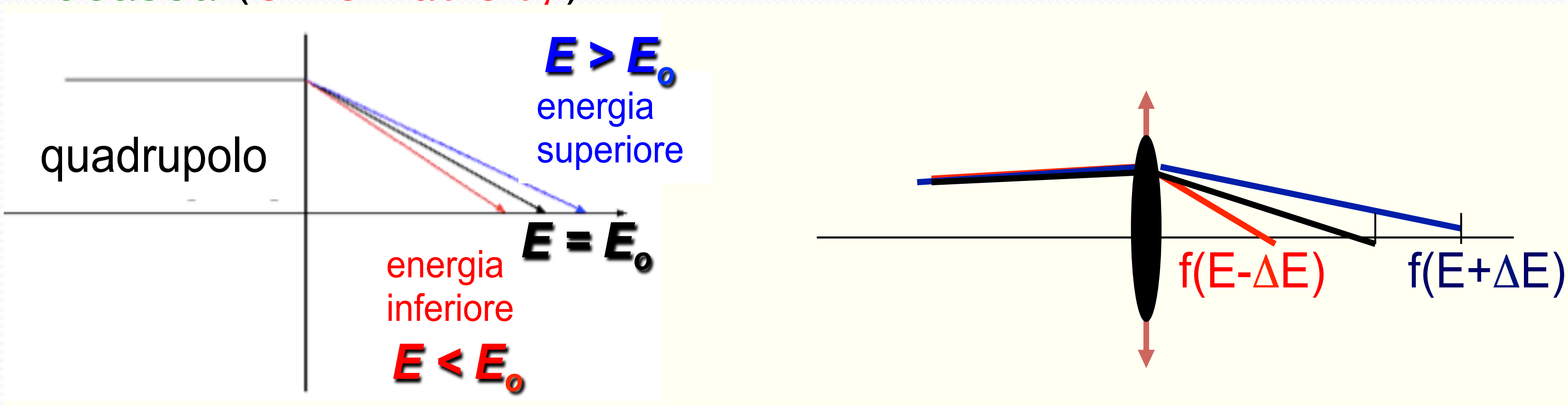


Chromatic effects

- A quadrupole acts as a focusing lens with focal length:

$$\boxed{f = \frac{1}{KL_q} = \frac{B\rho}{GL_q} = \frac{p}{e}\frac{1}{GL_q}}$$

- The focal length depends on the particle momentum
- Since the beam has an energy spread, the high energy particles will be under-focused and the low energy particles will be over-focused (chromaticity)

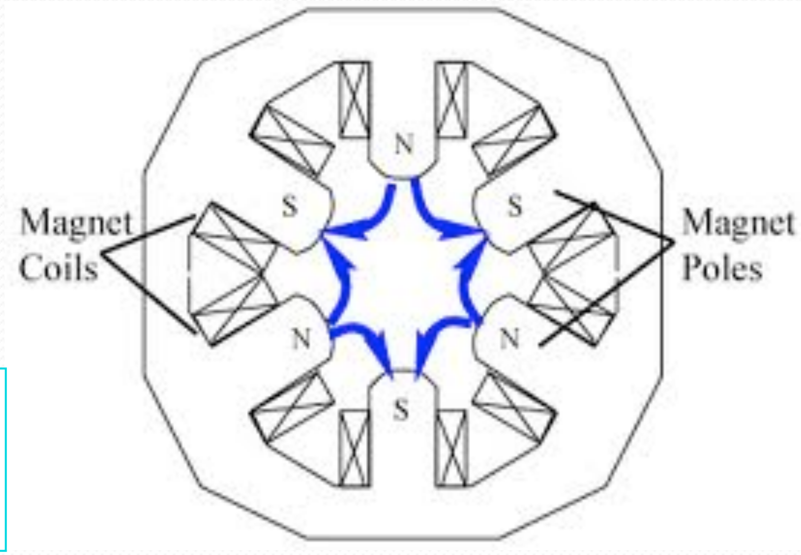


Solution: introduce sextupole magnets

Sextupole magnets

- Magnets with 6 poles with hyperbolic shape
- The field is zero at the center and varies quadratically with the transverse coordinate:

$$B_x = m\,xy$$
$$B_y = \frac{1}{2}m\left(x^2 - y^2\right)$$



- Normalized gradient:

$$m = \frac{1}{B\rho}\frac{d^2B_y}{dx^2} \quad [m^{-3}]$$

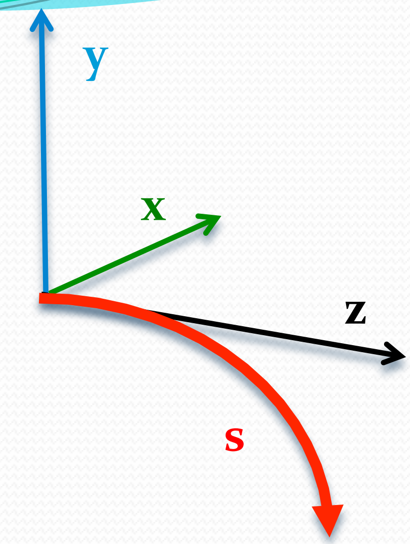
- Kick:

$$\Delta x' = -mx^2, \Delta y' = 2mxy$$

- A sextupole is like a quadrupole with a gradient proportional to the transverse diplacement

Transverse motion in a circular accelerator

Coordinate system



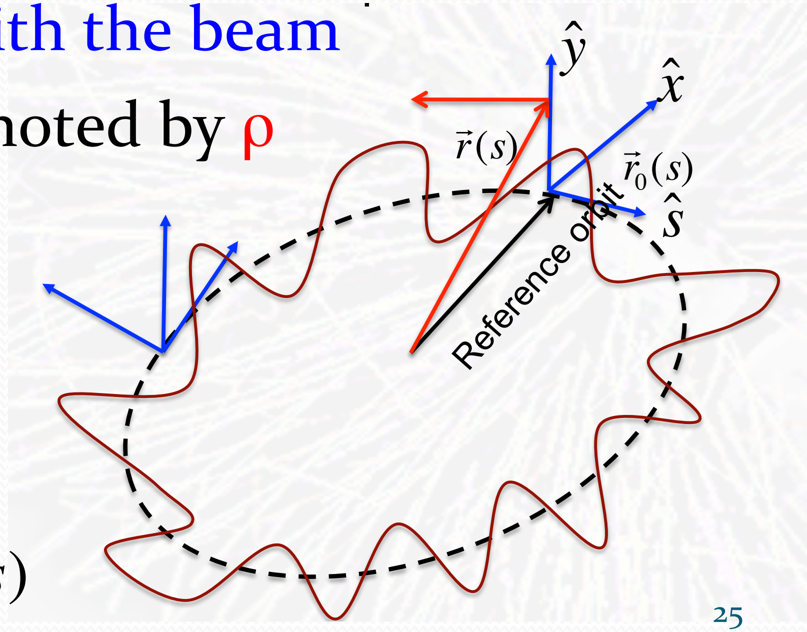
- For circular machines, it is convenient to convert to a curvilinear coordinate system (Frenet-Serret) and change the independent variable from time to the longitudinal abscissa "s", which is the reference orbit given by the bending magnets and is moving with the beam
- The local radius of curvature is denoted by ρ

The unit vectors $\hat{s},\hat{x},\hat{y}$ are the basis for the coordinate system

$$\hat{s}(s) = \frac{d\vec{r}_0(s)}{ds}$$

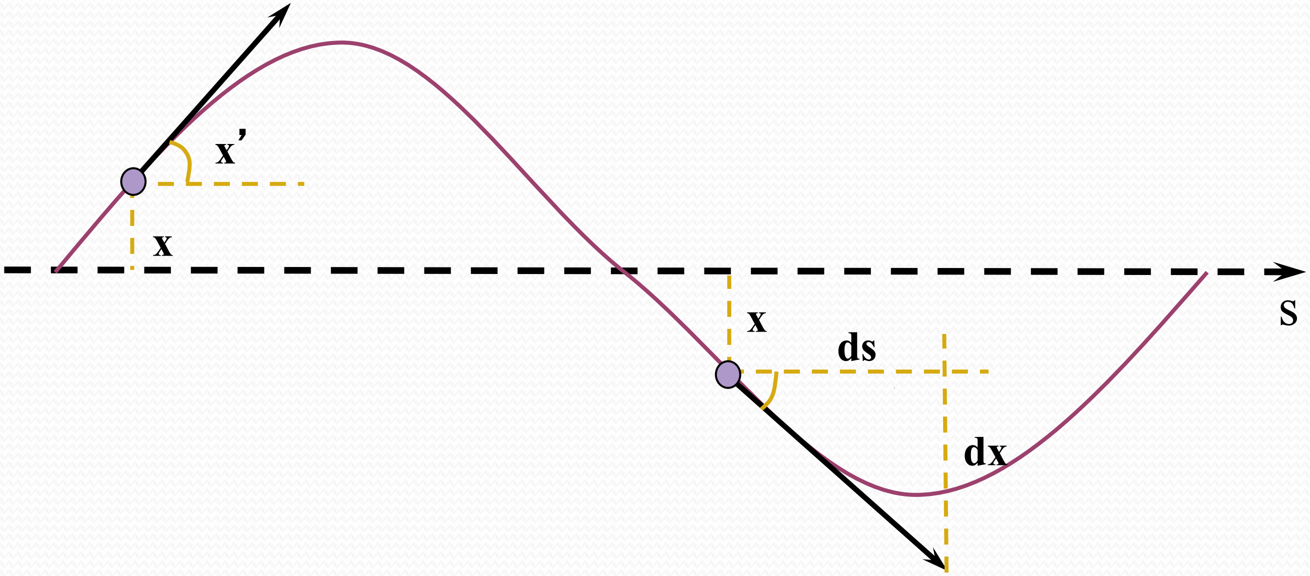
$$\hat{x}(s) = -\rho\frac{d\hat{s}(s)}{ds}$$

$$\hat{y}(s) = \hat{x}(s)\times\hat{s}(s)$$



Motion in a circular accelerator

- x and y are the particle coordinates representing *small amplitude motion* around the reference orbit
- In each plane (x,s) and (y,s) the motion of a particle in a transverse magnetic field is described by two variables:
  - Position *x(s)*, displacement perpendicular to the reference orbit
  - Angle *x'(s)= dx/ds* with respect to the reference orbit



- The motion is similar to that of an harmonic oscillator

Equations of motion

- Particle motion in electromagnetic fields is governed by the Lorentz force:

$$\boxed{\frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

$$\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t \qquad \vec{B} = \nabla \times \vec{A}$$

Φ = scalar potential
A = vector potential

- With the corresponding Hamiltonian in Cartesian coordinates:

$$\boxed{\begin{gathered}\mathcal{H} = c\left[m^2c^2 + \left(\vec{P} - e\vec{A}\right)^2\right]^{1/2} + e\Phi \\ \dot{x} = \frac{\partial\mathcal{H}}{\partial P_x}, \ \dot{P}_x = -\frac{\partial\mathcal{H}}{\partial x}, \ldots\end{gathered}}$$

Equations of motion (2)

- In absence of synchrotron motion, we can generate the equations of motion with:

$$\boxed{x' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{\mathcal{H}}}{\partial x}, \quad y' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_y}, \quad p_y' = -\frac{\partial \tilde{\mathcal{H}}}{\partial y}}$$

$$B_x = -\frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial y} \quad B_y = \frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial x}$$

- Which yields (top/bottom sign for +/- charge):

$$\boxed{x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2}$$

$$\boxed{y'' = \mp \frac{B_x}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2}$$

Equations of motion (Hill's Equation) (3)

- We next want to consider the equations of motion for a ring with only guiding (dipole) and focusing (quadrupole) elements:

$$\boxed{\begin{aligned} &x'' + K_x(s)\,x = 0, \qquad && K_x(s) = \frac{1}{\rho^2(s)} \mp k(s) \\ &y'' + K_y(s)\,y = 0, \qquad && K_y(s) = \pm k(s) \end{aligned}}$$

also commonly denoted as k_1

- K_x and K_y are periodic functions of s
- The period length L_p is the circumference, or a fraction of it in case the layout of dipole and quadrupoles in the accelerator has a periodic structure
- A horizontal bending dipole has $K_x = 1/\rho^2$ and $K_y = 0$
- In a quadrupole $1/\rho = 0$ and $K_x = -K_y$ is the focusing strength (a horizontally focusing quadrupole is vertically defocusing)

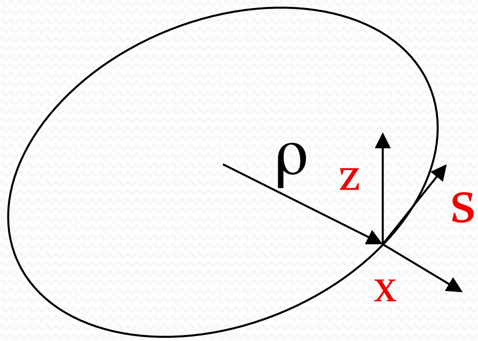
Betatron Motion

Reference trajectory

Mid-plane symmetry: magnetic field in the horizontal plane is perpendicular to the plane

$\rho(s)$ = local radius of curvature

K(s) = local focusing strength



Particles are kept on a nearly circular trajectory by bending and focusing magnetic fields

The reference trajectory is the equilibrium closed orbit for a particle of momentum p_o. It is a sequence of straight lines and circular arcs (in bending magnets)

Quadrupoles act as focusing systems which produce small *betatron oscillations* around the reference trajectory

General Solution to Hill's Equation

- The general solution to Hill's equation can be written:

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s)+\phi_0\right] \quad \text{where} \quad \psi_x(s) = \int_0^s \frac{ds}{\beta_x(s)}$$

with $\beta_x(s)$ a periodic function of s: $\beta_x(s+L_P) = \beta_x(s)$

- The linear "*betatron*" motion is like an harmonic oscillation with amplitude and phase varying along the ring as a function of *s*
- We can now define the *betatron tune* (number of betatron oscillations/turn) for a ring as:

$$Q_x = \nu_x = \frac{\Phi_{turn}}{2\pi} = \frac{1}{2\pi}\int_s^{s+C} \frac{ds}{\beta_x(s)}$$

C = ring circumference

Closed orbit (c.o.)

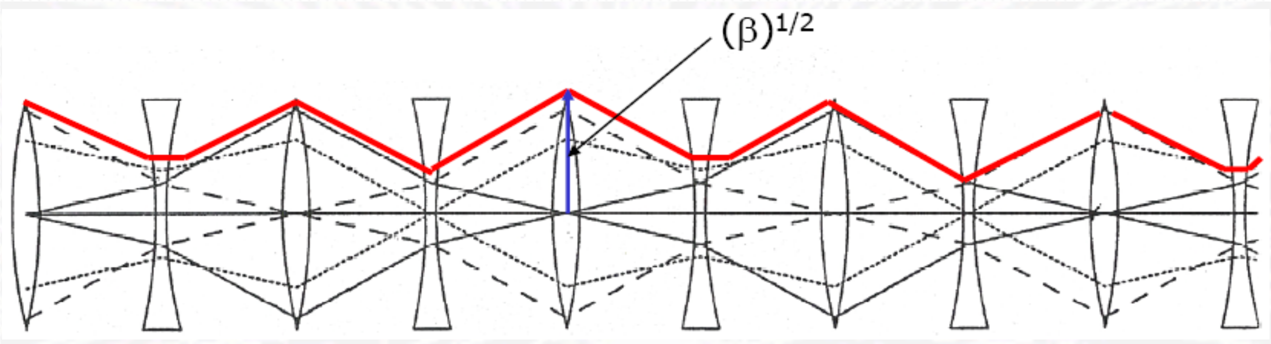
- The "closed orbit" is that trajectory which after one turn (C = circumference, M = one turn matrix) is closed on itself:

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = M(s+C,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Betatron oscillations

- The periodic function β(s) describes the envelope of the *betatron oscillations* that the particles perform with respect to the reference orbit given by guide field of the dipoles



- The oscillations are in both planes, x and y
- The number of betatron oscillations per turn, "betatron tune", or "phase advance", is an important ring parameter

$$\boxed{Q_x = \nu_x = \frac{\Psi_{turn}}{2\pi} = \frac{1}{2\pi}\int_s^{s+C}\frac{ds}{\beta_x(s)} = \frac{R}{\langle\beta_x\rangle}}$$

C = ring circumference
R = ring radius

Transfer matrices

Let $\mathbf{y}(s) = \begin{pmatrix} y(s) \\ y(s) \end{pmatrix}$ be the "position vector"

In any point along the accelerator we can define the particle's trajectory starting from its initial coordinates, through:

$$\mathbf{y}(s) = M(s|s_o)\,\mathbf{y}(s_o)$$

where $M(s|s_o)$ is the betatron transfer matrix

The passage through a magnetic element can be described by a 2x2 matrix, which transforms the "position vector" of a particle *before* the element to the position vector *after* it

Transfer Matrices

- We can write the solutions of the Hill's equations in transfer matrix form for each component with constant B field:

$$\mathbf{M}(s|s_0) = \begin{cases} \begin{pmatrix} \cos\left(\sqrt{k}\,\mathrm{l}\right) & \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,\mathrm{l}\right) \\ -\sqrt{k}\sin\left(\sqrt{k}\,\mathrm{l}\right) & \cos\left(\sqrt{k}\,\mathrm{l}\right) \end{pmatrix} & \text{Focusing Quadrupole } (k > 0) \\ \begin{pmatrix} 1 & \mathrm{l} \\ 0 & 1 \end{pmatrix} & \text{Drift Region } (k = 0) \\ \begin{pmatrix} \cosh\left(\sqrt{|k|}\,\mathrm{l}\right) & \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}\,\mathrm{l}\right) \\ \sqrt{|k|}\sinh\left(\sqrt{|k|}\,\mathrm{l}\right) & \cosh\left(\sqrt{|k|}\,\mathrm{l}\right) \end{pmatrix} & \text{Defocusing Quadrupole } (k < 0) \end{cases}$$

where $\boxed{\mathrm{l} = s - s_0.}$ $\boxed{k = \frac{1}{B\rho}\frac{\partial B}{\partial x}}$

All Matrices have Det = 1

Examples of transfer matrices

All Matrices have Det = 1

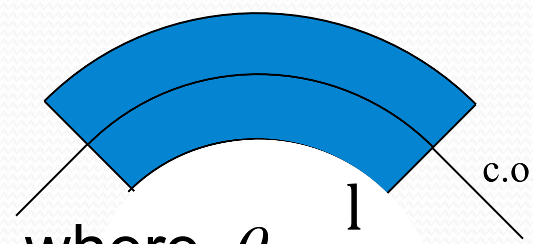
- Quadrupole in thin lens approximation:

$$l \to 0, \qquad f = \lim_{l \to 0} \frac{1}{|K|\, l}$$

$$\mathbf{M}_{\text{focusing}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \qquad \mathbf{M}_{\text{defocusing}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

- Sector dipole (entrance and exit faces ⊥ to closed orbit):

$$\mathbf{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & l \\ -\frac{l}{\rho^2} & 1 \end{pmatrix}$$



where $\theta = \dfrac{l}{\rho}$

One turn matrix

- A ring is a sequence of N elements, and it can be represented by the product of the matrices of each element

$$M = M_N \cdot M_{N-1} \cdot \ldots M_3 \cdot M_2 \cdot M_1$$

- The determinant of the matrices is: $\det(M_i)=1$
- Periodicity condition: $M(s_1+L|s_1) = M(s_1)$
- Map of m turns: $M(s_1)^m$
- Matrices are an easy tool to calculate beam trajectories and properties in an accelerator of no matter how complicated layout

One turn matrix & Twiss Parameters

- The generalized one turn matrix can be written as:

$$\boxed{\mathbf{M} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = \mathbf{I}\cos\Phi + \mathbf{J}\sin\Phi}$$

- This is the most general form of the matrix. α, β, and γ are known as either the Courant-Snyder or **Twiss parameters** (*note: they have nothing to do with the familiar relativistic parameters*) and Φ is the betatron phase advance. The matrix **J** has the properties:

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \qquad \mathbf{J}^2 = -\mathbf{I} \Leftrightarrow \beta\gamma = 1 + \alpha^2$$

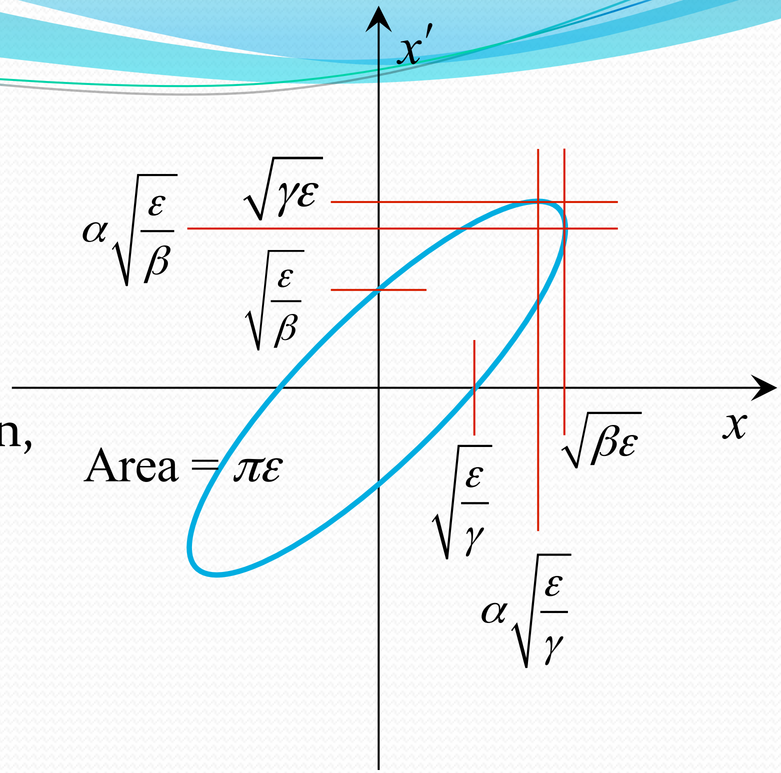
- The n-turn matrix can be expressed as: $\boxed{\mathbf{M}^n = \mathbf{I}\cos(n\Phi) + \mathbf{J}\sin(n\Phi)}$

which leads to the stability requirement for betatron motion:

$$\boxed{\left|\mathrm{Trace}(\mathbf{M})\right| = 2\cos\Phi \le 2}$$

Twiss Functions

$\phi(s)$ betatron phase advance
$\alpha(s)$, $\beta(s)$, $\gamma(s)$ Twiss functions



Since there are no friction terms in Hill's equation, the "energy" of the betatron oscillations is conserved during the motion.
It is important to stress that the Twiss functions (also called optical functions) are periodic $\beta(s+C)=\beta(s)$, their value depends only on the coordinate s along the ring, while the coordinates y(s) and y'(s) do not repeat after one revolution

The trajectory of a particle follows an ellipse described by the Courant-Snyder invariant

$$\boxed{\varepsilon = \frac{1}{\beta}\left[y^2 + (\alpha y + \beta y')^2\right] = \gamma y^2 + 2\alpha y y' + \beta y'^2}$$

The area of the ellipse is **$\pi\varepsilon$** , and it is constant along the ring

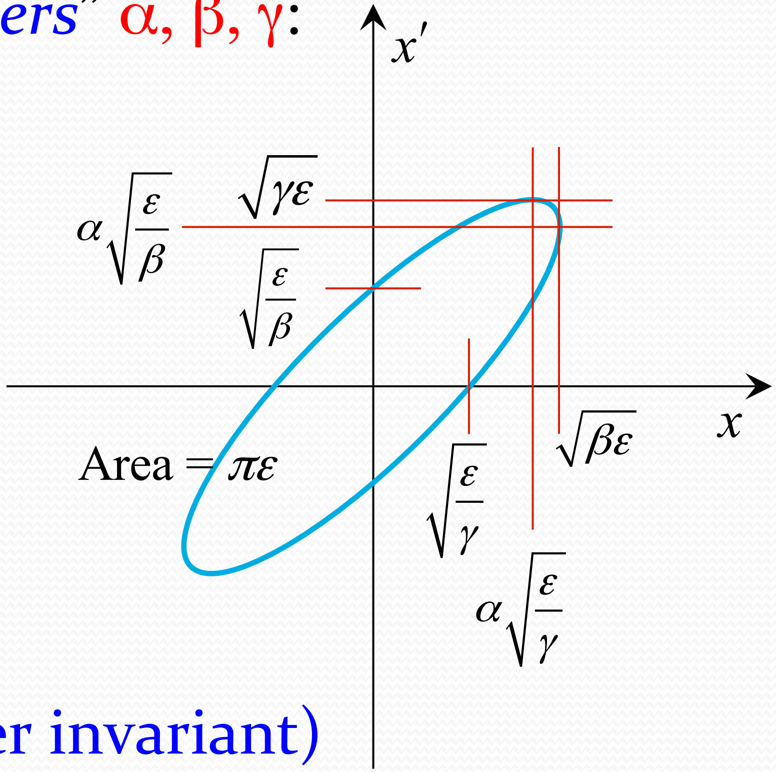
Phase space

- Phase Space is defined by the particles coordinates (*x,x'*) o (*y,y'*)
- In this space the betatron oscillation projects an ellipse defined by 3 quantities called "*Twiss parameters*" α, β, γ:

$$\boxed{\beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x' = \varepsilon}$$

where:

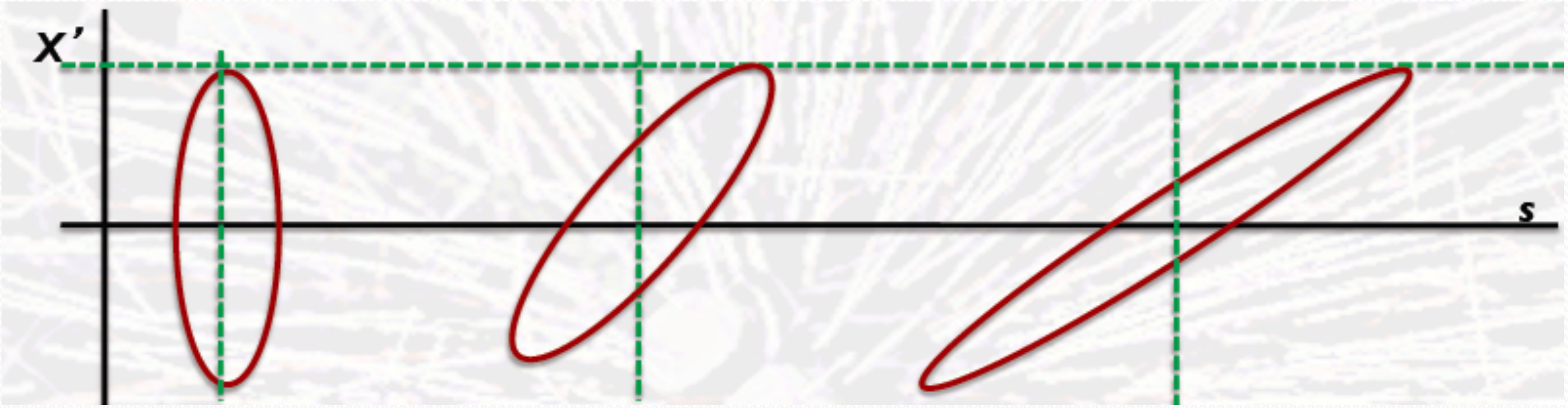
$$\boxed{\begin{aligned} \alpha_x &= -\frac{1}{2}\beta'_x \\ \beta_x\gamma_x &= 1+\alpha_x^2 \end{aligned}}$$



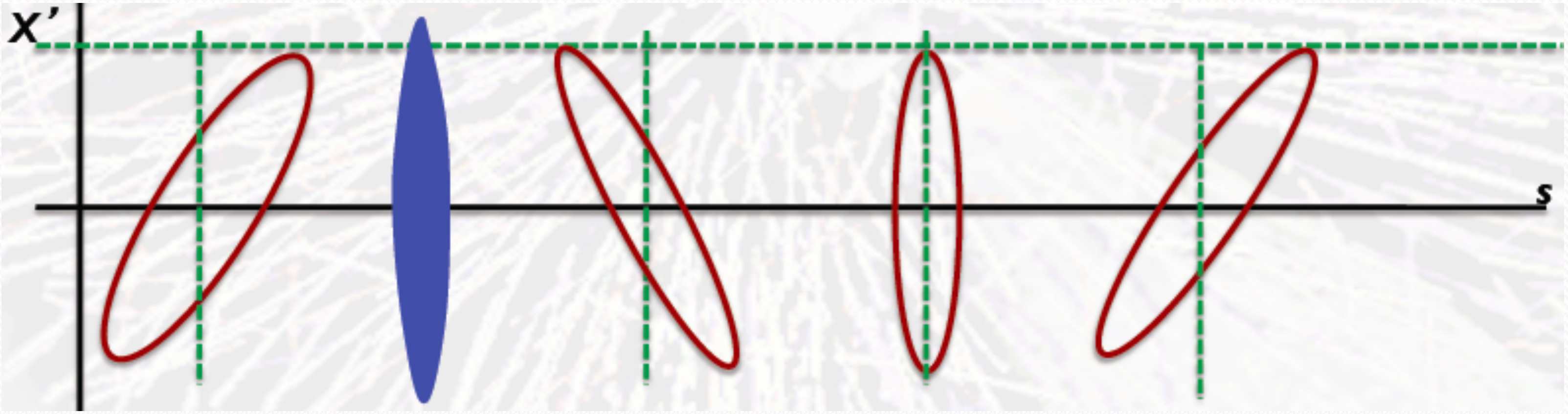
- The ellipse area is πε (Courant-Snyder invariant)
- Each turn each particle will cover different positions along the ring

Trasformations in phase space

- In a drift space (no fields) the particle's ellipse rotates as:



- Effect of a focusing quadrupole:



Beam emittance

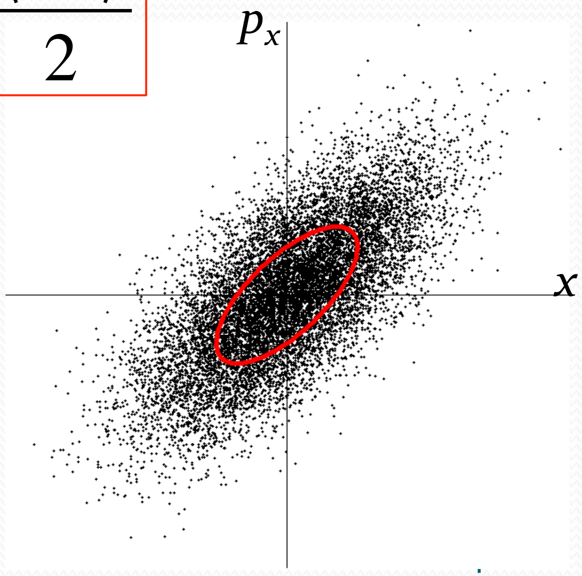
- The equation $$\boxed{\beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x' = \varepsilon}$$
  describes an ellipse of area $\pi\varepsilon$
- For an ensemble of particles, each describing its ellipse, we can define the beam rms emittance as the area enclosed by the ellipse of an *rms* particle:

$$\boxed{\varepsilon_x^{rms} = \sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} = \frac{\langle A^2\rangle}{2}}$$

x = horizontal, vertical coordinate,

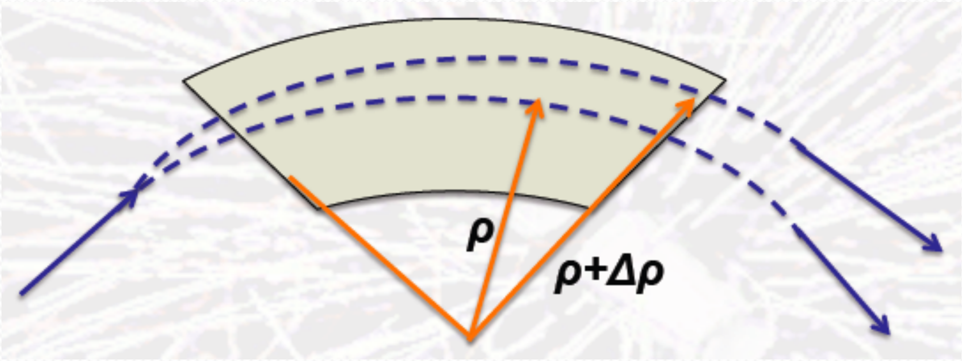
p_x = horizontal momentum normalized to the reference momentum $P_o \approx E/c$,

σ = beam sizes



Dispersion function

- In all magnets the transverse trajectory is a function of the particle momentum



$\Delta\theta = \theta\,\Delta p/p$

momentum spread

- The Dispersion D defines the trajectory of a particle whose energy differs by $\Delta p/p$ from the nominal one

$$\boxed{x_D(s) = D(s)\frac{\Delta p}{p}}$$

(x, D in [m])

- H-Dipoles generate D_x, V-dipoles (usually missing in rings, except for particular cases) generate D_y

Dispersion function (2)

A particle with a momentum deviation $\Delta p = p - p_0$ satisfies the equation:

$$\boxed{x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}}$$

The total deviation of the particle from the reference orbit is:

$$x(s) = x_\beta(s) + D(s)\Delta p/p_o$$

The particle with momentum deviation $\Delta p/p_o$ performs a betatron oscillation x_β with respect to an equilibrium orbit $D(s)\ \Delta p/p_o$

The dispersion function $D(s)$ satisfies the equation:

$$\boxed{D'' + K(s)D = \frac{1}{\rho}}$$

With periodic boundary conditions

$$D(s+L) = D(s)\ \ ;\ \ D'(s+L) = D'(s)$$

Momentum Compaction

The total path length for an off-momentum particle differs from that for the on-momentum closed orbit by:

$$\boxed{\Delta C = \oint \frac{x}{\rho} ds = \left[\oint \frac{D(s)}{\rho} ds \right] \frac{\Delta p}{p}}$$

The momentum compaction factor α_c takes into account this effect and is defined by

$$\boxed{\alpha_c = \frac{\Delta C / C}{\Delta p / p} = \frac{1}{C} \oint \frac{D(s)}{\rho} ds}$$

Integral performed in dipoles only

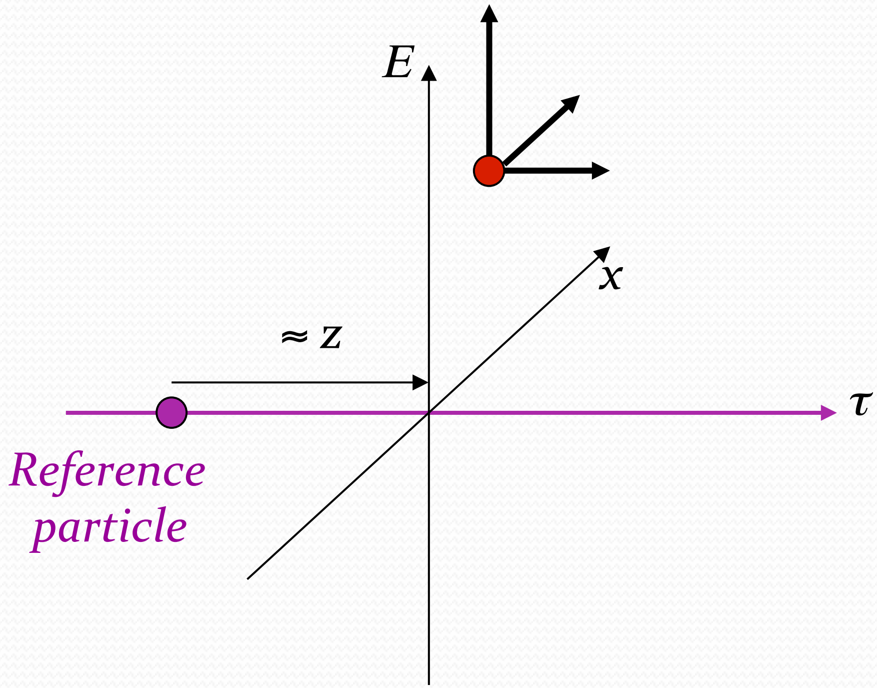
The α_c parameter governs the longitudinal motion in storage rings

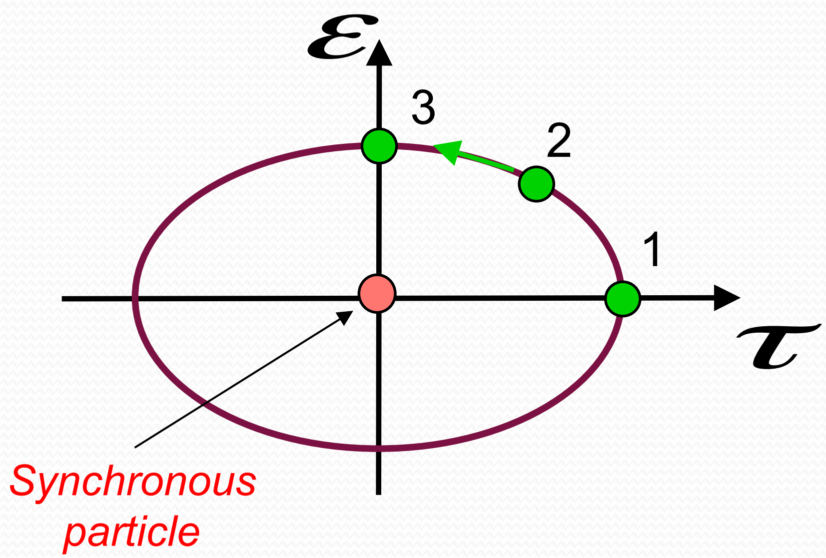
Longitudinal motion

- Only particles with the "nominal energy" (*synchronous*) whose orbit is exactly in the center of the magnets have a perfectly uniform motion
- Other beam particles (the most) have deviations from the nominal energy and do not follow this "ideal orbit", subject to forces producing an harmonic motion around the ideal orbit and the ideal energy → transverse oscillations and longitudinal oscillations

Longitudinal coordinate system

- Reference particle travels along reference (ideal) orbit with momentum P_0 and speed $\beta_0 c$
- If a particle arrives at time τ *before* reference particle then its longitudinal coordinate will be: $z = c\tau$





Synchrotron Radiation

- When a particle is bent in a dipole emits photons: Synchrotron Radiation (SR)
- Emission of SR exerts a strong influence on electron beam dynamics in storage rings
- Emission of SR leads to damping of synchrotron and betatron oscillations and determines the beam sizes
- These effects strongly affect the design of electron machines, while are negligible for proton machines
- The energy lost by SR at each turn (Uo) has to be replaced by means of the electric field in the Radio Frequency (RF) cavities
- The synchronous particle arrives at the RF cavity at time to so that the energy gained is equal to the energy Uo lost per turn by SR
- The arrival time for an off-momentum particle is given by the momentum compaction α_c
- Due to this spread in time and energy the beam will perform energy oscillations (synchrotron oscillations)

Synchrotron Radiation (e^+/e^-) (2)

- The instantaneous power radiated by a relativistic electron of energy E in a magnetic field B with curvature radius ρ is

$$P_\gamma = \frac{c\,C_\gamma E^4}{2\pi\rho} = \frac{e^2 c^3}{2\pi} C_\gamma E^2 B^2 \qquad (C_\gamma = 8.85x10^{-5}\ m/GeV^3)$$

- Integrating on one turn we get the energy loss/turn U_o

$$U_0 = \frac{C_\gamma E^4}{2\pi} \oint \frac{ds}{\rho^2} = \frac{C_\gamma E^4}{2\pi} I_2$$

- For a ring with uniform magnetic field (*iso-magnetic*)

$$\boxed{U_0[MeV] = 8.85x10^{-2} \frac{E^4[GeV]}{\rho[m]}}$$

- If this energy is not restored by the RF cavity particles will spiralize toward the inside and are lost on the beam pipe

Energy loss/turn

e^+e^- Colliders

	E (GeV)	ρ (m)	L (m)	T_o (μs)	U_o (dip)* (MeV)
ADONE	1.5	5	105	0.35	0.09
DAFNE	0.51	1.4	98	0.31	0.004
PEP-II LER	3.1	30.5	2200	13.6	0.27
PEP-II HER	9	165	2200	13.6	3.52
LEP	100	3100	27000	89	2855

Proton Collider

	E (GeV)	ρ (m)	L (m)	T_0 (s)	$U_{0,dip}$ (MeV)
LHC	7700	2568	$3\ 10^4$	89	.011

- dip = radiation from dipoles, excluding contribution of wigglers