Actions with gravity

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Gravity as geometry

General relativity (GR) is a *geometric* theory of gravitation whereby the gravitational field of an object is described by the curvature of spacetime in the neighbourhood of the object.

Three principles underlying GR:

- Principle of equivalence
- Principle of general covariance
- Geodesic principle

Equations of general relativity

Recall the Poisson equation:

$$\nabla^2 \varphi = 4\pi G \rho$$

Variables

- φ is the gravitational potential
- ρ is the matter density

- Equation is of the form:

$$\text{Geometry} = 4\pi G \times (\text{Matter}) \quad \rightarrow \quad \text{Geometry} \propto \text{Matter}$$

- Equation is valid for weak fields; GR extends this to strong fields

Curvature

- Object of interest in GR is a *differentiable manifold*, denoted $\mathcal{M}$
- Manifold $\rightarrow$ a collection of points that are connected to each other such that the neighbourhood of each point looks like $\mathbb{R}^n$, but globally is curved.
- $\mathcal{M}$ has three basic structures: vector, metric and curvature
- Vector: tangent to curve γ on $\mathcal{M}$
- Metric: distance between two points on $\mathcal{M}$
- Curvature: acceleration between two neighbouring geodesics

Einstein equations

- The Einstein field equations couple matter to spacetime geometry

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

Variables

- g_{ab} spacetime metric
- $R_{ab} = R^c{}_{acb}$ Ricci tensor and R_{abcd} Riemann curvature tensor
- $R = R_{ab} g^{ab}$ Ricci scalar
- T_{ab} stress-energy tensor
- Λ cosmological constant

- Classical theory works well on astrophysical scales, but breaks down where quantum effects in matter become important

Black hole thermodynamics

- Black holes have thermal properties

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Variables

- M mass
- r radial distance from center

- The Schwarzschild black hole has a temperature $T = 1/(8\pi M)$ and an entropy $\mathscr{S} = 4\pi M^2$
- In general a black hole has a temperature $T = \kappa/(2\pi)$ and an entropy $\mathscr{S} = A/4$; κ is the surface gravity and A is the surface area
- Example: the entropy of a one solar-mass Schwarzschild black hole is $\mathscr{S} = 2.895 \times 10^{54} \quad \mathrm{J} \cdot \mathrm{K}^{-1}$
- Quantum gravity will explain this entropy from first principles

Singularities

- Hawking and Penrose showed that a large class of spacetimes are geodesically incomplete; for a trapped surface this necessarily implies that a curve will terminate at a point where curvature diverges
- Singularities are present in strong gravitational fields
- Example: the Schwarzschild solution contains a curvature singularity at $r = 0$

Where classical general relativity breaks down, physics has to continue

Leading approaches to quantum gravity

- Loop quantum gravity:
  - Canonical quantization of Hamiltonian phase space with constraints
  - The resulting picture is that of a *discrete* spacetime with area and volume operators that have discrete eigenvalues
  - Difficulties with finding classical or semiclassical limit that would describe quantum fields over spacetime backgrounds
- Superstring theory:
  - Standard perturbative QFT methods used to quantize open and closed strings
  - The spectrum of the closed string includes a spin-two excitation; naturally includes quantum gravity!
  - Theory requires extra dimensions, supersymmetry etc for consistency
  - Theory describes QFT on fixed nondynamical backgrounds; a background independent formulation is necessary

Overview of path integral methods

Fundamental object here is the *amplitude* from initial configuration on manifold $\mathcal{M}_1$ with metric g_1 and matter field ϕ_1 to a final configuration on manifold $\mathcal{M}_2$ with metric g_2 and matter field ϕ_2:

$$\langle g_2, \phi_2; \mathcal{M}_2 | g_1, \phi_1; \mathcal{M}_1 \rangle = \int \mathcal{D}[g]\mathcal{D}[\phi]\exp(iI[g,\phi])$$

- Does not require a space-time split; does not require extra dimensions, supersymmetry...
- Disadvantages? See below!

Black-hole thermodynamics

Recall from statistical mechanics the partition function

$$\mathcal{Z} = \mathrm{Tr}\left[\exp\left(-\beta\hat{H}[\phi]\right)\right] \quad \longrightarrow \quad \mathcal{Z} = \int \mathcal{D}[\phi]\exp\left(-\tilde{I}[\phi]\right)$$

Variables

ϕ fields, β inverse temperature, $\hat{H}$ Hamiltonian and $\tilde{I}$ Euclidean action

- Typically hard to evaluate $\mathcal{Z}$ exactly so need approximation; standard trick from thermodynamics is to expand action around *on-shell* field ϕ_0
- To first-order in the Taylor expansion the partition function is:

$$\mathcal{Z} = \exp\left(-\tilde{I}[\phi_0]\right)$$

- From here ones finds average energy $\langle E\rangle = -\partial(\ln\mathcal{Z})/\partial\beta$ and entropy $\mathscr{S} = \beta\langle E\rangle + \ln\mathcal{Z}$

Perturbation theory

- Starting point is the formal path integral

$$Z[J] = \int \mathcal{D}[g] \exp\left\{ i \left(I_{\mathrm{Grav}} + \int g_{ab} J^{ab} d^4 V \right) \right\}$$

  with J an external source
- One can then take functional derivatives of Z with respect to J and get correlation functions
- Problem: Metric-based action (see below) is not suitable to define Z

Surface term

The *second-order* action for gravity on a four-dimensional (Lorentzian) manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ is given by:

$$I[g] = \frac{1}{16\pi}\int_{\mathcal{M}} (R - 2\Lambda) d^4V + \frac{1}{8\pi}\oint_{\partial\mathcal{M}} K d^3V$$

Variables

with $G = 1$, R Ricci scalar of spacetime metric g, K trace of extrinsic curvature of boundary, $\Lambda = -3/\ell^2$ the cosmological constant with ℓ the anti-de Sitter radius, d^4V volume element determined by g, and d^3V volume element determined by induced metric h on $\partial\mathcal{M}$

- Appearance of the boundary term can be traced to the fact that the action is second-order derivatives of metric; in the variational principle both g and its first derivative must be held fixed
- Prescription fails for asymptotic boundaries such as those of asymptotically flat and anti-de Sitter spacetimes

Counter-terms

- First suggested resolution was to isometrically embed $(\partial\mathcal{M}, h)$ in background spacetime, calculate the extrinsic curvature K_0 of $\partial\mathcal{M}$ defined by the background metric, and subtract resulting quantity from $\oint_{\partial\mathcal{M}} K d^3V$
- The action is thus

$$I[g] = \frac{1}{16\pi}\int_{\mathcal{M}}(R - 2\Lambda)d^4V + \frac{1}{8\pi}\oint_{\partial\mathcal{M}}(K - K_0)d^3V$$

- Resulting action is finite on spacetimes with asymptotic boundaries, but K_0 requires an isometric embedding into flat spacetime by definition and so the prescription cannot be applied to certain spacetimes, for example those containing nut charges

Asymptotically flat spacetimes

- Resolution to embedding problem is to define *local* counter-terms that are intrinsic to $\partial\mathcal{M}$; functions of Ricci tensor of boundary
- For boundaries with topology $S^n \times \mathbb{R}^{3-n}$ with $n \in [2, 3]$ (e.g. $S^2 \times \mathbb{R}^1$ etc) two local counter-terms:
  - Mann:
$$I_{\mathrm{CT}}[h] = -\frac{1}{8\pi}\oint_{\partial\mathcal{M}} \sqrt{\frac{n\mathcal{R}}{n-1}}\, d^3V$$
  - Kraus-Larsen-Siebelink:
$$I_{\mathrm{CT}}[h] = -\frac{1}{8\pi}\oint_{\partial\mathcal{M}} \frac{\mathcal{R}^{3/2}}{\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}}\, d^3V$$

Asymptotically anti-de Sitter spacetimes

- Lau-Mann prescription:

$$I_{\mathrm{CT}}[h] = -\frac{1}{4\pi\ell}\oint_{\partial\mathcal{M}}\sqrt{1+\frac{\ell^2\mathcal{R}}{2}}\,d^3V$$

- Balasubramanian-Kraus:

$$I_{\mathrm{CT}}[h] = -\frac{1}{4\pi\ell}\oint_{\partial\mathcal{M}}\left(1-\frac{\ell^2\mathcal{R}}{4}\right)d^3V$$

... Plus many other much more complicated counter-terms!

Summary so far...

- There is a plethora of counter-term prescriptions that renormalize *either* the action for asymptotically flat *or* anti-de Sitter spacetimes
- The question we are currently exploring is the following:

Is there a generic counter-term prescription that renormalizes quantities for both asymptotically flat and anti-de Sitter spacetimes???

A closer look at the Lagrangian

The first term in the second-order action is

$$R = R[\Gamma, \partial\Gamma] \sim \partial^2 g$$

$\rightarrow$ No quadratic term in Lagrangian so not suitable for perturbation theory

Hilbert-Palatini action

In the first-order formulation of general relativity the action is given by

$$I[e,A] = -\frac{1}{16\pi}\int_{\mathcal{M}} \star(e^i \wedge e^j) \wedge F_{ij} + 2\Lambda\epsilon + \frac{1}{16\pi}\oint_{\partial\mathcal{M}} \star(e^i \wedge e^j) \wedge A_{ij}$$

Variables

e^i coframe ($i,j \in [0,3]$), $A^i{}_j$ an $SO(3,1)$ connection, $F^i{}_j$ associated curvature, ϵ the volume four-form and $\star$ the internal Hodge dual operator

- Boundary term is the natural one on the configuration space $\mathcal{C} = \{e, A\}$ that is required by differentiablility; arises due to presence of exterior derivative dA in action
- Resulting action is both finite and does not make any reference to the embedding of boundary in flat space
- Same boundary term works for asymptotically anti-de Sitter spacetimes; employing Hamiltonian phase space techniques, one finds the Ashtekar-Magnon-Das conserved charges at $\mathscr{I}$

Transition to second-order action

What does first-order boundary term look like in second-order formalism???

- Use several algebraic identities plus definition of extrinsic curvature and find that

$$\star(e^i \wedge e^j) \wedge A_{ij} = e^{ai} A_{ai}{}^{j} n_j d^3V = (K - e^{ai}\partial_a n_i) d^3V$$

- The full action in second-order formalism is therefore

$$I[g] = \frac{1}{16\pi}\int_{\mathcal{M}} (R - 2\Lambda) d^4V + \frac{1}{8\pi}\int_{\partial\mathcal{M}} (K - e^{ai}\partial_a n_i) d^3V$$

- The surface integral is completely independent of the presence of Λ...

Therefore $e^{ai}\partial_a n_i$ is a good candidate for a generic counter-term for both asymptotically flat and ADS spacetimes!

Boundary stress tensor

- Boundary stress tensor was found to be

$$\mathscr{T}_{ab} = -\left[(K_{ab} - Kh_{ab}) - (h_a{}^c e_b{}^i \partial_c n_i - e^{ai} \partial_a n_i)\right]$$

- Work to be done? Show explicitly that $\mathscr{T}_{ab}$ is finite when boundary is pushed to infinity; and in particular derive the corresponding conserved charges

$$\mathscr{Q}_\xi = -\lim_{\Omega\to\infty} \oint_\Omega \mathscr{T}_{ab} \xi^a n^b d^3V$$

- Show that the resulting charges reproduce the correct quantities when evaluated for spacetimes such as Kerr, Taub-NUT etc with $\Lambda \leq 0$

Extension to supergravity

- Extension to supergravity would be of interest because Gibbons-Hawking-York term breaks supersymmetry
- Boundary action for 4D $N = 1$ supergravity in “1.5-order formalism” was found to be:

$$I_{\mathrm{CT}}[h, \psi]$$

$$= \frac{1}{8\pi} \oint_{\partial \mathcal{M}} \left(K - e^{ai} \partial_a n_i - n_i \nabla_a e^{ai} - \frac{1}{2} \tilde{\epsilon}^{abc} \bar{\psi}_a \gamma_i e_b{}^i \psi_c \right) d^3 V$$

with $\tilde{\epsilon}^{abc} = \epsilon^{abcd} n_d$

- Work to be done? Show that $I_{\mathrm{CT}}[h]$ is supersymmetric without imposing any boundary conditions on the configuration variables
- Work out boundary stress tensor, conserved charges etc...

"BF" type actions

Motivation from perturbation theory:

- Note that the lowest order term in the first-order action is cubic: $\star(e \wedge e) \wedge dA$; for perturbation theory ideal to have Lagrangian with lowest order term being quadratic
- Tetrad can be eliminated by defining a two form $B = \star(e \wedge e)$ and the action becomes

$$S = \int_{\mathcal{M}} B \wedge F - \frac{1}{2} \Phi B \wedge B$$

  Φ is a scalar matrix Lagrange multiplier imposing the constraint $B \wedge B = 0$ which has solution

$$B = \pm e \wedge e \quad \text{and} \quad B = \pm \star (e \wedge e)$$

- Starting point of so-called spin foam models; but not quite suitable for perturbation theory because of the contraint term Φ

Need an action that is quadratic and has an interaction term...

Resolution is the Freidel-Starodubtsev action:

$$S = \int_{\mathcal{M}} B^{IJ} \wedge F_{IJ} - \frac{\alpha}{4} B^{IJ} \wedge B^{KL} \epsilon_{IJKL4}$$

Variables

$A^I{}_J$ an $SO(4,1)$ connection with $I, J \in [0,4]$, $F^I{}_J$ associated curvature, α a constant

- Action describes breaking of $SO(4,1)$ down to $SO(3,1)$
- Action for general relativity is recovered in $4+1$ decomposition:

$$A^{ij} = \omega^{ij}\,, \quad R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}\,, \quad A^{i5} = \frac{1}{\ell} e^i$$

$$F^{ij} = R^{ij} - \frac{1}{\ell^2} e^i \wedge e^j\,, \quad F^{i5} = \frac{1}{\ell} \mathcal{D}_\omega e^i$$

with ℓ a constant of dimension length, together with the equation of motion for B^{5i}:

$$S = I[e,\omega] + \frac{1}{4\alpha} \epsilon_{ijkl} R^{ij} \wedge R^{kl}$$

Set-up for perturbation theory

- α is related to Newton constant via $G = \alpha\ell^2$; $\alpha = G\Lambda/3 \sim 10^{-120}$ which is suitable parameter for perturbative expansion
- One can write the action as

$$S = \int_{\mathcal{M}} B^{IJ} \wedge F_{IJ} - \frac{\alpha}{4} B_{IJ} \wedge J^{IJ}$$

  with $J^{IJ} = B_{KL}\epsilon^{IJKL4}$ a "source".
- This action contains quadratic term and an external source term; suitable for definition of generating functional and hence "perturbation theory over a topological background"