Time function: Classical and Quantum

Some ideas presented by:

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In collaboration with:

Paolo Aniello

Fabio Di Cosmo

Giuseppe Marmo

Juan Manuel Pérez-Pardo

Time is money, but Space is a very long Time

Time function: Classical and Quantum

The problem of Time in Quantum Mechanics

Is Time a quantum observable (self-adjoint operator)? →

- *Time of arrival;*
- *Time of occurrence;*
- *Tunneling Time.*

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$$[\mathbf{H}\,;\mathbf{T}] = -\imath\,\mathbb{I}$$

Pauli's theorem

- *Counterexamples;*
- *Maximally symmetric Time operators;*
- *Time POVMs.*

Time function: Classical and Quantum

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- *Counterexamples;*
- *Maximally symmetric Time operators;*
- *Time POVMs.*

Time in Q.M. is dynamical.

Self-adjoint operators are not enough.

Our proposal:

A Time function to define simultaneity.

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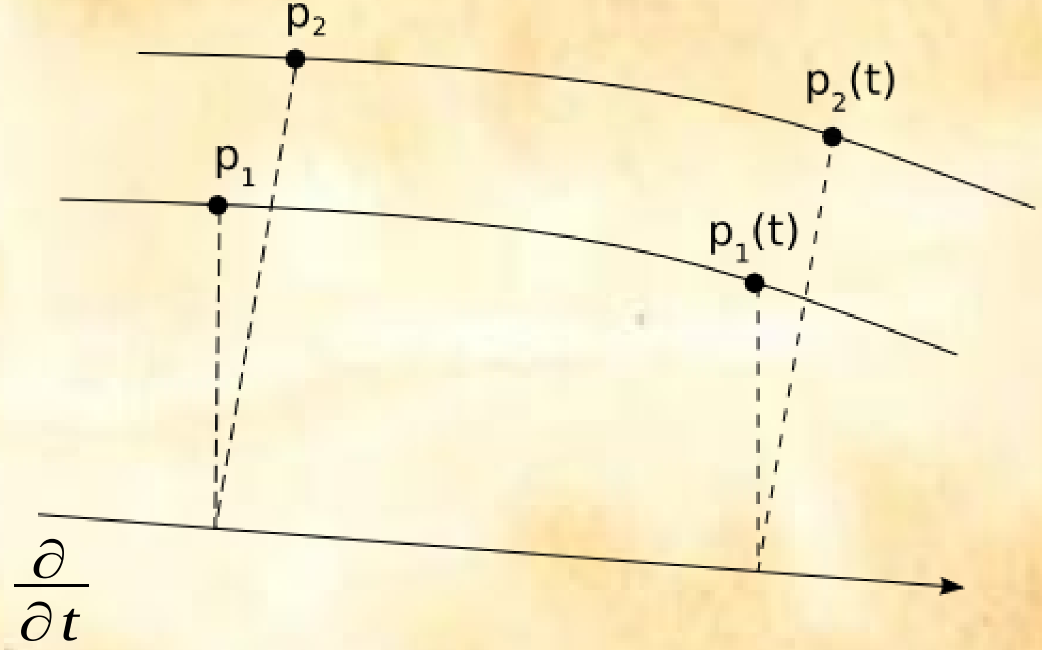
Time function: Classical and Quantum

Twofold role of Time

- Evolution parameter:

  Causality of physical phenomena



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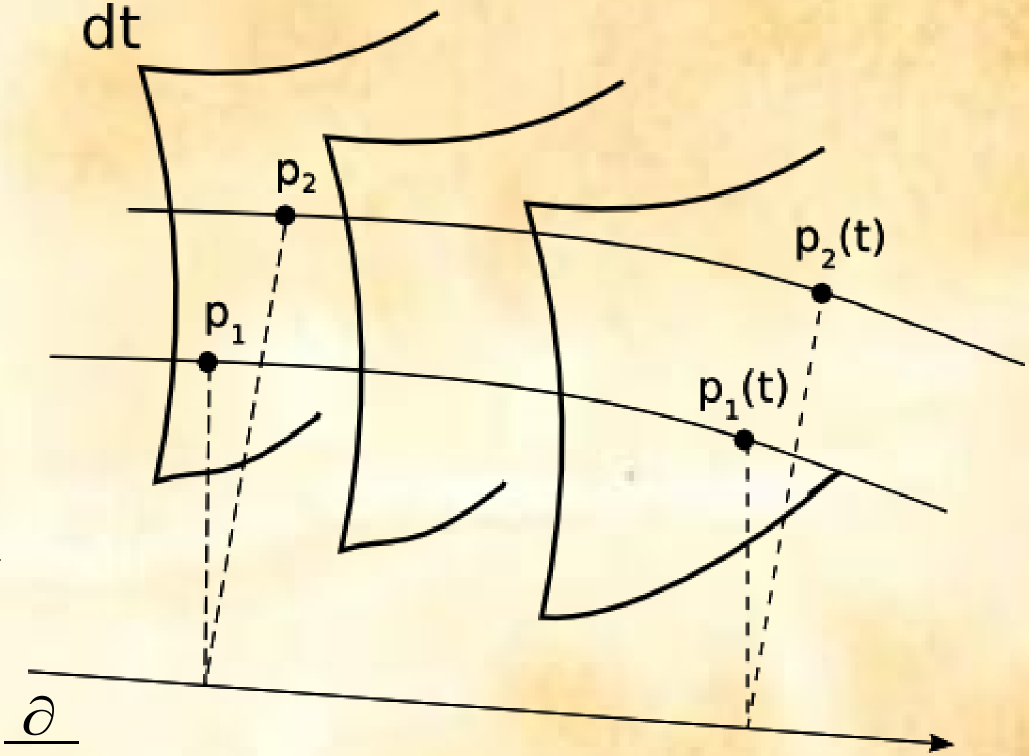
Twofold role of Time

- *Evolution parameter:*
  Causality of physical phenomena
- *Simultaneity relation:*
  Mutual relation between different events



$$\frac{\partial}{\partial t}$$

Compatibility between causality and simultaneity:

$$dt\left(\frac{\partial}{\partial t}\right) = 1$$

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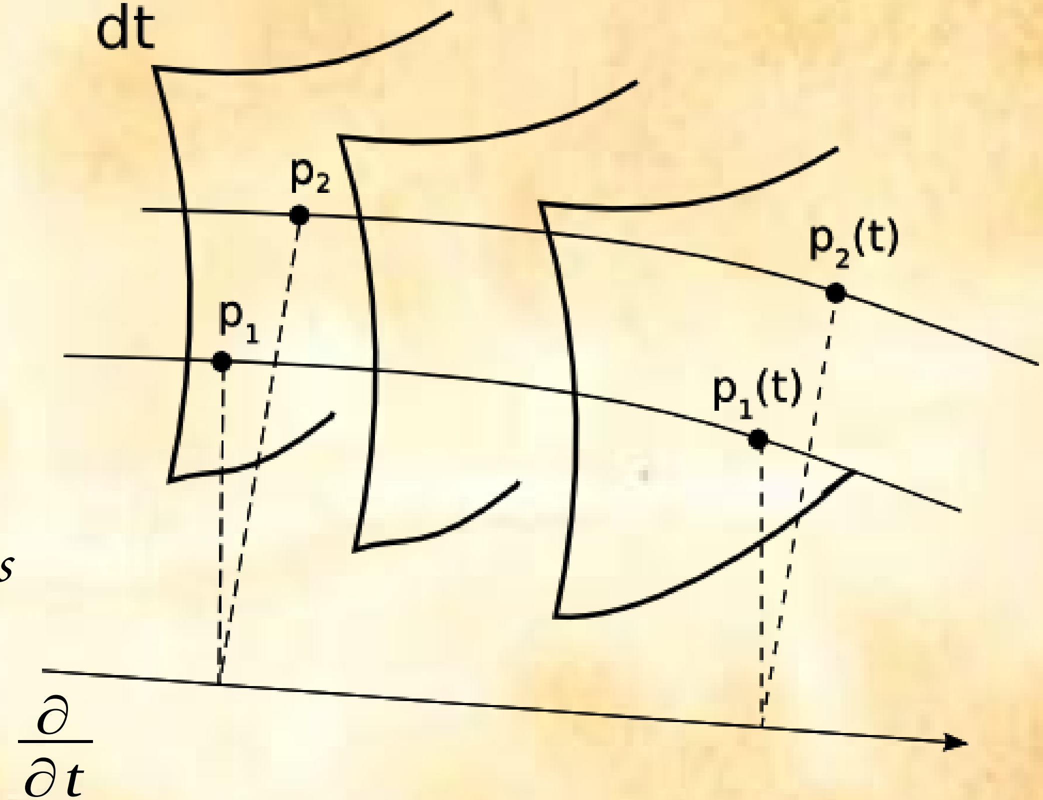
Twofold role of Time



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From a Kinematical setting to a Dynamical one.

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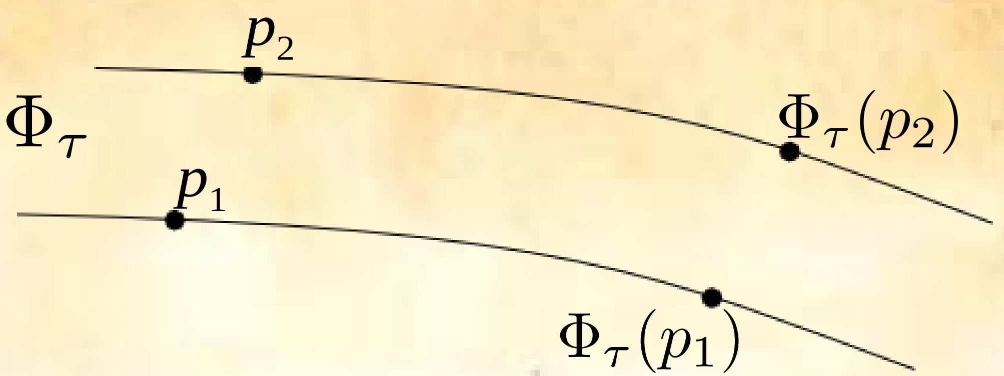
Time function: Classical and Quantum

The relevant dynamical objects are:

Simultaneity???

- *The space of states of the system;*
- *The dynamical evolution Φ_τ of the system.*



Time function: Classical and Quantum

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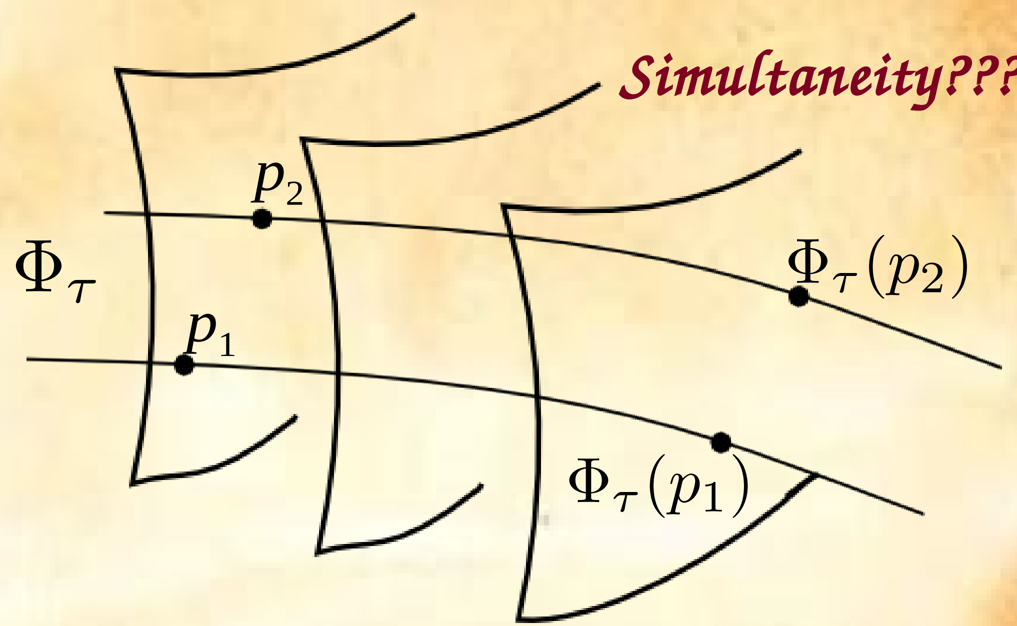

Time function:

- $T(p) \neq T(\Phi_\tau(p)) \quad \forall \tau \neq 0$
- $T(p_1) = T(p_2) \implies T(\Phi_\tau(p_1)) = T(\Phi_\tau(p_2))$

The level sets define the simultaneous states.

Time function: Classical and Quantum

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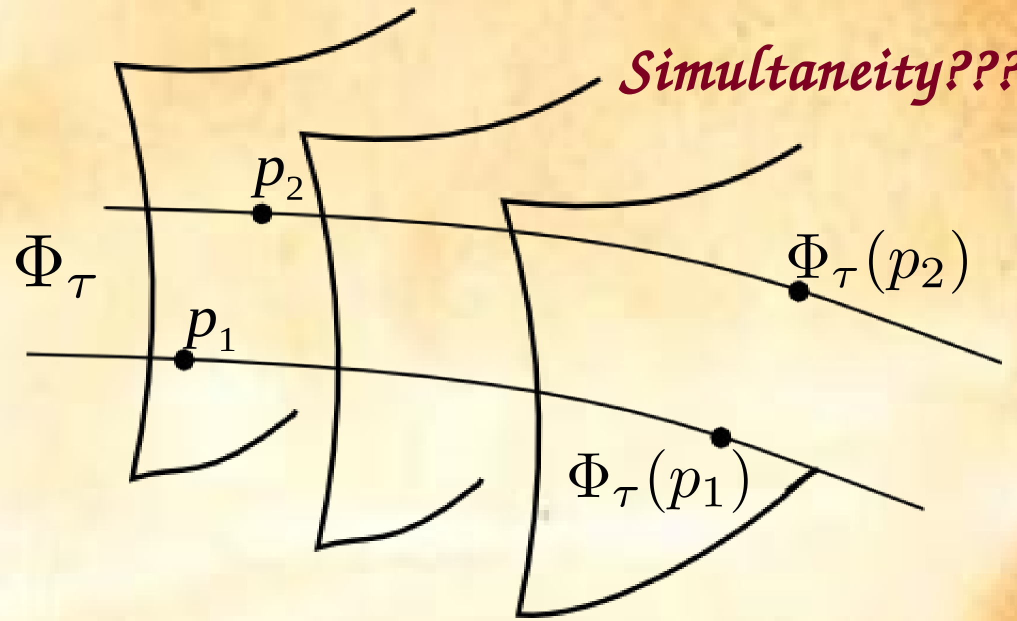

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The level sets define the simultaneous states.

- *The Time function is a dynamical object.*
  - *It can be defined for finite-level quantum systems.*
    - ***It is not associated to an operator.***

More or less like Entropy, Purity, Entanglement measures, ecc.

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Time function: Classical and Quantum

Classical Hamiltonian Mechanics:

- *The space of States is a symplectic manifold* $(\mathcal{P}\,;\omega)$*;*
- *The observables are smooth functions:*
- *The dynamical evolution* Φ_τ *is the flow of a vector field* Γ *associated to a Hamiltonian function* H *:*

$$i_\Gamma \omega = dH$$

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Time function: Classical and Quantum

3-D free point particle:

$$\mathcal{P} = T^*\mathbb{R}^3 \cong \mathbb{R}^6 \qquad \omega = \sum_{j=1}^{3} dp_j \wedge dq_j$$

$$H = \sum_{j=1}^{3} \frac{(p_j)^2}{2} \qquad \Gamma = \sum_{j=1}^{3} p_j \frac{\partial}{\partial q_j}$$

Dynamical trajectories:

$$\Phi_\tau(\vec{q};\vec{p}) = \begin{cases} p_j(\tau) = p_j \\ q_j(\tau) = p_j\tau + q_j \end{cases}$$

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The Time function is not defined on the fixed points of the dynamics (particle at rest).

Time function:

$$T(\vec{q};\vec{p}) = \frac{\vec{p}\cdot\vec{q}}{p^2} \implies T\circ\Phi_\tau(\vec{q};\vec{p}) = \tau + \frac{\vec{p}\cdot\vec{q}_0}{p^2}$$

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Time function: Classical and Quantum

1-D harmonic oscillator:

$$\mathcal{P} = T^*\mathbb{R} \cong \mathbb{R}^2 \qquad \omega = dp \wedge dq$$

$$H = \frac{p^2 + q^2}{2} \qquad \Gamma = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$$

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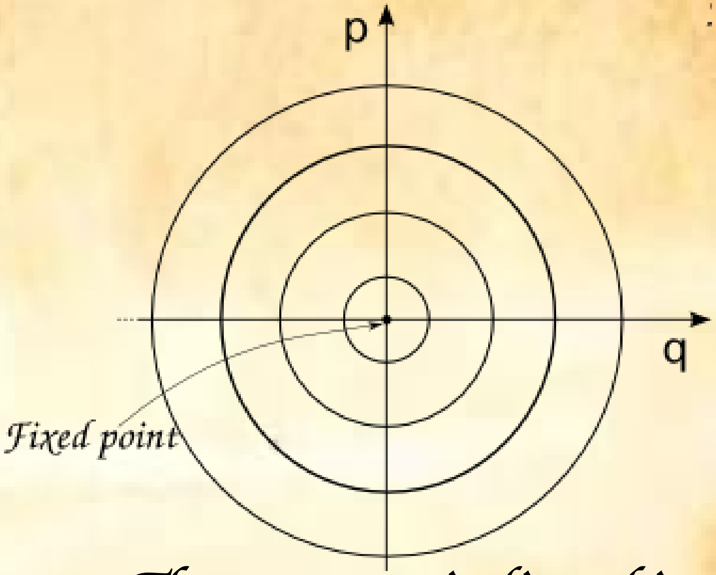

There are periodic orbits.

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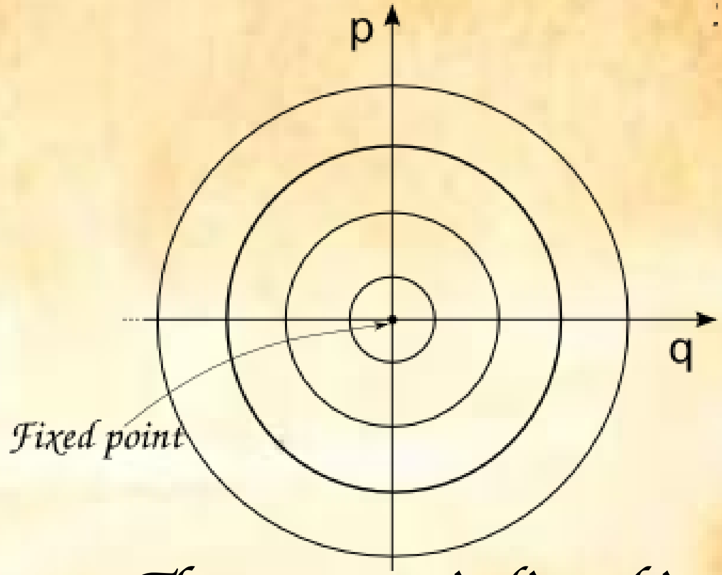

There are periodic orbits.

In presence of periodic orbits the simultaneity relation becomes periodic too.

Periodic Time function *(with period τ_T)*

$$T\left(\Phi_\tau\left(p\right)\right) = T\left(\Phi_{\tau + k\tau_T}\left(p\right)\right) \ \forall k \in \mathbb{Z}$$

$$T\left(p_1\right) = T\left(p_2\right) \implies T\left(\Phi_\tau\left(p_1\right)\right) = T\left(\Phi_\tau\left(p_2\right)\right)$$

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Time function: Classical and Quantum

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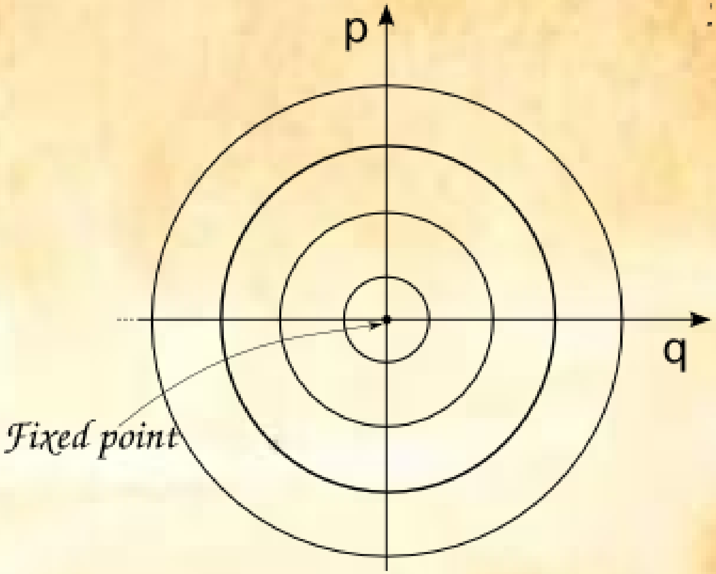

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The reduced space of states is diffeomorphic to the cilinder:

$$\Psi : \mathcal{P}_* \cong \mathbb{R}^2 - \{(0,0)\} \to S^1 \times \mathbb{R}^+$$

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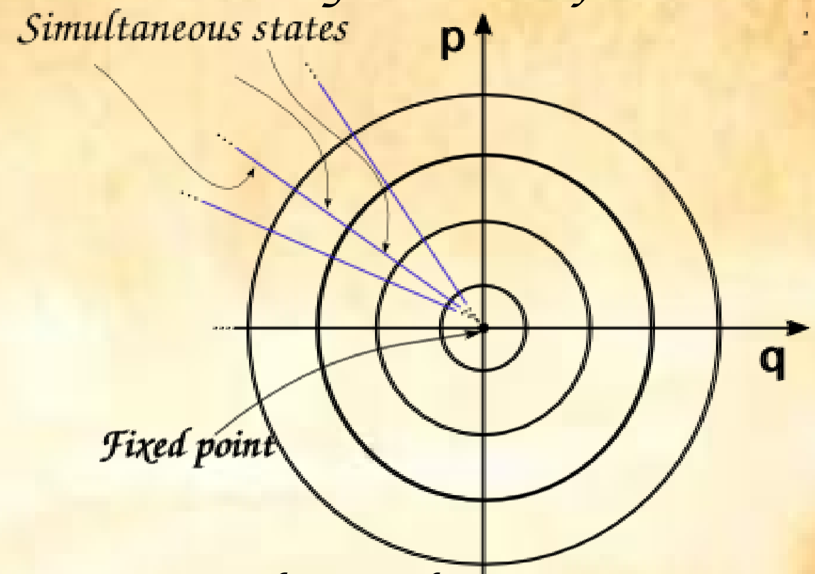

The simultaneous states are points on a radial line.

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Time function: Classical and Quantum

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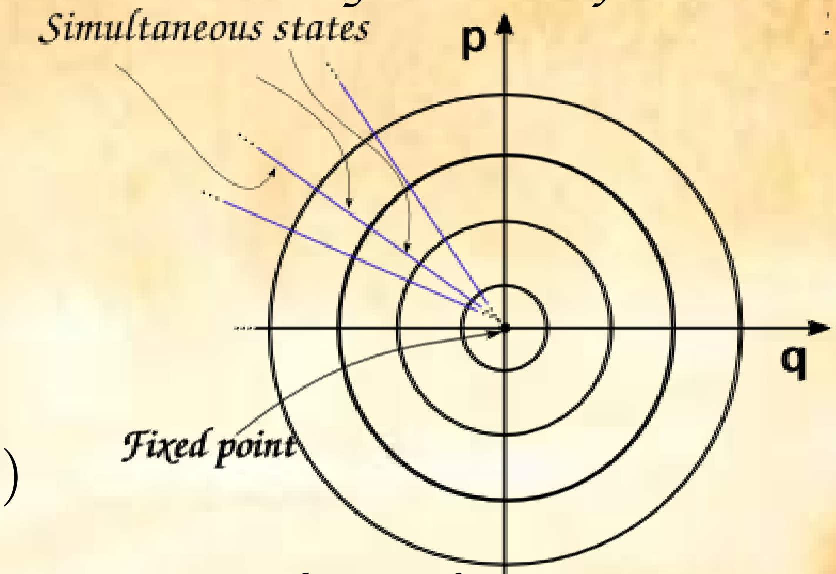

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Simultaneity one-form:

$$\Theta = T^*\theta = \frac{pdq - qdp}{p^2 + q^2} \qquad \Theta(\Gamma) = 1$$

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Time function: Classical and Quantum

Geometric formulation of Quantum Mechanics

The carrier space is a Hilbert space

$$\mathcal{H} \cong \mathbb{C}^n$$

The observables are self-adjoint linear operators on the Hilbert space:

$$\mathcal{B}(\mathcal{H}) \ni \mathbf{A} : \ \mathbf{A}^\dagger = \mathbf{A}$$

The pure states p_ψ are rays of the Hilbert space

$$|\psi\rangle \sim |\phi\rangle \longleftrightarrow |\psi\rangle = r e^{\imath\theta} |\phi\rangle \,, \quad r \in \mathbb{R}^+ \,, \ e^{\imath\theta} \in U(1)$$

The space of pure states is the complex projective space $CP(n-1)$ which is a Kahler manifold:

$$g(X\,;Y) = \omega(J(X)\,;Y) \qquad \forall X, Y \in \mathfrak{X}(CP(n-1))$$

$$\begin{array}{ll} \mathcal{H} - \{\mathbf{0}\} & \\ \downarrow & \mathbb{R}^+ \\ S^{2n-1} & \\ \downarrow & U(1) \\ CP(n-1) & \end{array}$$

Observables are represented by expectation value functions:

$$e_{\mathbf{A}}(p_\psi) := \frac{\langle\psi|\mathbf{A}|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$e_{a\mathbf{A}+b\mathbf{B}} = a e_{\mathbf{A}} + b e_{\mathbf{B}}$$

$$\{e_{\mathbf{A}}\,; e_{\mathbf{B}}\} = e_{\imath[\mathbf{A}\,;\mathbf{B}]}$$

Unitary evolutions are generated by a Hamiltonian operator:

$$i_\Gamma \omega = d e_{\mathbf{H}}$$

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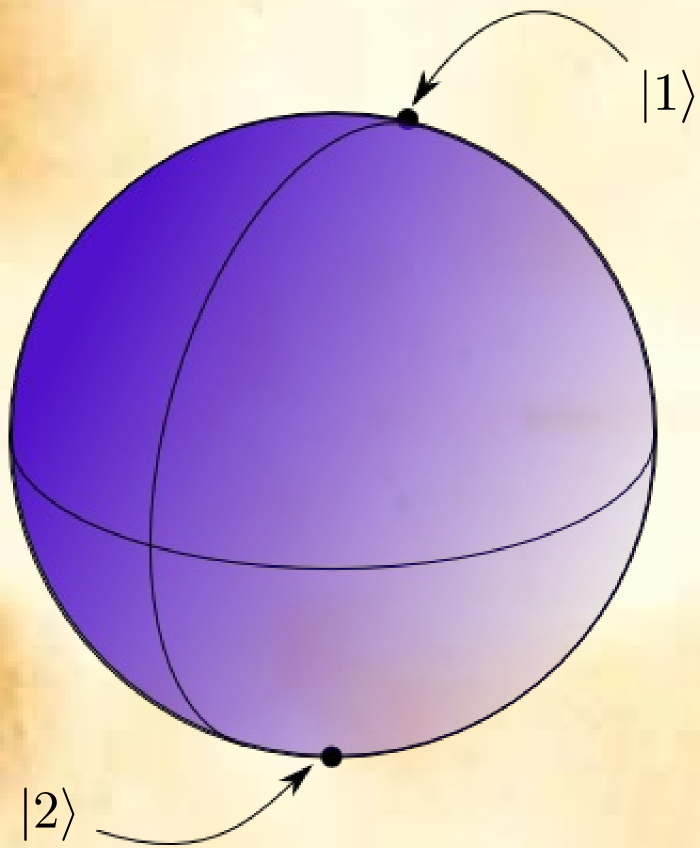
Time function: Classical and Quantum

The Qubit case

The complex projective space $CP(1)$ *is diffeomorphic to a 2-D sphere:*



Hamiltonian operator $\mathbf{H} = \nu_1 \, |1\rangle\langle 1| + \nu_2 \, |2\rangle\langle 2|$

Eigenvalues $\nu_1 \, , \; \nu_2$ *Eigenvectors* $|1\rangle \, , |2\rangle$

Eigenprojectors $\mathbf{E}_1 = |1\rangle\langle 1| \, , \; \mathbf{E}_2 = |2\rangle\langle 2| \, , \; \; \mathbf{E}_1 + \mathbf{E}_2 = \mathbb{I}$

Time function: Classical and Quantum

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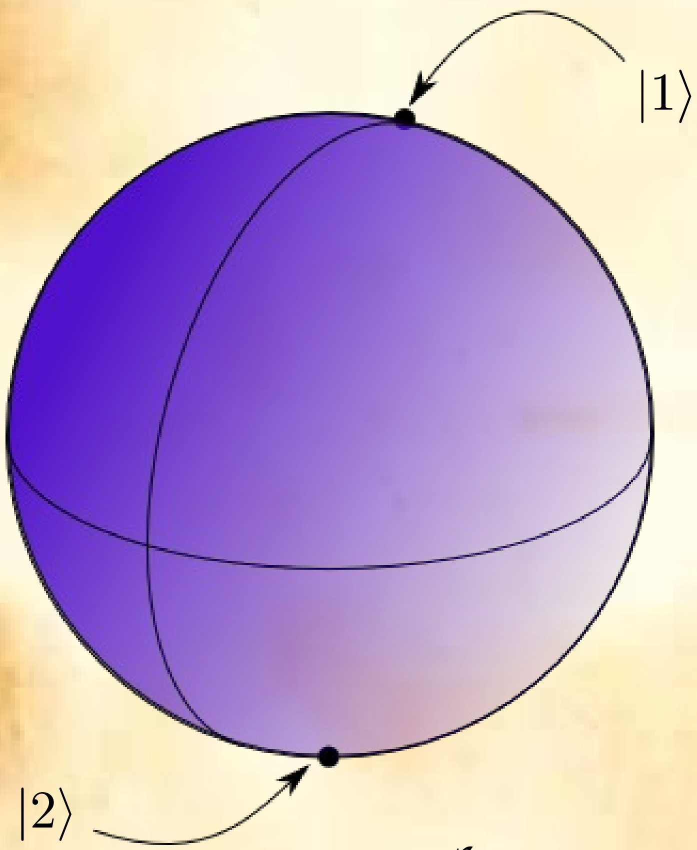

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Expectation value functions

$$e_1 := e_{\mathbf{E}_1} \,,\; e_2 := e_{\mathbf{E}_2} \,,\; e_{\mathbf{H}} = \nu_1 e_1 + \nu_2 e_2$$

These are constants of the motion in involution

$$\{e_1 \,; e_2\} = 0 \,,\; \{e_{\mathbf{H}} \,; e_1\} = 0 \,,\; \{e_{\mathbf{H}} \,; e_2\} = 0$$

They are not linearly independent:

$$e_1 + e_2 = 1 \implies e_2 = 1 - e_1$$

The same is true for their associated Hamiltonian vector fields:

$$[X_1 \,; X_2] = 0 \,, [X_1 \,; \Gamma] = 0 \,, [X_2 \,; \Gamma] = 0 \,,\; X_1 + X_2 = 0$$

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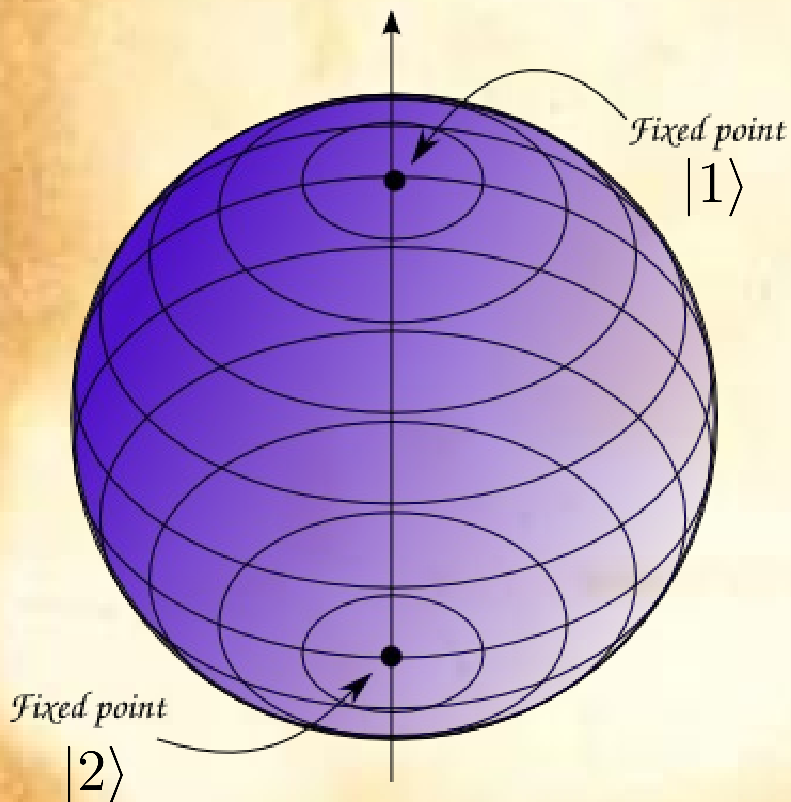
The Qubit case



The dynamical vector field reads:

$$\Gamma = \nu_1 X_1 + \nu_2 X_2 = (\nu_1 - \nu_2)\, X_1$$

The dynamical trajectories are circles with center on the z-axis.
The North and South poles are fixed points (the eigenvectors of the Hamiltonian operator).

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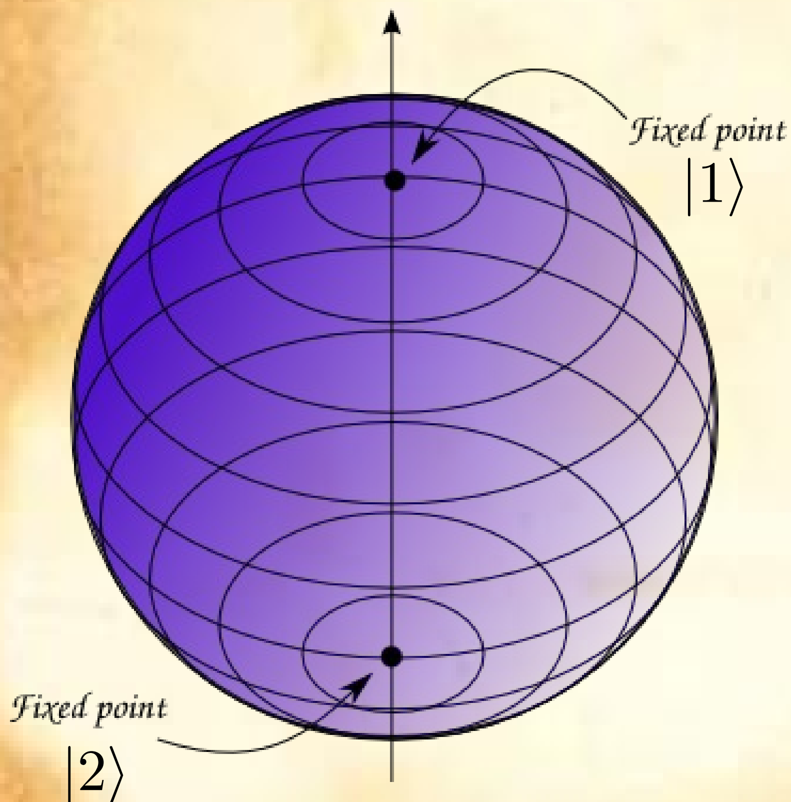

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The reduced space of states is diffeomorphic to an open finite cilinder:

$$\Psi \colon \mathcal{P}_* \to S^1 \times I$$

Normalized vector in the Hilbert space:

$$|\psi\rangle = \frac{1}{\sqrt{(r_1)^2 + (r_2)^2}} \left(r_1\, \mathrm{e}^{\imath\theta_1}\, |1\rangle + r_2\, \mathrm{e}^{\imath\theta_2}\, |2\rangle \right) = \frac{r_2 \mathrm{e}^{\imath\theta_2}}{\sqrt{(r_1)^2 + (r_2)^2}} \left(\frac{r_1}{(r_1)^2 + (r_2)^2}\, \mathrm{e}^{\imath(\theta_1 - \theta_2)}\, |1\rangle + |2\rangle \right)$$

The diffeomorphism reads: $\Psi\left(p_\psi\right) = \left(\mathrm{e}^{\imath(\theta_1 - \theta_2)}\, ;\, \frac{r_1}{(r_1)^2 + (r_2)^2} \right)$

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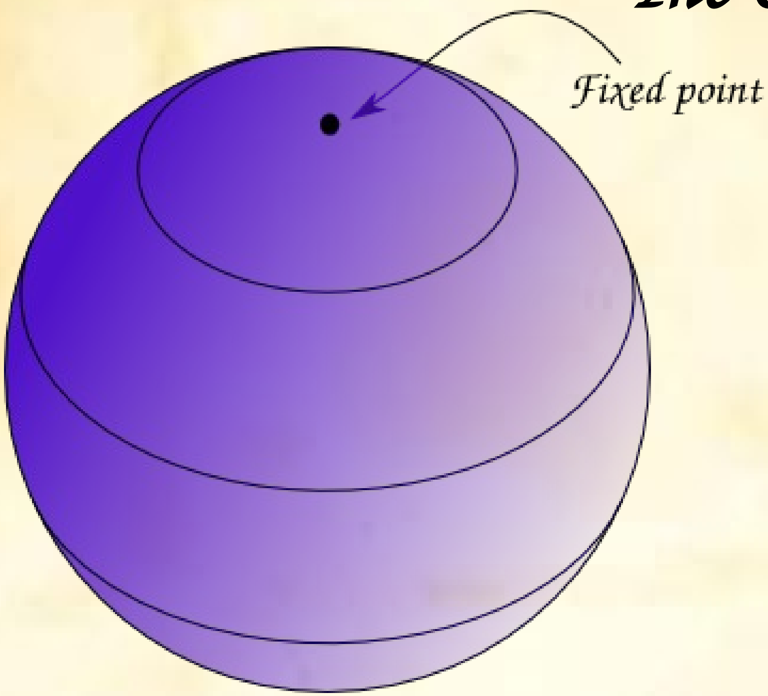

The qubit case is mathematically equivalent to the 1-D harmonic oscillator!!!

Time function: Classical and Quantum

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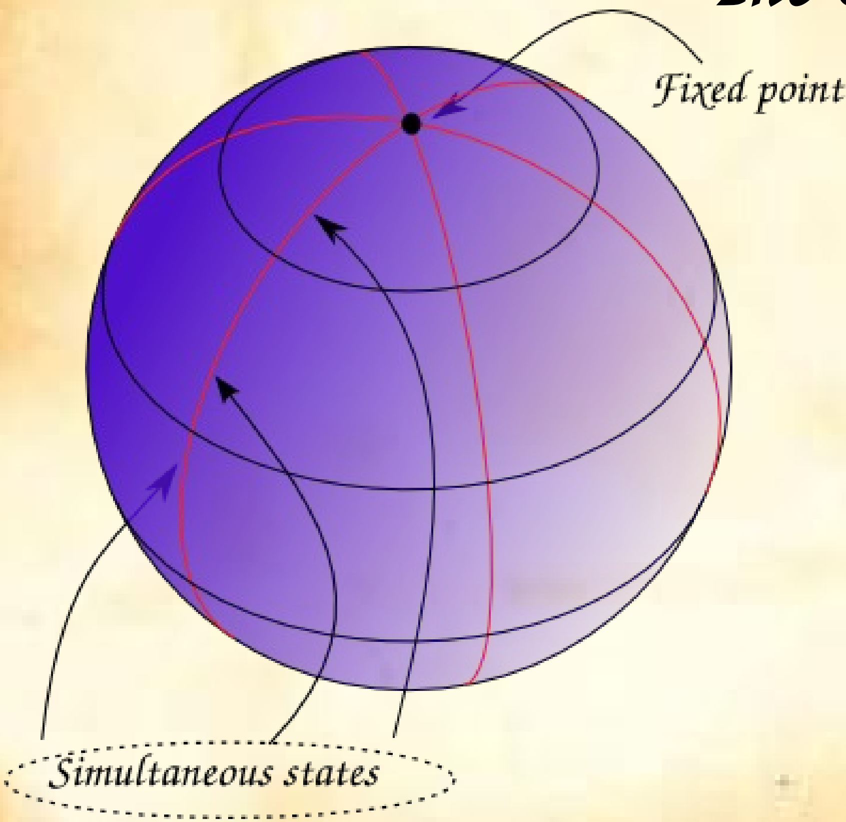

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The Time function is:

$$T = pr_{S^1} \circ \Psi \qquad T\left(p_\psi\right) = \mathrm{e}^{\imath(\theta_1 - \theta_2)}$$

$$T \circ \Phi_\tau \left(p_\psi\right) = \mathrm{e}^{\imath(\theta_1 - \theta_2)}\, \mathrm{e}^{-\imath(\nu_1 - \nu_2)\tau}$$

The sets of simultaneous states are the meridians on the 2-D sphere.

The periodic Time function is not the expectation value function of some self-adjoint linear operator.

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Time function: Classical and Quantum

Higher-dimensional generalization

Hamiltonian operator $$\mathbf{H} = \sum_{j=1}^{n} \nu_j \, \mathbf{E}_j$$

Expectation value functions $e_j := e_{\mathbf{E}_j} \, , \; \{e_j \, ; e_k\} = 0 = \{e_j \, ; e_{\mathbf{H}}\}$

There are (n-1) linearly independent constants of the motion.

These constants of the motion are functionally independent on the reduced space of states:

$$\mathcal{P}_* := \{p_\psi \in CP(n-1) : \; \langle j|\psi\rangle \neq 0 \;\; \forall j = 1, ..., n\}$$

The reduced space of states is diffeomorphic to a product: $\Psi : \mathcal{P}_* \to \left(S^1\right)^{(n-1)} \times I^{(n-1)}$

$$\Psi\left(p_\psi\right) = \left(\mathrm{e}^{\imath(\theta_1 - \theta_n)} \, ; \cdots \, ; \mathrm{e}^{\imath(\theta_{n-1} - \theta_n)} \, ; \frac{r_1}{N^2} \, ; \cdots \, ; \frac{r_{n-1}}{N^2}\right)$$

There is a family of periodic Time functions:

$$T_j \circ \Phi_\tau \left(p_\psi\right) = \mathrm{e}^{\imath(\theta_j - \theta_n)} \;\, \mathrm{e}^{\imath(\nu_n - \nu_j)\tau}$$

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Conclusions:

- *The simultaneity aspect of Time in Q.M. is captured by the Time function;*
- *The Time function is intimately connected to the dynamics of the system;*
- *In general, the Time function of a quantum system is not associated to an operator;*
- *The Time function is well-defined for finite-level quantum systems, whereas the Time operator is not;*
- *The definition of a Time function is well-suited for both Classical and Quantum systems without the need to invoke a quantization procedure;*
- *In principle, the Time function can be defined for dissipative systems.*

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Thank You for Your attention

Time in Quantum Mechanics

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"Without music to decorate it,
Time is just a bunch of boring production deadlines or dates by which bills must be paid."
Frank Zappa

Simultaneity and reference frames

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