Some applications of the Mellin transform: Perturbative and non-perturbative calculation

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Calculations in phenomenology of QFT

Mellin-Barnes representation may be of use for at least two important questions in the phenomenology of QFT:

1. Perturbative calculations in QFT imply to evaluate numerous Feynman diagrams often with several masses and momenta. How can we evaluate them analytically ?
2. Perturbative expressions are series in powers of the coupling constants. How can we have non-perturbative (*i.e.* exponentially suppressed) informations ?

Why using Mellin-Barnes representation?

- The first observation is that the Mellin transform has the following scale property

$$\mathcal{M}\left[f(ax)\right](s) = a^{-s}\mathcal{M}\left[f(x)\right](s)$$

- Renormalization Group solutions lead to consider asymptotic behaviours of the diagrams as $a^{\alpha} \ln^{\beta} a$. The Mellin transformation kernel is the most pertinent to obtain this type of asymptotic behaviour.

- The Mellin-Barnes representation allows expansion in several parameters and it also gives an explicit formula for the remainder that permits a control on perturbative and (in some cases) non-perturbative expansions.

Mellin-Barnes representation

and

Feynman diagrams calculation

One dimensional Mellin Transform

The Mellin transform of a function f and its inverse transform are defined as

$$\mathcal{M}[f(x)](s) \doteq \int_0^\infty dx\, x^{s-1} f(x) \quad \longleftrightarrow \quad f(x) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi}\, x^{-s} \mathcal{M}[f(x)](s)$$

If and only if

$$c \doteq \mathrm{Re}\, s \in]\alpha, \beta[\qquad \text{written} \qquad \langle \alpha, \beta \rangle \qquad \textit{Fundamental strip}$$

It corresponds to the behaviours

$$f(x) \underset{x\to 0^+}{=} \mathcal{O}(x^{-\alpha}) \qquad \& \qquad f(x) \underset{x\to +\infty}{=} \mathcal{O}(x^{-\beta})$$

$(1+x)^{-1}$	$\longleftrightarrow$	$\dfrac{\pi}{\sin \pi s}$	$\langle 0, 1 \rangle$
$(1+x)^{-\nu}$	$\longleftrightarrow$	$\dfrac{\Gamma(\nu - s)\Gamma(s)}{\Gamma(\nu)}$	$\langle 0, \mathrm{Re}\,\nu \rangle$
$\ln(1+x)$	$\longleftrightarrow$	$\dfrac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$

Idea

The singularities in the complex Mellin's plan govern completely the asymptotic behaviour of the associated function

We need to define the singular expansion

From the Laurent series of a function φ in p

$$\varphi(s) = \frac{A_{-n}}{(s-p)^n} + \cdots + \frac{A_{-1}}{s-p} + A_0 + A_1(s-p) + \cdots$$

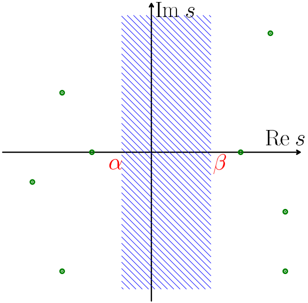
one can build the formal series by summing all over the poles of φ of the singular part:

$$\sum_p \left[\frac{A_{-n}}{(s-p)^n} + \cdots + \frac{A_{-1}}{s-p}\right]$$

this is the singular expansion of φ and it is written as

$$\varphi(s) \asymp \sum_p \left[\frac{A_{-n}}{(s-p)^n} + \cdots + \frac{A_{-1}}{s-p}\right]$$

Converse Mapping Theorem



If f satisfies the condition to have a Mellin transform in the fundamental strip $\langle\alpha,\beta\rangle$ and $\mathcal{M}\left[f\right](s) = \mathcal{O}\left[|s|^{-\eta}\right]$ for $\eta > 1$.

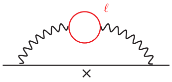
Converse Mapping Theorem

$$\mathcal{M}\left[f(x)\right]_{\text{right}}(s) \asymp \sum_{p>\beta,n} \frac{c_{p,n}}{(s-p)^n} \quad \leftrightarrow \quad f(x) \underset{x\to+\infty}{\sim} \sum_{p>\beta,n} \frac{(-1)^n c_{n,p}}{(n-1)!} x^{-p} \ln^{n-1} x$$

$$\mathcal{M}\left[f(x)\right]_{\text{left}}(s) \asymp \sum_{p<\alpha,n} \frac{d_{p,n}}{(s+p)^n} \quad \leftrightarrow \quad f(x) \underset{x\to 0}{\sim} \sum_{p<\alpha,n} \frac{(-1)^{n-1} d_{n,p}}{(n-1)!} x^{p} \ln^{n-1} x$$

A physical and practical example: $g-2$

Lautrup and de Rafael (1964) **Friot, Greynat and de Rafael (2005)**



$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{dx}{x}(1-x)\Pi_R^{(\ell)}\left(-\frac{x^2}{1-x}m_\mu^2\right)$$
$$= \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{dx}{x}(1-x)(2-x)\int_0^1 dy \frac{y(1-y)}{1+\frac{m_\ell^2}{m_\mu^2}\frac{1-x}{x^2 y(1-y)}}$$

Inverse Mellin Representation:
$$\frac{1}{1+X} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} X^{-s} \frac{\pi}{\sin \pi s} \qquad \langle 0,1\rangle$$

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi}\left(\frac{m_\ell^2}{m_\mu^2}\right)^{-s} \frac{\pi}{\sin\pi s}\int_0^1 dx\, x^{2s-1}(1-x)^{1-s}(2-x)\int_0^1 dy\, y^{1+s}(1-y)^{1+s}$$
$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi}\left(\frac{m_\ell^2}{m_\mu^2}\right)^{-s}\left(\frac{\pi}{\sin\pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)}$$

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} \left(\frac{m_\ell^2}{m_\mu^2}\right)^{-s} \left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)}$$

- For $m_\ell = m_\tau$ then $\frac{m_\ell^2}{m_\mu^2} \gg 1$: Right side of the fundamental strip

$$\left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)}$$
$$\asymp \left(-\frac{1}{45}\right)\frac{1}{s-1} + \left(-\frac{1}{140}\right)\frac{1}{(s-2)^2} + \left(-\frac{9}{19600}\right)\frac{1}{s-2} + \cdots$$

then

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{45}\frac{m_\mu^2}{m_\ell^2} + \frac{1}{140}\left(\frac{m_\mu^2}{m_\ell^2}\right)^2 \ln\left(\frac{m_\mu^2}{m_\ell^2}\right) + \frac{9}{19600}\left(\frac{m_\mu^2}{m_\ell^2}\right)^2 + \cdots\right]$$

- For $m_\ell = m_e$ then $\frac{m_\ell^2}{m_\mu^2} \ll 1$: Left side of the fundamental strip

$$\left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \asymp \frac{1}{6}\frac{1}{s^2} + \left(-\frac{25}{36}\right)\frac{1}{s} + \frac{\pi^2}{4}\frac{1}{s+\frac{1}{2}} + \cdots$$

then

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{6}\ln\left(\frac{m_\mu^2}{m_\ell^2}\right) - \frac{25}{36} + \frac{9}{19600}\left(\frac{m_\mu^2}{m_\ell^2}\right)^2 + \frac{\pi^2}{4}\frac{m_\ell}{m_\mu} + \cdots\right]$$

Exact solution

Flajolet et al. (1994) Friot et Grunberg JHEP 0709:002 (2007)

Those two expansions of the anomaly are the exact representation for $r = \frac{m_\ell^2}{m_\mu^2} \gg 1$ and $r = \frac{m_\ell^2}{m_\mu^2} \ll 1$ because in the rest of the two asymptotic series

$$a_\mu = \sum_{n,p}^{\lfloor T \rfloor} c_{np}\, r^n \ln^p r + R(T)$$

there are no exponential corrections

$$|R(T)| = \left| \int_{\pm T - iT}^{\pm T + iT} \frac{ds}{2i\pi}\, r^{-s} \left(\frac{\pi}{\sin \pi s} \right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \right|$$

$$\leqslant r^{\pm T}\, 2T\, \pi^2 \left| \frac{1 \pm T}{(2 \pm T)(1 \pm 2T)(3 \pm 2T)} \right|$$

$$R(T) \underset{T \to \infty}{=} o\left(r^{\pm T} \right)$$

Therefore the resummation of all the contributions from each poles is convergent and give the exact function.

Resummation

It is easy to perform resummations with the Mellin representation, in our example on right-side

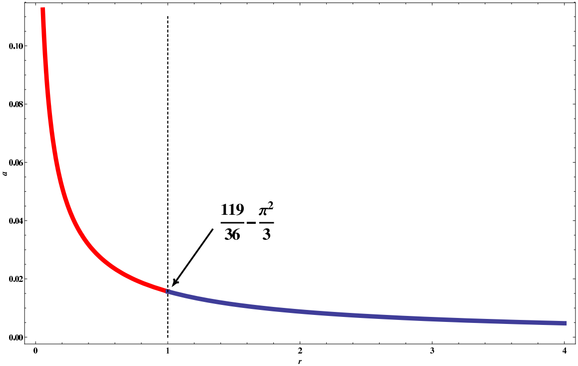
$$\left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \asymp \sum_{p=1}^{\infty}\left[\frac{1}{2+n}+\frac{1}{4}\frac{1}{\frac{1}{2}+n}-\frac{5}{4}\frac{1}{\frac{3}{2}+n}\right]\frac{1}{(s-p)^2} + \sum_{p=1}^{\infty}\left[\frac{1}{(2+n)^2}+\frac{1}{4}\frac{1}{\left(\frac{1}{2}+n\right)^2}-\frac{5}{4}\frac{1}{\left(\frac{3}{2}+n\right)^2}\right]\frac{1}{s-p}$$

And using the *Converse Mapping* Theorem (with $r=\frac{m_\ell^2}{m_\mu^2} \gg 1$) we have the convergent expression by identification of the series as "usual" functions

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2\left[-\frac{1}{4}-r+\frac{\Phi\left(\frac{1}{r},2,\frac{3}{2}\right)}{4r}-\frac{5\Phi\left(\frac{1}{r},2,\frac{5}{2}\right)}{4r}+\frac{1}{2}\sqrt{r}\mathrm{ArcCoth}\left(\sqrt{r}\right)\ln r-\frac{\ln r}{6} + \frac{3}{2}r\ln(r)-\frac{5}{2}r^{3/2}\mathrm{ArcCoth}\left(\sqrt{r}\right)\ln r - r^2\ln\left(1-\frac{1}{r}\right)\ln r + r^2\mathrm{Li}_2\left(\frac{1}{r}\right)\right]$$

Using the *Converse Mapping* Theorem (with $r = \frac{m_\ell^2}{m_\mu^2} \ll 1$) we have the expression

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[-\frac{25}{36} - \frac{\pi^2}{4} r^{1/2} + 3r - \frac{5\pi^2}{4} r^{3/2} + \left(\frac{44}{9} + \frac{\pi^2}{3}\right) r^2 + \frac{5}{4} r^3 \Phi\left(r, 2, \frac{3}{2}\right) \right.$$
$$- \frac{1}{4} r^3 \Phi\left(r, 2, \frac{5}{2}\right) - \frac{1}{6} \ln r + \frac{3}{2} r \ln r + \frac{1}{2}\sqrt{r}\,\mathrm{ArcTanh}\left(\sqrt{r}\right)$$
$$\left. - \frac{5}{2} r^{3/2} \mathrm{ArcTanh}\left(\sqrt{r}\right) \ln r - r^2 \ln(1-r) \ln r + \frac{1}{2} r^2 \ln^2 r - r^2 \mathrm{Li}_2(r) \right]$$



Multi-dimensional Mellin Transform

We define the *n*-dimensional Mellin transform of a function *f* as

$$\mathcal{M}[f](s_1, \ldots, s_n) \doteq \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \;\; x_1^{s_1-1} \cdots x_n^{s_n-1} \; f(x_1, \ldots, x_n)$$

and its inverse transformation

$$f(x_1, \ldots, x_n) \doteq \int\limits_{c_1+i\mathbb{R}} \frac{ds_1}{2i\pi} \cdots \int\limits_{c_n+i\mathbb{R}} \frac{ds_n}{2i\pi} \;\; x_1^{-s_1} \cdots x_n^{-s_n} \; \mathcal{M}[f](s_1, \ldots, s_n)$$

This inversion formula is of course valid in the fundamental polyhedra defined as all the constraints on $\mathbf{c} \doteq {}^T(c_1, \ldots, c_n)$ where the Mellin transform is completely analytic.

If we want to extend the *Converse Mapping* Theorem to the multi-dimensional case we need to introduce "briefly" the Grothendieck Residue theory.

A few words on Grothendieck Residues theory

P. Griffiths, J.Harris, *Principles of Algebraic Geometry*, Wyley NYC 1978

A.K. Tsikh *et al.*, hep-th 9609215

To simplify, we are considering the case of only 2 different scales i.e. 2 complex variables: s and t

One way to see the residues in multi-dimensional complex analysis is to consider the quantity (for any h completely analytic)

$$\mathrm{Res}_{(0,0)} \frac{h(s,t)}{\varphi_1(s,t)\ \varphi_2(s,t)} ds \wedge dt = \oint_{\mathbf{0}} \frac{h(s,t)}{\varphi_1(s,t)\ \varphi_2(s,t)} \frac{ds}{2i\pi} \wedge \frac{dt}{2i\pi} \doteq \oint_{\mathbf{0}} \omega$$

All the curves, the divisors, in the 4-dimension complex space given by – $j \in [\![1,2]\!]$

$$D_j \doteq \left\{(s,t) \in \mathbb{C}^2, \varphi_j(s,t) = 0\right\}$$

have intersections points in this space. They provide the calculation of the residue in a summation over

$$\bigcap_{j \in [\![1,2]\!]} D_j$$

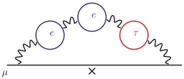
Multi-dimensional Converse Mapping Theorem

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Phys. Rev. D **77**, 093010 (2008)

Idea

Idea: If you combine the calculation of the Grothendieck residues and the *multi-dimensional Jordan lemma* you can define sectors in complex space where the x_j are bigger or smaller than 1, these sectors then allowing to generate the complete asymptotic behaviour in each variables.

Let us consider the following example

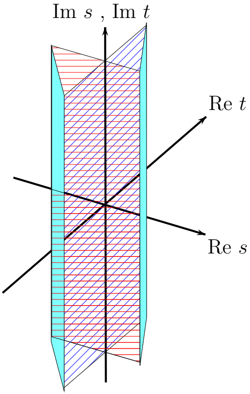


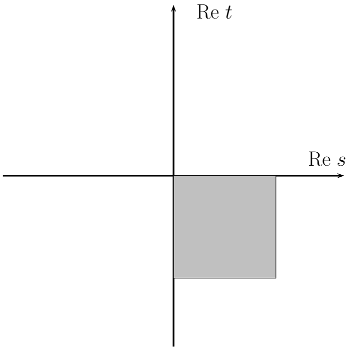
$$a_\mu^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \int_{c_s+i\mathbb{R}} \int_{c_t+i\mathbb{R}} \omega$$

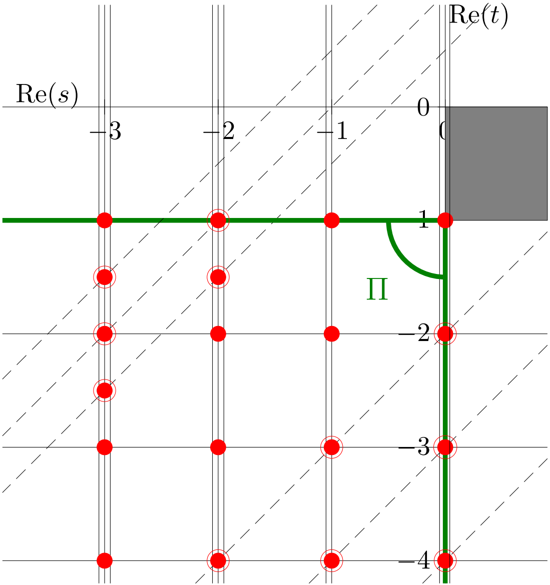
$$\omega = \frac{2\sqrt{\pi}}{3}\left(\frac{4m_e^2}{m_\mu^2}\right)^{-s}\left(\frac{m_\mu^2}{m_\tau^2}\right)^{-t}\frac{(6+13s+4s^2)}{s^3(2+s)(3+s)}\frac{\Gamma(t)\,\Gamma(1-t)\,\Gamma^2(2-t)}{t\,\Gamma(4-2t)}$$

$$\times\ \frac{\Gamma^2(s+1)\,\Gamma(2-s)}{\Gamma(s+\frac{3}{2})}\frac{\Gamma(1+2s-2t)\,\Gamma(2-s+t)}{\Gamma(3+s-t)}\ ds\wedge dt$$

Fundamental polyhedra and pertinent sector






Re(t)
Re(s)
0
−3
−2
−1
0
1
Π
−2
−3
−4

There are two kinds of singularities

The first kind: only vertical and horizontal divisors (at least 2 or more)

The 2-form ω can be rewritten as (n and m are positive integers)

$$\omega = \frac{h(s,t)}{s^n\ t^m} ds \wedge dt$$

Therefore, using the Cauchy formula, we have obviously

$$\mathrm{Res}_{(0,0)}\ \omega = \frac{1}{(n-1)!(m-1)!} \left.\frac{\partial^{n+m-2}\ h(s,t)}{\partial s^{n-1} \partial t^{m-1}}\right|_{(0,0)}$$

The second kind: at least one oblique divisor

The 2-form ω can be rewritten as (for example)

$$\omega = \frac{h(s,t)}{s^n\ t^m\ (-s+t)}\ ds \wedge dt$$

For calculating the residue in this case we need more: the Transformation Law.

The Transformation Law

The Transformation Law

For $U \subset \mathbb{C}^2$ an open set containing $(0,0)$

If $\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1(s,t) \\ \varphi_2(s,t) \end{pmatrix}$ and $\boldsymbol{g} = \begin{pmatrix} g_1(s,t) \\ g_2(s,t) \end{pmatrix}$ analytic mappings from U to $\mathbb{C}^2$

and $\boldsymbol{\varphi}^{-1}(0,0) = \boldsymbol{g}^{-1}(0,0) = 0$

If it exists an analytic matrix A such that: $\boldsymbol{g} = A\,\boldsymbol{\varphi}$ then

$$\mathrm{Res}_{(0,0)}\ \frac{h(s,t)}{\varphi_1(s,t)\ \varphi_2(s,t)} ds \wedge dt = \mathrm{Res}_{(0,0)}\ \frac{h(s,t)\det A(s,t)}{g_1(s,t)\ g_2(s,t)} ds \wedge dt$$

Example:

$$\mathrm{Res}_{(0,0)}\ \frac{h(s,t)}{s^3 t(-s+t)} ds \wedge dt$$

we take

$$\underbrace{\begin{pmatrix} s^3 \\ t^5 \end{pmatrix}}_{\doteq \mathbf{g}} = \underbrace{\begin{pmatrix} 1-t^2+st & s^3 \\ st & (t^2+s^2)(s+t)t \end{pmatrix}}_{\doteq A} \underbrace{\begin{pmatrix} s^3 \\ t(-s+t) \end{pmatrix}}_{\doteq \boldsymbol{\varphi}}$$

$$\det A = s^3 + s^2 t + s t^2 + t^3 - t^5$$

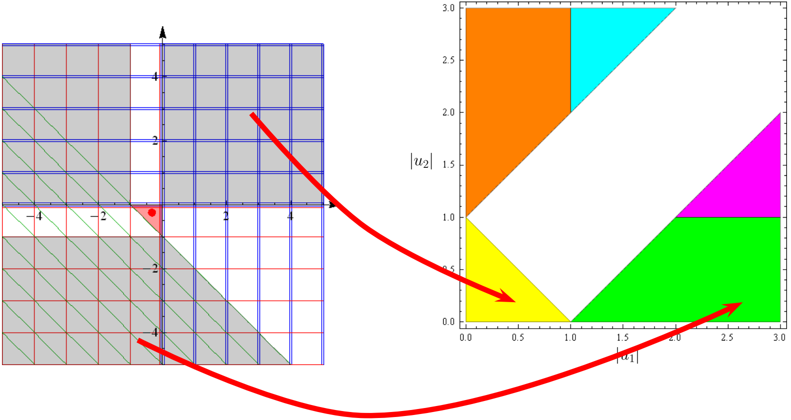
Therefore

$$
\begin{aligned}
&\mathrm{Res}_{(0,0)}\ \frac{h(s,t)}{s^3 t(-s+t)} ds\wedge dt \\
&= \mathrm{Res}_{(0,0)}\ h(s,t)\frac{s^3+s^2t+st^2+t^3-t^5}{s^3t^5}\ ds\wedge dt \\
&= \mathrm{Res}_{(0,0)}\ h(s,t)\left(-\frac{1}{s^3}+\frac{1}{t^5}+\frac{1}{st^4}+\frac{1}{s^2t^3}+\frac{1}{s^3t^2}\right)\ ds\wedge dt
\end{aligned}
$$

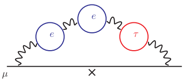
$$
\mathrm{Res}_{(0,0)}\ \frac{h(s,t)}{s^3 t(-s+t)}\ ds\wedge dt = \frac{1}{2}\frac{\partial^3 h(s,t)}{\partial s^2\partial t}\bigg|_{(0,0)} + \frac{1}{2}\frac{\partial^3 h(s,t)}{\partial s\partial t^2}\bigg|_{(0,0)} + \frac{1}{6}\frac{\partial^3 h(s,t)}{\partial t^3}\bigg|_{(0,0)}.
$$

Convergence sectors

$$R(u_1,u_2)=\int_{c+i\mathbf{R}}\int_{d+i\mathbf{R}}\frac{dz_1}{2i\pi}\wedge\frac{dz_2}{2i\pi}\;u_1^{z_1}\,u_2^{z_2}\;\Gamma^2(-z_1)\,\Gamma(z_1)\,\Gamma^2(-z_2)\,\Gamma(z_2)\,\Gamma(z_1+z_2+1)$$



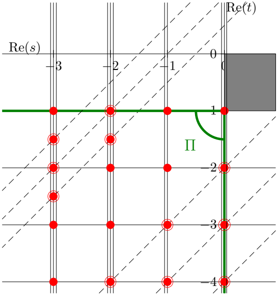
Coming back to our physical example



$$a_\mu^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \int_{c_s+i\mathbb{R}} \int_{c_t+i\mathbb{R}} \omega$$

Along the line $t = -1$:

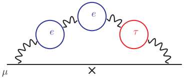
$$\omega = \frac{h_{(0,-1)}(s,t)}{s^3 t}\, ds \wedge dt$$



$$h_{(0,-1)}(s,t) = \frac{2\sqrt{\pi}}{3}\left(\frac{4m_e^2}{m_\mu^2}\right)^{-s}\left(\frac{m_\mu^2}{m_\tau^2}\right)^{1-t}\frac{\Gamma(3+2s-2t)\,\Gamma(1-s+t)}{\Gamma(4+s-t)}$$
$$\times \frac{\Gamma^2(1+s)\Gamma(2-s)(6+13s+4s^2)}{(2+s)(s+3)\Gamma\left(\frac{3}{2}+s\right)}\;\frac{\Gamma^2(3-t)\Gamma(1+t)\Gamma(2-t)}{(-1+t)^2\;\Gamma(6-2t)}$$

$$\mathrm{Res}_{(0,-1)}\,\omega = \frac{1}{2}\left.\frac{\partial^2 h_{(0,-1)}}{\partial s^2}\right|_{(0,0)}$$

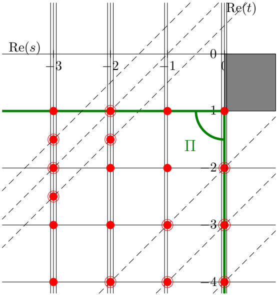
$$a_\mu^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4\left(\frac{m_\mu^2}{m_\tau^2}\right)\left[\frac{1}{135}\log^2\frac{m_\mu^2}{m_e^2} - \frac{1}{135}\log\frac{m_\mu^2}{m_e^2} - \frac{61}{2430} + \frac{2}{405}\pi^2\right]$$



$$a_\mu^{(eee\tau)} = \left(\frac{\alpha}{\pi}\right)^5 \int_{c_1+i\mathbb{R}} \int_{c_2+i\mathbb{R}}$$

Along the line $t = -2$:

$$\omega = \frac{h_{(0,-2)}(s,t)}{s^3 t(-s+t)}\, ds \wedge dt$$



$$h_{(0,-2)}(s,t) = \left(\frac{4m_e^2}{m_\mu^2}\right)^{-s} \left(\frac{m_\mu^2}{m_\tau^2}\right)^{2-t} \frac{2\sqrt{\pi}}{3}\, \frac{\Gamma(5+2s-2t)\Gamma(1-s+t)}{\Gamma(5+s-t)}$$

$$\times \frac{(6+13s+4s^2)\Gamma^2(1+s)\Gamma(2-s)}{(2+s)(3+s)\Gamma\left(\frac{3}{2}+s\right)}\, \frac{\Gamma(1+t)\Gamma(3-t)}{(-1+t)(-2+t)^2}\, \frac{\Gamma^2(4-t)}{\Gamma(8-2t)}$$

$$\mathrm{Res}_{(0,-2)}\omega = \frac{1}{2}\frac{\partial^3 h_{(0,-2)}(s,t)}{\partial s^2 \partial t}\bigg|_{(0,0)} + \frac{1}{2}\frac{\partial^3 h_{(0,-2)}(s,t)}{\partial s \partial t^2}\bigg|_{(0,0)} + \frac{1}{6}\frac{\partial^3 h_{(0,-2)}(s,t)}{\partial t^3}\bigg|_{(0,0)}$$

Easily now the contribution to the anomaly is

$$
\begin{aligned}
a_\mu^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \left(\frac{m_\mu^2}{m_\tau^2}\right)^2 \Bigg[& \frac{1}{1260}\log^3\frac{m_\mu^2}{m_\tau^2} - \left(\frac{1}{420}\log\frac{m_\mu^2}{m_e^2} + \frac{37}{44100}\right)\log^2\frac{m_\mu^2}{m_\tau^2} \\
& + \left(\frac{1}{420}\log^2\frac{m_\mu^2}{m_e^2} + \frac{37}{22050}\log\frac{m_\mu^2}{m_e^2} + \frac{40783}{4630500}\right)\log\frac{m_\mu^2}{m_\tau^2} \\
& + \frac{3}{19600}\log^2\frac{m_\mu^2}{m_e^2} + \left(\frac{\pi^2}{630} - \frac{229213}{12348000}\right)\log\frac{m_\mu^2}{m_e^2} \\
& + \frac{\pi^2}{1512} - \frac{30026659}{5186160000}\Bigg] \\
& + \cdots + \mathcal{O}\left[\left(\frac{m_\mu^2}{m_\tau^2}\right)^4 \log\frac{m_\tau^2}{m_e^2}\log\frac{m_\mu^2}{m_e^2}\log\frac{m_\tau^2}{m_\mu^2}\right] \\
= \left(\frac{\alpha}{\pi}\right)^4 & 0.002\,748\,6(9)
\end{aligned}
$$

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Phys.Rev.**D 77** (2008).

Mellin-Barnes representation

and

asymptotic improvement of perturbative expansion

Open Problems

- In QFT we have to deal with presumably divergent non-Borel summable series and expected asymptotic.
- But asymptotic to what ?
- One more difficulty: the general terms of the expansions are unknown.
- Consequence: theoretical errors are not under control then how can we find new physics effects in precision physics ?

Introduction to the method

Framework

The general term of a divergent (and supposed asymptotic) perturbative series is known.

Aim

Find non perturbative effects directly from the perturbative series.

Tools

- Terminant functions theory `Dingle 1973`
- Exponential improvement of asymptotic series : MB hyperasymptotic theory. **Paris *et al.*'90, Berry 1989**

0−dimensional euclidean example

Let us consider the 0−dimensional euclidean action S with a unit mass

$$S \doteq \frac{1}{2!}\phi^2 + \frac{\lambda}{4!}\phi^4$$

and the associated generating functional

$$Z(j) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\phi \; \mathrm{e}^{-S+j\phi} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2!}\phi^2-\frac{\lambda}{4!}\phi^4+j\phi}$$

with $\lambda \in \mathbb{C}$ and $\mathrm{Re}\,\lambda > 0$

We will focus on the vacuum-vacuum transitions

$$Z(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2!}\phi^2-\frac{\lambda}{4!}\phi^4}$$

Perturbative expansion

Taking λ small, one has the perturbative expansion of $Z(0)$

$$Z(0) \underset{\lambda\to 0}{\sim} \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{1}{2}+2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k = 1 - \frac{1}{8}\lambda + \frac{35}{384}\lambda^2 - \frac{385}{3072}\lambda^3 + \mathcal{O}(\lambda^4)$$

Yet

$$\forall \lambda, \quad \left| \frac{(-1)^k \Gamma\left(\frac{1}{2}+2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k \right| \underset{k\to\infty}{\longrightarrow} \infty$$

then this is a divergent series.

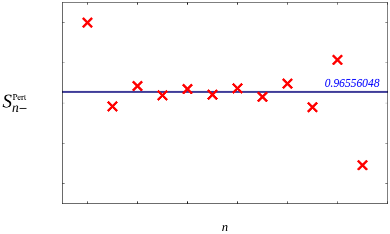
Let us define $S_n^{\text{Pert.}}$ and R_n

$$Z(0) \underset{\lambda\to 0}{\sim} \underbrace{\frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2}+2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k}_{\doteq S_{n-1}^{\text{Pert.}}} + \underbrace{\frac{1}{\sqrt{\pi}} \sum_{k=n}^{\infty} \frac{(-1)^k \Gamma\left(\frac{1}{2}+2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k}_{\doteq R_n}$$

Numerical analysis

From now, for all numerical calculations, we fix $\lambda = \frac{1}{3}$. Mathematica gives with *8 decimal precision*

$$Z(0)|_{\lambda=\frac{1}{3}} \approx 0.96556048...$$



Choosing a resummation procedure for the perturbative series leads to the property that $|R_n| < |u_n|$ and $|R_n| < |u_{n-1}|$. One then has

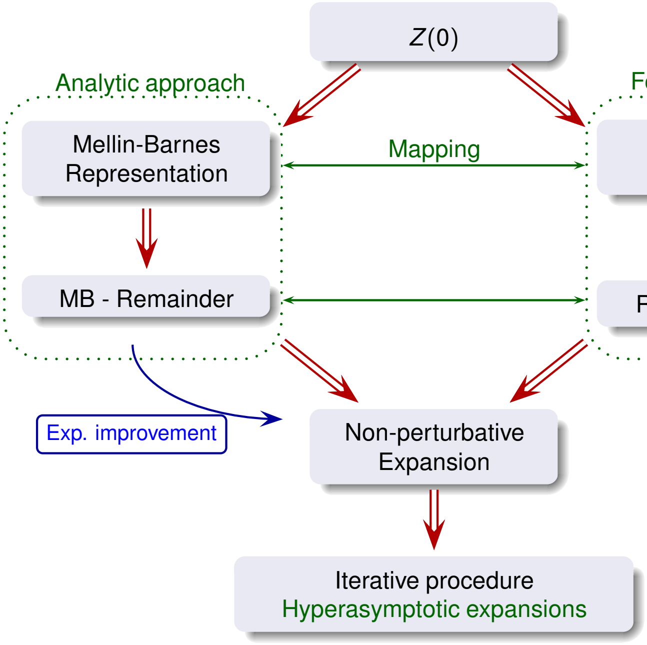
$$Z(0) = \sum_{k=0}^{\eta-1} u_k + \frac{1}{2}|u_\eta| \pm \frac{1}{2}|u_\eta| \ ,$$

here

$$Z(0)|_{\lambda=\frac{1}{3}} = 0.96555187 \pm 0.00140990$$

The central value corresponds to the standard Stieltjes approximation which is given, for the rank η defined $\forall n, \ |u_\eta| \leqslant |u_n|$

$$Z(0) = \sum_{k=0}^{\eta-1} u_k + \frac{1}{2}|u_\eta| \ .$$


$Z(0)$
Analytic approach
Formal approach
Mellin-Barnes Representation
Mapping
Perturbative Expansion
MB - Remainder
Remainder (tail)
Exp. improvement
Non-perturbative Expansion
Iterative procedure
Hyperasymptotic expansions

Analytic approach

Mellin-Barnes Hyperasymptotic Theory

Main Idea

Construct an exponentially small remainder with its inverse Mellin-Barnes representation (non perturbative in λ).

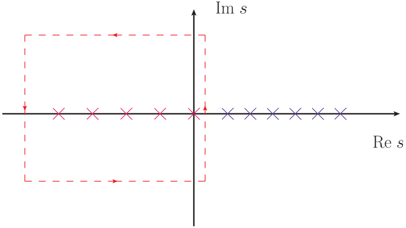
The first step is to obtain an inverse Mellin-Barnes representation of $Z(0)$:

$$e^{-\frac{\lambda}{4!}\phi^4} = \int_{c+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{4!}\phi^4\right)^{-s} \Gamma(s)\ ,$$

with $\langle 0, +\infty \rangle$ and $|\arg \lambda| < \frac{\pi}{2}$. Then we obtain the following results

$$Z(0) = \frac{1}{\sqrt{\pi}} \int_{c+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s)\Gamma\left(\frac{1}{2} - 2s\right)$$

with $\langle 0, \frac{1}{4} \rangle$ and $|\arg \lambda| < \frac{3\pi}{2}$ (Analytic continuation). `Paris et al. 90's`



Applying the Cauchy's theorem on the integral over s inside a rectangle horizontally sized n, one can prove that

$$Z(0) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k + \frac{1}{\sqrt{\pi}} \int_{c-n+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s)\Gamma\left(\frac{1}{2} - 2s\right)$$

therefore

$$R_n = \frac{1}{\sqrt{\pi}} \int_{c-n+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s)\Gamma\left(\frac{1}{2} - 2s\right)$$

Using the Stirling's formula: $|\Gamma(\sigma + i\tau)| \underset{\tau\to\pm\infty}{=} \mathcal{O}\left(|\tau|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|\tau|}\right)$

and majoring the integrand one has **Paris *et al.* 90's**

$$|R_n| \underset{n\to\infty}{=} \left|\frac{2\lambda}{3}\right|^{n-c} \mathcal{O}\left(e^{-n}\, n^{n-c}\right)$$

Superasymptotic Theorem

When $|\lambda|$ is small, there exist $a_0 > 0$ and $|b_0| < \infty$ so that if

$$n = \frac{a_0}{|\lambda|} + b_0$$

then R_n is exponentially small in λ (non perturbative).

Here,

$$|R_n| \underset{|\lambda|\to 0}{=} \mathcal{O}\left(e^{-\frac{a_0}{|\lambda|}} \left(\frac{2a_0}{3}\right)^{\frac{a_0}{|\lambda|}}\right)$$

and for $a_0 = \frac{3}{2}$, we have the Optimal Truncation Scheme *i.e.* the smallest remainder

$$|R_n| \underset{n\to\infty}{=} \mathcal{O}\left(e^{-\frac{3}{2|\lambda|}}\right) .$$

The iterative procedure

Main Idea

It is possible to construct more and more exponentially small remainders order by order in the expansion of R_n.

With the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

we can rewrite the remainder as

$$R_n = \frac{1}{\pi\sqrt{2}} \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \frac{\Gamma\left(t+\frac{1}{4}\right)\Gamma\left(t+\frac{3}{4}\right)}{\Gamma(t+1)}$$

We introduce now the main tool : the inverse factorial expansion.

The inverse factorial expansion

Barnes' lemma

$$\frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+c)} = \int\limits_{i\,\mathbb{R}} \frac{dt}{2i\pi}\, \frac{\Gamma(t+c-a)\Gamma(t+c-b)\Gamma(s+\vartheta-t)\Gamma(-t)}{\Gamma(c-a)\Gamma(c-b)}$$

with $\vartheta = a+b-c$.

Inverse factorial expansion

$$\frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+c)} = \sum_{j=0}^{M-1} \frac{(-1)^j}{j!}\,(c-a)_j\,(c-b)_j\,\Gamma(s+\vartheta-j)$$
$$+ \int\limits_{M+i\,\mathbb{R}} \frac{dt}{2i\pi}\, \frac{\Gamma(t+c-a)\Gamma(t+c-b)\Gamma(s+\vartheta-t)\Gamma(-t)}{\Gamma(c-a)\Gamma(c-b)}$$

$M > \mathrm{Re}\,(a-c)+\delta$, $M > \mathrm{Re}\,(b-c)+\delta$, $M > \mathrm{Re}\,(s+\vartheta)$
for $0 < \delta < 1$ and $|\arg s| < \frac{\pi}{2}$.

Olver 1995

One has in our example,

$$\frac{\Gamma\left(t+\frac{1}{4}\right)\Gamma\left(t+\frac{3}{4}\right)}{\Gamma(t+1)} = \sum_{j=0}^{m-1}(-1)^j\, A_j\, \Gamma(t-j) + \int\limits_{c+m+i\,\mathbb{R}} \frac{ds}{2i\pi}\, f(s)\, \Gamma(t-s)$$

$$\text{where } A_j = \frac{1}{j!}\left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j \quad \text{and} \quad f(s) = \frac{\Gamma(-s)\Gamma\left(s+\frac{1}{4}\right)\Gamma\left(s+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

The remainder is then

$$R_n = -\frac{1}{\pi\sqrt{2}}\sum_{j=0}^{m-1}(-1)^j\, A_j \int\limits_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi}\left(\frac{3}{2\lambda}\right)^{-t}\frac{\pi}{\sin\pi t}\Gamma\left(t-j\right) + R_{n,m}$$

with

$$R_{n,m} = -\frac{1}{\pi\sqrt{2}} \int\limits_{\substack{(-c+n+i\mathbb{R})\\ \times(-c+m+i\mathbb{R})}} \frac{dt}{2i\pi}\wedge\frac{ds}{2i\pi}\left(\frac{3}{2\lambda}\right)^{-t}\frac{\pi}{\sin\pi t}\, f(s)\Gamma(t-s)$$

Yet, we have $(m \leqslant n)$, $|R_{n,m}| \underset{n\to\infty}{=} \mathcal{O}\left(\left|\frac{3}{2\lambda}\right|^{-n-c} e^{-n}(n-m)^{n-m} m^{m+c-\frac{1}{2}}\right)$

Optimal Truncation Scheme

For $m = \frac{a_1}{|\lambda|} + b_1$, we have

Paris *et al.* 90's

$$|R_{n,m}| \underset{|\lambda|\to 0}{=} \mathcal{O}\left(\sqrt{|\lambda|}\mathrm{e}^{-\frac{a_0+\ln 3-\ln 2}{|\lambda|}} a_1^{\frac{a_1}{|\lambda|}} (a_0-a_1)^{\frac{a_0-a_1}{|\lambda|}}\right)$$

$$|R_n| \underset{|\lambda|\to 0}{=} \mathcal{O}\left(\mathrm{e}^{-\frac{a_0}{|\lambda|}} \left(\frac{2a_0}{3}\right)^{\frac{a_0}{|\lambda|}}\right)$$

We can optimize the remainders in two ways, first satisfying the condition on R_n and then optimizing $R_{n,m}$ or optimizing directly $R_{n,m}$:

			$R_n = \mathcal{O}(\cdot)$	$R_{n,m} = \mathcal{O}(\cdot)$
OTS 1	$a_0 = \frac{3}{2}$	$a_1 = \frac{3}{4}$	$\mathrm{e}^{-\frac{3}{2\|\lambda\|}}$	$\sqrt{\|\lambda\|}\mathrm{e}^{-\frac{3}{2\|\lambda\|}(1+\ln 2)}$
OTS 2	$a_0 = 3$	$a_1 = \frac{3}{2}$	–	$\sqrt{\|\lambda\|}\mathrm{e}^{-\frac{3}{\|\lambda\|}}$

Hyperasymptotic expansion of $Z(0)$ at first hyperasymptotic level

$$Z(0) = \sum_{k=0}^{n-1} (-1)^k A_k \left(\frac{3}{2\lambda}\right)^k$$

$$- \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j)$$

$$- \frac{1}{\pi\sqrt{2}} \int_{\substack{(-c+n+i\mathbb{R}) \\ \times(-c+m+i\mathbb{R})}} \frac{dt}{2i\pi} \wedge \frac{ds}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} f(s)\Gamma(t-s)$$

OTS1 at first hyperasymptotic level

$$Z(0) = \sum_{k=0}^{\left[\frac{3}{2|\lambda|}\right]-1} (-1)^k A_k \left(\frac{3}{2\lambda}\right)^k - \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+\left[\frac{3}{2|\lambda|}\right]+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j) + \mathcal{O}\left(\sqrt{|\lambda|} e^{-\frac{3(1+\ln 2)}{2|\lambda|}}\right)$$

OTS2 at first hyperasymptotic level

$$Z(0) = \sum_{k=0}^{\left[\frac{3}{|\lambda|}\right]-1} (-1)^k A_k \left(\frac{3}{2\lambda}\right)^k - \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{\left[\frac{3}{2|\lambda|}\right]-1} (-1)^j A_j \int_{-c+\left[\frac{3}{|\lambda|}\right]+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j) + \mathcal{O}\left(\sqrt{|\lambda|} e^{-\frac{3}{|\lambda|}}\right)$$

Iterations...

We have shown that it is possible to write the remainder as

$$R_n = -\frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma\left(t-j\right)$$

$$- \frac{1}{\pi\sqrt{2}} \int_{\substack{(-c+n+i\mathbb{R}) \\ \times(-c+m+i\mathbb{R})}} \frac{dt}{2i\pi} \wedge \frac{ds}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} f(s)\Gamma(t-s)$$

where $f(s) = \frac{\Gamma(-s)\Gamma\left(s+\frac{1}{4}\right)\Gamma\left(s+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = -\frac{1}{\pi\sqrt{2}} \frac{\Gamma\left(s+\frac{1}{4}\right)\Gamma\left(s+\frac{3}{4}\right)}{\Gamma(s+1)} \frac{\pi}{\sin \pi s}$

Yet applying the Inverse factorial expansion one has

Functional equation

$$f(s) = -\frac{1}{\pi\sqrt{2}} \frac{\pi}{\sin \pi s} \left[\sum_{l=0}^{m'-1} (-1)^l A_l \, \Gamma(s-l) + \int_{c+m'-i\,\mathbb{R}} \frac{dt}{2i\pi} \, f(t)\, \Gamma(s-t) \right]$$

Then using this expression of f in R_n one may get straightforwardly a hyperasymptotic expansions at an arbitrary hyperasymptotic level.

Formal approach

The starting point is now the expression of $Z(0)$ as a divergent series

$$Z(0) \underset{\lambda\to 0}{\sim} \underbrace{\frac{1}{\sqrt{\pi}}\sum_{k=0}^{n-1}\frac{(-1)^k\Gamma\left(\frac{1}{2}+2k\right)}{k!}\left(\frac{\lambda}{6}\right)^k}_{\doteq S^{\mathrm{Pert.}}_{n-1}} + \underbrace{\frac{1}{\sqrt{\pi}}\sum_{k=n}^{\infty}\frac{(-1)^k\Gamma\left(\frac{1}{2}+2k\right)}{k!}\left(\frac{\lambda}{6}\right)^k}_{\doteq R_n}$$

Using the duplication formula,

$$R_n = \frac{1}{\pi\sqrt{2}}\sum_{k=n}^{\infty}\frac{\Gamma\left(k+\frac{1}{4}\right)\Gamma\left(k+\frac{3}{4}\right)}{\Gamma(k+1)}\left(-\frac{2\lambda}{3}\right)^k$$

We can apply the Inverse factorial expansion for

$$\frac{\Gamma\left(k+\frac{1}{4}\right)\Gamma\left(k+\frac{3}{4}\right)}{\Gamma(k+1)} = \sum_{j=0}^{m-1}(-1)^j A_j\,\Gamma(k-j) + \int_{c+m+i\,\mathbb{R}}\frac{ds}{2i\pi}\,f(s)\,\Gamma(k-s)$$

where $A_j = \frac{1}{j!}\left(\frac{1}{4}\right)_j\left(\frac{3}{4}\right)_j$ and $f(s) = \frac{\Gamma(-s)\Gamma\left(s+\frac{1}{4}\right)\Gamma\left(s+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$

Resummation and terminant function

The remainder is then

$$R_n = \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \sum_{k=n}^{\infty} \Gamma(k-j) \left(\frac{-2\lambda}{3}\right)^k$$

$$+ \frac{1}{\pi\sqrt{2}} \int_{c+m+i\,\mathbb{R}} \frac{ds}{2i\pi}\, f(s) \sum_{k=n}^{\infty} \Gamma(k-s) \left(\frac{-2\lambda}{3}\right)^k$$

One can now perform a Borel resummation on the two infinite sums in R_n,

$$\mathcal{B}\left[\sum_{k=n}^{\infty} \Gamma(k-j) \left(\frac{-2\lambda}{3}\right)^k\right] = \Gamma(n-j) \left(-\frac{2\lambda}{3}\right)^n \Lambda_{n-j-1}\left(\frac{3}{2\lambda}\right)$$

The function Λ_ℓ is known as a terminant function, here Dingle 1973

$$\Lambda_\ell(x) = x^{\ell+1} \mathrm{e}^{\,x} \Gamma(-\ell, x)$$

where

$$\Gamma(a,x) = \int_x^\infty dy\; y^{a-1}\, \mathrm{e}^{-y}$$

is the incomplete Gamma function.

Therefore performing the summation, one has the expansion ($m \leqslant n$)

$$Z(0) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2}+2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k$$
$$+ \frac{(-1)^n}{\pi\sqrt{2}}\, e^{\frac{3}{2\lambda}} \sum_{j=0}^{m-1} (-1)^j A_j\, \Gamma(n-j) \left(\frac{2\lambda}{3}\right)^j \Gamma\left(-n+j+1, \frac{3}{2\lambda}\right)$$
$$+ \frac{(-1)^n}{\pi\sqrt{2}}\, e^{\frac{3}{2\lambda}} \int\limits_{c+m+i\,\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{2}{3\lambda}\right)^{-s} f(s)\, \Gamma(n-s)\, \Gamma\left(-n+s+1, \frac{3}{2\lambda}\right)$$

where the sub-dominant terms appear explicitly through the expression of the tail of the series.

Using the MB representation of the incomplete Gamma function we have exactly the hyperasymptotic expansion of $Z(0)$ at first level obtained in the analytic approach.

Resurgence phenomenon

If we consider a few terms in the perturbative expansion of $Z(0)$

$$Z(0) = 1 - \frac{1}{8}\lambda + \frac{35}{384}\lambda^2 - \frac{385}{3072}\lambda^3 + \cdots$$

we have the same coefficient in the non-perturbative expansion of R_n as

$$R_n = -\frac{(-1)^n}{\pi\sqrt{2}} e^{\frac{3}{2\lambda}}$$

$$\times \left\{ 1\ \Gamma(5)\Gamma\left(-4, \frac{3}{2\lambda}\right) - \frac{1}{8}\lambda\ \Gamma(4)\Gamma\left(-3, \frac{3}{2\lambda}\right) + \frac{35}{384}\lambda^2\ \Gamma(3)\Gamma\left(-2, \frac{3}{2\lambda}\right) - \frac{385}{3072}\lambda^3\ \Gamma(2)\Gamma\left(-1, \frac{3}{2\lambda}\right) + \cdots \right\}$$

This resurgence phenomenon also appears in higher order hyperasymptotic levels.

Numerical analysis

		n	m	m'	m''
OTS 1	1	$[\frac{3}{2\vert\lambda\vert}]=4$	$[\frac{3}{4\vert\lambda\vert}]=2$		
	2	$[\frac{3}{2\vert\lambda\vert}]=4$	$[\frac{3}{4\vert\lambda\vert}]=2$	$[\frac{3}{8\vert\lambda\vert}]=1$	
	3	$[\frac{3}{2\vert\lambda\vert}]=4$	$[\frac{3}{4\vert\lambda\vert}]=2$	$[\frac{3}{8\vert\lambda\vert}]=1$	$[\frac{3}{8\vert\lambda\vert}]=0$
OTS 2	1	$[\frac{3}{\vert\lambda\vert}]=9$	$[\frac{3}{2\vert\lambda\vert}]=4$		
	2	$[\frac{9}{2\vert\lambda\vert}]=13$	$[\frac{3}{\vert\lambda\vert}]=9$	$[\frac{3}{2\vert\lambda\vert}]=4$	
	3	$[\frac{6}{\vert\lambda\vert}]=18$	$[\frac{9}{2\vert\lambda\vert}]=13$	$[\frac{3}{\vert\lambda\vert}]=9$	$[\frac{3}{2\vert\lambda\vert}]=4$

		$Z(0)$	S_n	S_m	$S_{m'}$	$S_{m''}$
Mathematica		0.965560481..				
Pertur. expa.		0.96555187 $\pm$0.001410990				
OTS 1	1	0.96552297	0.9638	0.0017		
	2	0.96556492	0.9638	0.0017	0.000041	
	3	0.96556492	0.9638	0.0017	0.000041	0
OTS 2	1	0.965562911	0.9696	-0.0040		
	2	0.965560477	1.0573	-0.0917	0.0000061	
	3	0.965560486	-27.696	28.662	-0.0001292	9×10^{-9}

- Numerical stability Olver *et al.* '95

It has been proven that it is possible to obtain numerical stability by choosing another OTS, taking

$$a_j = \frac{n+1-j}{n+1}$$

For the third hyperasymptotic level we have

$$n = 8 \qquad m = 6 \qquad m' = 4 \qquad m'' = 2$$

and

$$Z(0) = 0.9655604480$$

- An numerical evaluation on the Riemann surface

For $\lambda = e^{i\pi}\frac{1}{3}$, we have

Mathematica	$1.05995021 - 0.00758472\,i$
Stieljes	$1.06098837 + 0\,i \quad \pm 0.00140990$
$S_5 + S_2^{NP}$	$1.05990083 - 0.00752794\,i$

Conclusions

- The Mellin Barnes representation is a perfect tool to obtain asymptotic expansions of Feynman diagrams containing one or several scales. It assures the calculation at any order and under certain conditions the exact representation.

- The Mellin Barnes representation is also a powerful instrument to deal with perturbative divergent series and their resummation. It allows also through the hyperasymptotic method to get non perturbative contributions directly from the perturbative expansion itself.