# Three-dimensional fragmentation functions and semi-inclusive $e^+e^-$-annihilation at high energies

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Based on

K.B. Chen, S.Y. Wei, and ZTL, Front. Phys. 10, 101204 (2015) (short review);
K.B. Chen, W.H. Yang, S.Y. Wei, and ZTL, PRD94, 034003 (2016);
K.B. Chen, W.H. Yang, Y.J. Zhou, & ZTL, arXiv:1609.07001 [hep-ph] (2016).

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Transverse momentum dependent fragmentation functions (TMD FFs) defined via quark-quark correlator

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- Spin and angular dependences and structure functions
- Azimuthal asymmetries and polarizations

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- The hadronic tensor and structure functions up to twist-3
- Azimuthal asymmetries and polarizations
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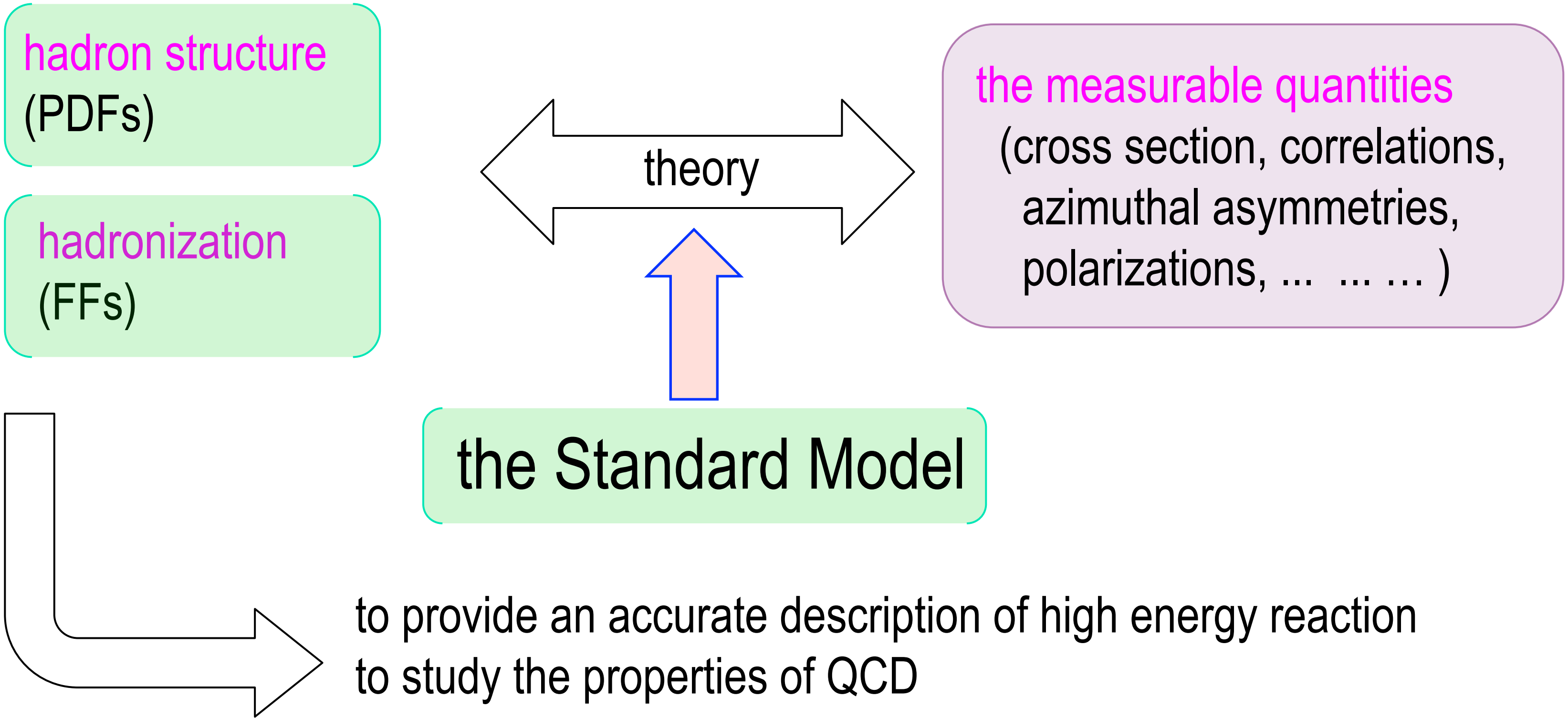
# Introduction: QCD and hadron physics

Two important quantities: parton distribution function (PDF) ⟷ hadron structure

fragmentation function (FF) ⟷ hadronization



hadron structure (PDFs)

hadronization (FFs)

theory

the measurable quantities (cross section, correlations, azimuthal asymmetries, polarizations, ... ... … )

the Standard Model

to provide an accurate description of high energy reaction

to study the properties of QCD

to test the standard model precisely

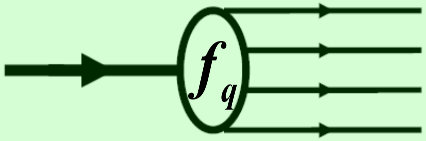
to provide “initial conditions” to search for new physics

………………………

# Introduction: Intuitive definition of PDFs

One-dimensional PDFs:

Parton model: A fast moving proton $=$ A beam of partons



$f_q(x)$: number density of parton in proton, known as parton distribution function (PDF)

$x = k / p$: fractional momentum carried by the parton

Including spin ⇒ spin dependent one-dimensional PDFs:

$$f_1(x, s_q; S) = f_1(x) + \lambda_q \lambda \, g_{1L}(x) + \vec{s}_{Tq} \cdot \vec{S}_T \; h_{1T}(x)$$

helicity distribution transversity

Including transverse momentum ⇒ three-dimensional (or TMD) PDFs:

Sivers function

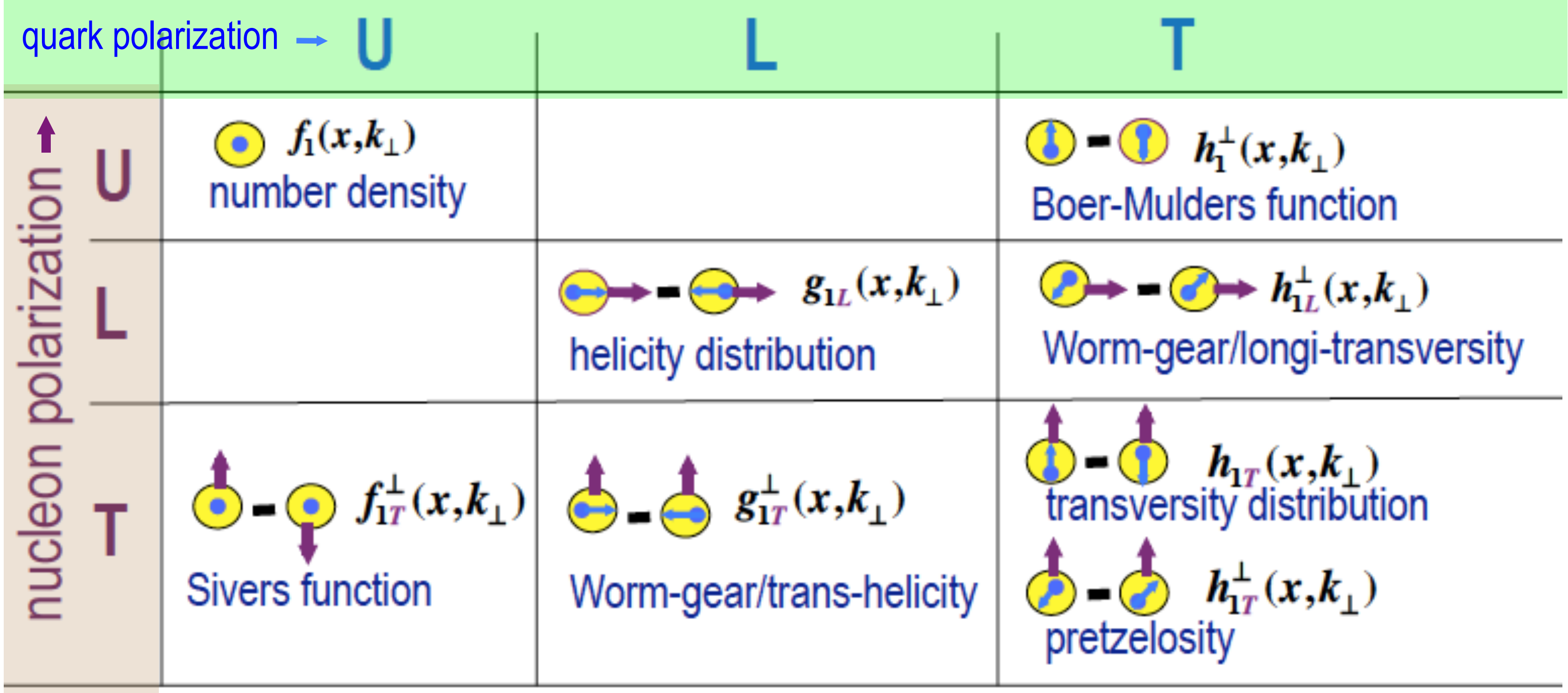
$$f\,(x, k_\perp, s_q; p, S) = f_1\,(x, k_\perp) + \frac{1}{M}\vec{S}_T \cdot (\hat{p} \times \vec{k}_\perp) f_{1T}^\perp(x, k_\perp) + \lambda_q \lambda g_{1L}(x, k_\perp) + \lambda_q \frac{1}{M}(\vec{S}_T \cdot \vec{k}_\perp) g_{1T}^\perp(x, k_\perp)$$

$$+ \frac{1}{M}\vec{s}_{\perp q} \cdot (\hat{p} \times \vec{k}_\perp) h_1^\perp(x, k_\perp) + \vec{s}_{\perp q} \cdot \vec{S}_T h_{1T}(x, k_\perp) + \frac{1}{M^2}(\vec{s}_{\perp q} \cdot \vec{k}_\perp)(\vec{S}_T \cdot \vec{k}_\perp) h_{1T}^\perp(x, k_\perp) + \frac{1}{M}(\vec{s}_{\perp q} \cdot \vec{k}_\perp) \lambda h_{1L}^\perp(x, k_\perp)$$

Boer-Mulders function

# Introduction: Leading twist TMD PDFs

## The 8 three-dimensional (or TMD) PDFs



| quark polarization → | | U | L | T |
|---|---|---|---|---|
| nucleon polarization ↑ | U | $f_1(x,k_\perp)$<br>number density | | $h_1^\perp(x,k_\perp)$<br>Boer-Mulders function |
| | L | | $g_{1L}(x,k_\perp)$<br>helicity distribution | $h_{1L}^\perp(x,k_\perp)$<br>Worm-gear/longi-transversity |
| | T | $f_{1T}^\perp(x,k_\perp)$<br>Sivers function | $g_{1T}^\perp(x,k_\perp)$<br>Worm-gear/trans-helicity | $h_{1T}(x,k_\perp)$<br>transversity distribution<br>$h_{1T}^\perp(x,k_\perp)$<br>pretzelosity |

In quantum field theory, they are defined via Lorentz decomposition of the quark-quark correlator

$$\hat{\Phi}^{(0)}(k;p,S)=\int d^4z e^{ikz}\langle p,S|\bar{\psi}(0)\mathcal{L}(0,z)\psi(z)|p,S\rangle$$

# TMD PDFs defined via quark-quark correlator

The quark-quark correlator: $\hat{\Phi}^{(0)}(k;p,S)=\int d^4z e^{ikz}\langle p,S|\bar{\psi}(0)\mathcal{L}(0,z)\psi(z)|p,S\rangle$

integrate over $k^-$ : $\hat{\Phi}^{(0)}(x,k_\perp;p,S)=\int dz^- d^2z_\perp e^{i(xp^+z^- -\vec{k}_\perp\cdot\vec{z}_\perp)}\langle p,S|\bar{\psi}(0)\mathcal{L}(0,z)\psi(z)|p,S\rangle$

Expansion in terms of the Γ-matrices

$$
\begin{aligned}
\hat{\Phi}^{(0)}(x,k_\perp;p,S)=\frac{1}{2}\Big[\ &\Phi^{(0)}(x,k_\perp;p,S) && \text{scalar}\\
&+i\gamma_5\ \tilde{\Phi}^{(0)}(x,k_\perp;p,S) && \text{pseudo-scalar}\\
&+\gamma^\alpha\ \Phi^{(0)}_\alpha(x,k_\perp;p,S) && \text{vector}\\
&+\gamma_5\gamma^\alpha\ \tilde{\Phi}^{(0)}_\alpha(x,k_\perp;p,S) && \text{axial vector}\\
&+i\gamma_5\sigma^{\alpha\beta}\ \Phi^{(0)}_{\alpha\beta}(x,k_\perp;p,S)\Big] && \text{tensor}
\end{aligned}
$$

e.g.: $\Phi^{(0)}_\alpha(x,k_\perp;p,S)=\frac{1}{2}\mathrm{Tr}\left[\gamma_\alpha\hat{\Phi}^{(0)}(x,k_\perp;p,S)\right]$

$$=\int d^4z e^{ikz}\langle p,S|\bar{\psi}(0)\mathcal{L}(0,z)\frac{\gamma_\alpha}{2}\psi(z)|p,S\rangle$$

# TMD PDFs defined via quark-quark correlator

**The Lorentz decomposition**

totally 8(twist 2)+16(twist 3)+8(twist 4) components

$$\Phi_S^{(0)}(x,k_\perp;p,S) = Me(x,k_\perp) + (\tilde{k}_\perp \cdot S_T) e_T^\perp(x,k_\perp)$$

twist-3

$$\Phi_\alpha^{(0)}(x,k_\perp;p,S) = \frac{p^+}{M}\bar{n}_\alpha \left[ Mf_1(x,k_\perp) + (\tilde{k}_\perp \cdot S_T) f_{1T}^\perp(x,k_\perp) \right]$$

twist-2

$$+ \ k_{\perp\alpha} f^\perp(x,k_\perp) + M\tilde{S}_{T\alpha} f_T(x,k_\perp) + \frac{\tilde{k}_{\perp\alpha}}{M}\left[ \lambda M f_L^\perp(x,k_\perp) + (k_\perp \cdot S_T) f_T^\perp(x,k_\perp) \right]$$

$$+ \frac{M}{p^+} n_\alpha \left[ Mf_3(x,k_\perp) + (\tilde{k}_\perp \cdot S_T) f_{3T}^\perp(x,k_\perp) \right]$$

twist-4

E.g.: $f_1(x,k_\perp) = \int dz^- d^2 z_\perp e^{ikz} \langle p,S | \bar{\psi}(0) \mathcal{L}(0,z) \frac{\gamma^+}{2} \psi(z) | p,S \rangle$

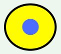
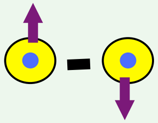
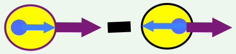
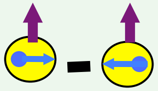
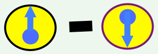
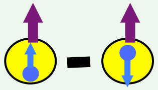
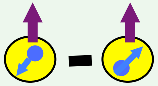
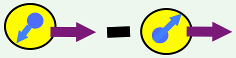
See e.g., K. Goeke, A. Metz, M. Schlegel, PLB 618, 90 (2005);
P. J. Mulders, lectures in 17th Taiwan nuclear physics summer school, August, 2014.

$$p = p^+ \bar{n} + \frac{M^2}{2p^+} n, \quad S = \lambda \frac{p^+}{M} \bar{n} + S_T - \lambda \frac{M^2}{2p^+} n, \quad \tilde{k}_{\perp\alpha} \equiv \varepsilon_{\perp\rho\alpha} k_\perp^\rho \equiv \varepsilon_{\perp k_\perp \alpha}, \quad A_{[\alpha} B_{\beta]} \equiv A_\alpha B_\beta - A_\beta B_\alpha$$

# Twist-2 TMD PDFs defined via quark-quark correlator

Leading twist (twist 2)

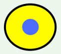
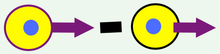
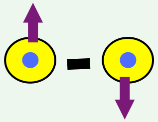
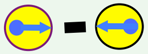
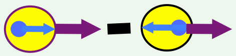
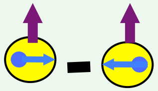
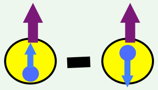
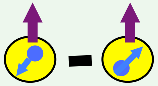
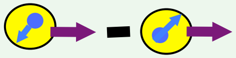
$f, g, h$: quark un-, longitudinally, transversely polarized

| polarization | | | TMD PDFs (8) | if no gauge link | integrated over $k_\perp$ | name |
|---|---|---|---|---|---|---|
| quark | nucleon | pictorially | | | | |
| $U$ | $U$ |  | $f_1(x,k_\perp)$ | | $q(x)$ | number density |
| | $T$ |  | $f_{1T}^\perp(x,k_\perp)$ | 0 | × | Sivers function |
| $L$ | $L$ |  | $g_{1L}(x,k_\perp)$ | | $\Delta q(x)$ | helicity distribution |
| | $T$ |  | $g_{1T}^\perp(x,k_\perp)$ | | × | worm gear/trans-helicity |
| $T$ | $U$ |  | $h_1^\perp(x,k_\perp)$ | 0 | × | Boer-Mulders function |
| | $T(//)$ |  | $h_{1T}(x,k_\perp)$ | | $\delta q(x)$ | transversity distribution |
| | $T(\perp)$ |  | $h_{1T}^\perp(x,k_\perp)$ | | | pretzelocity |
| | $L$ |  | $h_{1L}^\perp(x,k_\perp)$ | | × | worm gear/<br>longi-transversity |

# Twist-3 TMD PDFs defined via quark-quark correlator

## Next to the leading twist (twist-3)

they are NOT probability distributions but include the quantum interference effects.

| polarization | | | TMD PDFs (16) | if no gauge link | integrated over $k_\perp$ | name |
|---|---|---|---|---|---|---|
| quark | nucleon | pictorially | | | | |
| $U$ | $U$ |  | $e(x,k_\perp)$, $f^\perp(x,k_\perp)$ | 0 $\frac{f_1(x,k_\perp)}{x}$ | $e(x)$, × | number density |
| | $L$ |  | $f_L^\perp(x,k_\perp)$ | 0 | × | Sivers function |
| | $T$ |  | $e_T^\perp(x,k_\perp)$,<br>$f_T(x,k_\perp)$, $f_T^\perp(x,k_\perp)$ | 0<br>0 0 | ×<br>$f_T(x)$ | |
| $L$ | $U$ |  | $g^\perp(x,k_\perp)$ | 0 | × | |
| | $L$ |  | $e_L(x,k_\perp)$, $g_L^\perp(x,k_\perp)$ | 0 $\frac{g_{1L}(x,k_\perp)}{x}$ | $e_L(x)$, × | helicity distribution |
| | $T$ |  | $e'^\perp_T(x,k_\perp)$,<br>$g_T(x,k_\perp)$, $g_T^\perp(x,k_\perp)$ | 0<br>0 $\frac{g_{1T}(x,k_\perp)}{x}$ | ×<br>$g'_T(x)$ | worm gear/trans-helicity |
| $T$ | $U$ |  | $h(x,k_\perp)$ | 0 | $h(x)$ | Boer-Mulders function |
| | $T$(//) |  | $h_T^\perp(x,k_\perp)$ | $\frac{h_{1T}^\perp(x,k_\perp)}{x}$ | × | transversity distribution |
| | $T(\perp)$ |  | $h_T^{\perp'}(x,k_\perp)$ | $\frac{k_\perp^2 h_{1T}^\perp(x,k_\perp)}{M^2 x}$ | × | pretzelocity |
| | $L$ |  | $h_L(x,k_\perp)$ | $\frac{k_\perp^2 h_{1L}^\perp(x,k_\perp)}{M^2 x}$ | $h_L(x)$ | worm gear/ longi-transversity |

# Twist-3 TMD PDFs defined via quark-gluon-quark correlator

The quark-gluon-quark correlator involved:

$$\hat{\varphi}^{(1)}_{\rho}(x,k_\perp;p,S)=\int dz^- d^2z_\perp e^{i(xp^+z^- -\vec{k}_\perp\cdot\vec{z}_\perp)}\langle p,S|\bar{\psi}(0)\mathcal{L}(0,z)D_\rho(z)\psi(z)|p,S\rangle$$

$$D_\rho(z)=-i\partial_\rho+gA_\rho(z)$$

a subscript "d" to denote that they are from D-type quark-gluon-quark correlator

The Lorentz decompositions, e.g.:

$$\varphi^{(1)}_{\rho\alpha}(x,k_\perp;p,S)=p^+\bar{n}_\alpha\Big\{k_{\perp\rho}f_d^\perp(x,k_\perp)+M\varepsilon_{\perp\rho\sigma}S_T^\sigma f_{dT}(x,k_\perp)$$
$$+\frac{\tilde{k}_{\perp\rho}}{M}\left[\lambda M f_{dL}^\perp(x,k_\perp)+(k_\perp\cdot S_T)f_{dT}^\perp(x,k_\perp)\right]+\ldots\ldots$$

$$\tilde{\varphi}^{(1)}_{\rho\alpha}(x,k_\perp;p,S)=p^+\bar{n}_\alpha\Big\{\tilde{k}_{\perp\rho}g_d^\perp(x,k_\perp)-MS_{T\rho}g_{dT}(x,k_\perp)$$
$$-\frac{k_{\perp\rho}}{M}\left[\lambda M g_{dL}^\perp(x,k_\perp)+(k_\perp\cdot S_T)g_{dT}^\perp(x,k_\perp)\right]+\ldots\ldots$$

totally 16 twist-3 TMD PDFs are defined via quark-gluon-quark correlator, the same number as those defined via quark-gluon-quark correlator!

# Twist-3 TMD PDFs defined via quark-gluon-quark correlator

(continued)

However, they are NOT independent!

QCD equation of motion $\gamma\cdot D(z)\psi(z)=0$

chiral even:

$$f_{dS}^{K}(x,k_\perp)+g_{dS}^{K}(x,k_\perp)=x\left[f_S^K(x,k_\perp)+ig_S^K(x,k_\perp)\right]$$

$K=\text{null},\ S=T;\qquad K=\perp,\ S=\text{null}, L, \text{ or } T$

chiral odd:

$$h_{dS}^{K}(x,k_\perp)+\frac{k_\perp^2}{2M^2}h_{dS}^{K'}(x,k_\perp)=\frac{1}{2}x\left[h_S^K(x,k_\perp)-ie_S^K(x,k_\perp)\right]$$

$(K,K')=(\text{null},\perp),\ S=\text{null or } L;\quad (K,K')=(\perp,\perp') \text{ or } ('\perp,'\perp'),\ S=T.$

Exactly 16 (real) equations!

the twist-3 TMD PDFs defined via quark-gluon-quark correlator
DO NOT appear explicitly in the final expressions of the cross section.

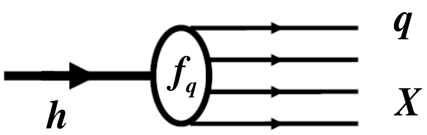
See e.g., Y.K. Song, J.H. Gao, ZTL & X.N. Wang, PRD (2014); …….

# Fragmentation Function v.s. Parton Distribution Function

TMDs = TMD PDFs + TMD FFs

## Parton distribution functions (PDFs)：

$h \to q + X$



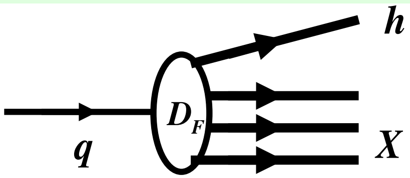
a hadron ⟶ a beam of partons

number density of parton in the beam

$$\hat{\Phi}(k;p,S)=\sum_X\int d^4z e^{ikz}$$
$$\times\langle h|\bar{\psi}(0)|X\rangle\langle X|\mathcal{L}(0,z)\psi(z)|h\rangle$$

## Fragmentation functions (FFs)：

$q \to h + X$



a quark ⟶ a jet of hadrons

number density of hadron in the jet

$$\hat{\Xi}(k_F;p,S)=\sum_X\int d^4\xi e^{ik_F\xi}$$
$$\times\langle 0|\mathcal{L}(0,\xi)\psi(\xi)|hX\rangle\langle hX|\bar{\psi}(0)|0\rangle$$

“conjugate” to each other

⇨ Studies on FFs and PDFs should keep pace with each other.

# Description of polarization of particles with different spins

**Spin 1/2 hadrons:**

The spin density matrix is 2x2: $\rho = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} = \frac{1}{2}(1 + \vec{S}\cdot\vec{\sigma})$

Vector polarization: $S^{\mu} = (0, \vec{S}_T, \lambda)$

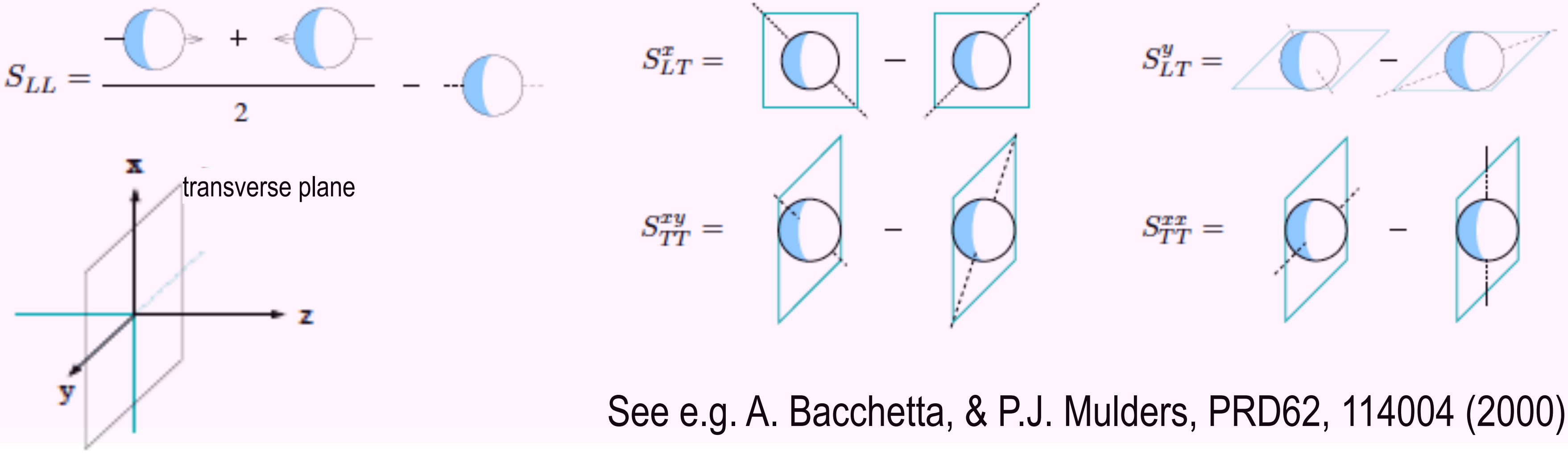
**Spin 1 hadrons:**

The spin density matrix is 3x3: $\rho = \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix} = \frac{1}{3}(1 + \frac{3}{2}\vec{S}\cdot\vec{\Sigma} + 3T^{ij}\Sigma^{ij})$

Vector polarization: $S^{\mu} = (0, \vec{S}_T, \lambda)$ 3

Tensor polarization: $S_{LL}$, $S_{LT}^{\mu} = (0, S_{LT}^{x}, S_{LT}^{y}, 0)$, $S_{TT}^{x\mu} = (0, S_{TT}^{xx}, S_{TT}^{xy}, 0)$ 5

$\left.\begin{matrix} 3 \\ 5 \end{matrix}\right\} 8$ independent components.



See e.g. A. Bacchetta, & P.J. Mulders, PRD62, 114004 (2000).

# Measurements of different components of polarization

Both { hyperon (vector) polarization $P_h$ &
all five components of tensor polarization of vector meson } can be measured.

- ✧ E.g.: $\Lambda \to p\pi^-$ $$\frac{dN}{d\Omega} = \frac{1}{4\pi}(1+\alpha P\cos\theta)$$
- ✧ E.g.: $K^* \to K\pi$ $$\frac{dN}{d\Omega} = \frac{3}{8\pi}\Big[\frac{2}{3} - \frac{2}{3}S_{LL}(\cos^2\theta + \cos 2\theta) - S_{LT}^x \sin 2\theta\cos\varphi - S_{LT}^y \sin 2\theta \sin\varphi - S_{TT}^{xx}\sin^2\theta\cos 2\varphi - S_{TT}^{xy}\sin^2\theta\sin 2\varphi\Big]$$

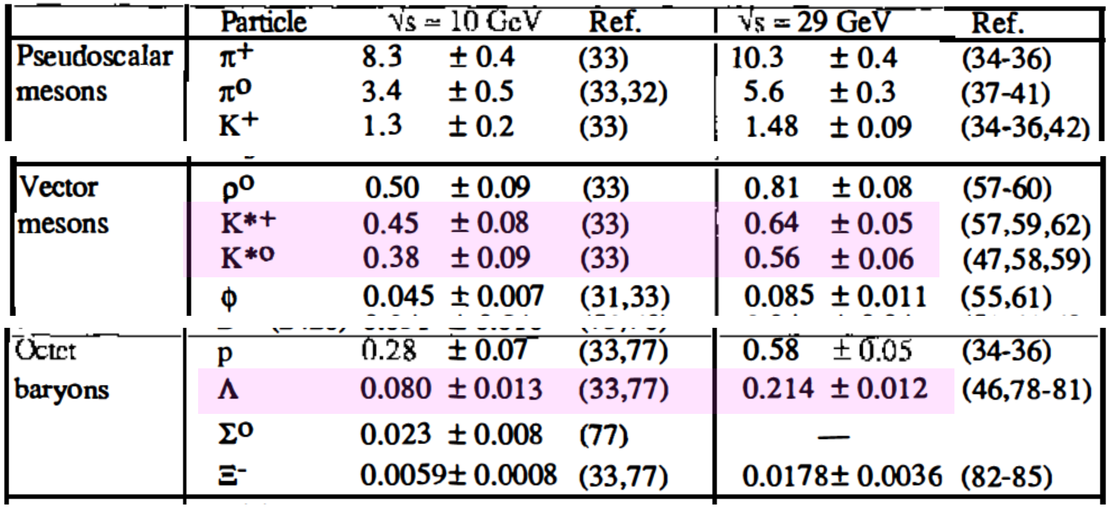
Advantages to study vector mesons:

(1) production rates are usually much higher than those of hyperons.

(2) decay contributions are negligible, much less than those of Λ.

Average yields of hadrons
in $e^+e^-$ annihilations

W. Hoffmann, Ann. Rev. Nucl. Part. Sci. 38, 279-322 (1988).



| | Particle | $\sqrt{s} \simeq 10$ GeV | Ref. | $\sqrt{s} = 29$ GeV | Ref. |
|---|---|---|---|---|---|
| Pseudoscalar mesons | $\pi^+$ | 8.3 ± 0.4 | (33) | 10.3 ± 0.4 | (34-36) |
| | $\pi^0$ | 3.4 ± 0.5 | (33,32) | 5.6 ± 0.3 | (37-41) |
| | $K^+$ | 1.3 ± 0.2 | (33) | 1.48 ± 0.09 | (34-36,42) |
| Vector mesons | $\rho^0$ | 0.50 ± 0.09 | (33) | 0.81 ± 0.08 | (57-60) |
| | $K^{*+}$ | 0.45 ± 0.08 | (33) | 0.64 ± 0.05 | (57,59,62) |
| | $K^{*0}$ | 0.38 ± 0.09 | (33) | 0.56 ± 0.06 | (47,58,59) |
| | $\phi$ | 0.045 ± 0.007 | (31,33) | 0.085 ± 0.011 | (55,61) |
| Octet baryons | p | 0.28 ± 0.07 | (33,77) | 0.58 ± 0.05 | (34-36) |
| | $\Lambda$ | 0.080 ± 0.013 | (33,77) | 0.214 ± 0.012 | (46,78-81) |
| | $\Sigma^0$ | 0.023 ± 0.008 | (77) | — | |
| | $\Xi^-$ | 0.0059 ± 0.0008 | (33,77) | 0.0178 ± 0.0036 | (82-85) |

# TMD FFs defined via quark-quark correlator (for spin-1/2 hadrons)

The quark-quark correlator $\hat{\Xi}^{(0)}(k_F;p,S)=\frac{1}{2\pi}\sum_X\int d^4\xi e^{-ik_F\xi}\langle hX|\bar{\psi}(\xi)\mathcal{L}(\xi,\infty)|0\rangle\langle 0|\mathcal{L}^\dagger(0,\infty)\psi(0)|hX\rangle$

integrate over $k_F^-$ : $\hat{\Xi}^{(0)}(z,k_{F\perp};p,S)=\frac{1}{2\pi}\sum_X\int p^+d\xi^-d^2\xi_\perp e^{-i(p^+\xi^-/z-\vec{k}_{F\perp}\cdot\vec{\xi}_\perp)}\langle hX|\bar{\psi}(\xi)\mathcal{L}(\xi,\infty)|0\rangle\langle 0|\mathcal{L}^\dagger(0,\infty)\psi(0)|hX\rangle$

## Expansion in terms of the Γ-matrices

$\hat{\Xi}^{(0)}(z,k_{F\perp};p,S)=\Xi^{(0)}(z,k_{F\perp};p,S)$ ———— scalar

$+i\gamma_5\ \tilde{\Xi}^{(0)}(z,k_{F\perp};p,S)$ ———— pseudo-scalar

$+\gamma^\alpha\ \Xi_\alpha^{(0)}(z,k_\perp;p,S)$ ———— vector

$+\gamma_5\gamma^\alpha\ \tilde{\Xi}_\alpha^{(0)}(z,k_{F\perp};p,S)$ ———— axial vector

$+i\gamma_5\sigma^{\alpha\beta}\ \Xi_{\alpha\beta}^{(0)}(z,k_{F\perp};p,S)$ ———— tensor

e.g.:
$$\Xi_\alpha^{(0)}(z,k_{F\perp};p,S)=\frac{1}{4}\mathrm{Tr}\left[\gamma_\alpha\hat{\Xi}^{(0)}(z,k_{F\perp};p,S)\right]$$
$$=\frac{1}{2\pi}\sum_X\int p^+d\xi^-d^2\xi_\perp e^{-i(p^+\xi^-/z-\vec{k}_{F\perp}\cdot\vec{\zeta}_\perp)}\langle hX|\bar{\psi}(\xi)\mathcal{L}(\xi,\infty)\frac{\gamma_\alpha}{4}|0\rangle\langle 0|\mathcal{L}^\dagger(0,\infty)\psi(0)|hX\rangle$$

# TMD FFs defined via quark-quark correlator (for spin-1/2 hadrons)

**The Lorentz decomposition**

totally 8(twist 2)+16(twist 3)+8(twist 4) components

$$\Xi_S^{(0)}(z,k_{F\perp};p,S)=ME(z,k_{F\perp})+(\tilde{k}_{F\perp}\cdot S_T)E_T^{\perp}(z,k_{F\perp})$$

twist-3

$$\Xi_\alpha^{(0)}(z,k_{F\perp};p,S)=\frac{p^+}{M}\bar{n}_\alpha\Big[MD_1(z,k_{F\perp})+(\tilde{k}_{F\perp}\cdot S_T)D_{1T}^{\perp}(z,k_{F\perp})\Big]$$

twist-2

$$+\ k_{F\perp\alpha}D^{\perp}(z,k_{F\perp})+M\tilde{S}_{T\alpha}D_T(z,k_{F\perp})+\frac{\tilde{k}_{F\perp\alpha}}{M}\Big[\lambda MD_L^{\perp}(z,k_{F\perp})+(k_{F\perp}\cdot S_T)D_T^{\perp}(z,k_{F\perp})\Big]$$

$$+\frac{M}{p^+}n_\alpha\Big[MD_3(z,k_{F\perp})+(\tilde{k}_{F\perp}\cdot S_T)D_{3T}^{\perp}(z,k_{F\perp})\Big]$$

twist-4

$$p=p^+\bar{n}+\frac{M^2}{2p^+}n,\qquad S=\lambda\frac{p^+}{M}\bar{n}+S_T-\lambda\frac{M^2}{2p^+}n,\qquad \tilde{k}_{\perp\alpha}\equiv\varepsilon_{\perp\rho\alpha}k_\perp^\rho$$

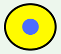
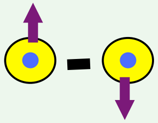
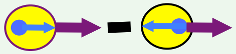
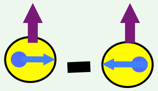
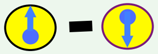
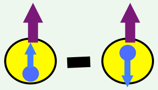
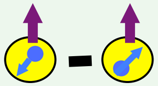
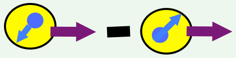
E.g.: $$D_1(z,k_{F\perp})=\frac{1}{2\pi}\sum_X\int d\xi^- d^2\xi_\perp e^{-i(p^+\xi^-/z-\vec{k}_{F\perp}\cdot\vec{\zeta}_\perp)}\langle hX|\bar{\psi}(\xi)\mathcal{L}(\xi,\infty)\frac{\gamma^+}{4}|0\rangle\langle 0|\mathcal{L}^\dagger(0,\infty)\psi(0)|hX\rangle$$

See e.g., K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD94, 034003 (2016).

# Twist-2 TMD FFs defined via quark-quark correlator (spin-1/2)

Leading twist (twist 2)

$D, G, H:$ quark un-, longitudinally, transversely polarized

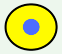
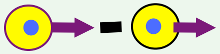
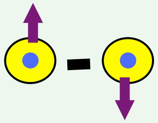
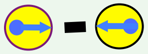
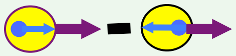
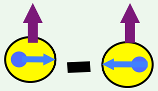
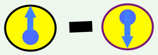
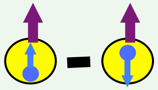
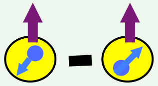
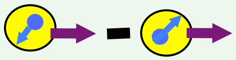
| | polarization | | TMD FFs (8) | integrated over $k_{F\perp}$ | name |
|---|---|---|---|---|---|
| quark | hadron | pictorially | | | |
| $U$ | $U$ |  | $D_1(z,k_{F\perp})$ | $D_1(z)$ | number density |
| | $T$ |  | $D_{1T}^{\perp}(z,k_{F\perp})$ | × | Sivers-type function |
| $L$ | $L$ |  | $G_{1L}(z,k_{F\perp})$ | $G_{1L}(z)$ | spin transfer (longitudinal) |
| | $T$ |  | $G_{1T}^{\perp}(x,k_{\perp})$ | × | |
| $T$ | $U$ |  | $H_1^{\perp}(z,k_{F\perp})$ | × | Collins function |
| | $T(//)$ |  | $H_{1T}(z,k_{F\perp})$ | $H_{1T}(z)$ | spin transfer (transverse) |
| | $T(\perp)$ |  | $H_{1T}^{\perp}(z,k_{F\perp})$ | | |
| | $L$ |  | $H_{1L}^{\perp}(z,k_{F\perp})$ | × | |

See e.g., K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD94, 034003 (2016).

# Twist-3 TMD FFs defined via quark-quark correlator (spin-1/2)

## Next to the leading twist (twist-3)

they are NOT probability distributions but include the quantum interference effects.

| polarization | | | TMD FFs (16) | integrated over $k_{F\perp}$ | name |
|---|---|---|---|---|---|
| quark | hadron | pictorially | | | |
| $U$ | $U$ |  | $E(z,k_{F\perp})$, $D^{\perp}(z,k_{F\perp})$ | $E(z)$, × | number density |
| | $L$ |  | $D_L^{\perp}(z,k_{F\perp})$ | × | Sivers-type function |
| | $T$ |  | $E_T^{\perp}(z,k_{F\perp})$,<br>$D_T(z,k_{F\perp})$, $D_T^{\perp}(z,k_{F\perp})$ | ×<br>×<br>$D_T(z)$ | |
| $L$ | $U$ |  | $G^{\perp}(z,k_{F\perp})$ | × | |
| | $L$ |  | $E_L(z,k_{F\perp})$, $G_L^{\perp}(z,k_{F\perp})$ | $E_L(z)$, × | spin transfer (longitudinal) |
| | $T$ |  | $E'^{\perp}_T(z,k_{F\perp})$,<br>$G_T(z,k_{F\perp})$, $G_T^{\perp}(z,k_{F\perp})$ | ×<br>$G_T(z)$ | |
| $T$ | $U$ |  | $H(z,k_{F\perp})$ | $H(z)$ | Collins function |
| | $T(//)$ |  | $H_T^{\perp}(z,k_{F\perp})$ | × | spin transfer (transverse) |
| | $T(\perp)$ |  | $H_T(z,k_{F\perp})$ | × | |
| | $L$ |  | $H_L(z,k_{F\perp})$ | $H_L(z)$ | |

# TMD FFs defined via quark-quark correlator (T-dep. part)

## The Lorentz decomposition

totally 10(twist-2)+20(twist-3)+10(twist-4) components

The tensor polarization dependent part

$$z\Xi_S^{T(0)}(z,k_{F\perp};p,S) = MS_{LL}E_{LL}(z,k_{F\perp}) + (k_{F\perp}\cdot S_{LT})E_{LT}^{\perp}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M}E_{TT}^{\perp}(z,k_{F\perp})$$

twist-3

$$z\tilde{\Xi}_S^{T(0)}(z,k_{F\perp};p,S) = (\tilde{k}_{F\perp}\cdot S_{LT})E'^{\perp}_{LT}(z,k_{F\perp}) + \frac{S_{TT}^{\tilde{k}_F k_F}}{M}E'^{\perp}_{TT}(z,k_{F\perp})$$

$$z\Xi_\alpha^{T(0)}(z,k_{F\perp};p,S) = \frac{p^+}{M}\bar{n}_\alpha\left[MS_{LL}D_{1LL}(z,k_{F\perp}) + (k_{F\perp}\cdot S_{LT})D_{1LT}^{\perp}(z,k_{F\perp})\right]$$

twist-2

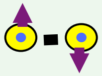
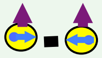
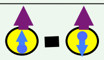
$$+ MS_{LT\alpha}D_{LT}(z,k_{F\perp}) + S_{TT\alpha}D'^{\perp}_{TT}(z,k_{F\perp}) + \frac{k_{F\perp\alpha}}{M}\left[MS_{LL}D_{LL}^{\perp}(z,k_{F\perp}) + (k_{F\perp}\cdot S_{LT})D_{LT}^{\perp}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M}D_{TT}^{\perp}(z,k_{F\perp})\right]$$

$$+ \frac{M}{p^+}n_\alpha\left[MS_{LL}D_{3LL}(z,k_{F\perp}) + (k_{F\perp}\cdot S_{LT})D_{3LT}^{\perp}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M}D_{3TT}^{\perp}(z,k_{F\perp})\right]$$

twist-4

See e.g., K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD94, 034003 (2016).

# Twist-2 TMD FFs defined via quark-quark correlator (spin-1)

| Quark pol | Hadron pol | | TMD FFs (2+6+10=18) | integrated over $k_{F\perp}$ | name |
|---|---|---|---|---|---|
| $U$ | $U$ |  | $D_1(z,k_{F\perp})$ | $D_1(z)$ | number density |
| | $T$ |  | $D_{1T}^{\perp}(z,k_{F\perp})$ | × | Sivers-type function |
| | $LL$ | | $D_{1LL}(z,k_{F\perp})$ | $D_{1LL}(z)$ | spin alignment |
| | $LT$ | | $D_{1LT}^{\perp}(z,k_{F\perp})$ | × | |
| | $TT$ | | $D_{1TT}^{\perp}(z,k_{F\perp})$ | × | |
| $L$ | $L$ |  | $G_{1L}(z,k_{F\perp})$ | $G_{1L}(z)$ | spin transfer (longitudinal) |
| | $T$ |  | $G_{1T}^{\perp}(z,k_{F\perp})$ | × | |
| | $LT$ | | $G_{1LT}^{\perp}(z,k_{F\perp})$ | × | |
| | $TT$ | | $G_{1TT}^{\perp}(z,k_{F\perp})$ | × | |
| $T$ | $U$ |  | $H_1^{\perp}(z,k_{F\perp})$ | × | Collins function |
| | $T(//)$ |  | $H_{1T}(z,k_{F\perp})$ | $H_{1T}(z)$ | spin transfer (transverse) |
| | $T(\perp)$ |  | $H_{1T}^{\perp}(z,k_{F\perp})$ | | |
| | $L$ |  | $H_{1L}^{\perp}(z,k_{F\perp})$ | × | |
| | $LL$ | | $H_{1LL}^{\perp}(z,k_{F\perp})$ | × | |
| | $LT$ | | $H_{1LT}(z,k_{F\perp}),\ H_{1LT}^{\perp}(z,k_{F\perp})$ | $H_{1LT}(z)$ | |
| | $TT$ | | $H_{1TT}^{\perp}(z,k_{F\perp}),\ H'^{\perp}_{1TT}(z,k_{F\perp})$ | ×, × | |

See e.g., K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD94, 034003 (2016).

# Twist-3 TMD FFs defined via quark-quark correlator (spin-1)

| Quark pol | Hadron pol | TMD FFs (4+12+20=36) | integrated over $k_{F\perp}$ | name |
|---|---|---|---|---|
| $U$ | $U$ | $E(z,k_{F\perp}),\ D^{\perp}(z,k_{F\perp})$ | $E(z)$, × | number density |
| | $L$ | $D_L^{\perp}(z,k_{F\perp})$ | × | Sivers-type function |
| | $T$ | $E_T^{\perp}(z,k_{F\perp}),\ D_T(z,k_{F\perp}),\ D_T^{\perp}(z,k_{F\perp})$ | ×, $D_T(z)$ | |
| | $LL$ | $E_{LL}(z,k_{F\perp}),\ D_{LL}^{\perp}(z,k_{F\perp})$ | $E_{LL}(z)$, × | spin alignment |
| | $LT$ | $E_{LT}^{\perp}(z,k_{F\perp}),\ D_{LT}(z,k_{F\perp}),\ D_{LT}^{\perp}(z,k_{F\perp})$ | ×, $D_{LT}(z)$ | |
| | $TT$ | $E_{TT}^{\perp}(z,k_{F\perp}),\ D_{TT}^{\perp}(z,k_{F\perp}),\ D'^{\perp}_{TT}(z,k_{F\perp})$ | ×, ×, × | |
| $L$ | $U$ | $G^{\perp}(z,k_{F\perp})$ | × | |
| | $L$ | $E_L(z,k_{F\perp}),\ G_L^{\perp}(z,k_{F\perp})$ | $E_L(z)$, × | spin transfer (longitudinal) |
| | $T$ | $E'^{\perp}_T(z,k_{F\perp}),\ G_T(z,k_{F\perp}),\ G_T^{\perp}(z,k_{F\perp})$ | ×, $G_T(z)$ | |
| | $LL$ | $G_{LL}^{\perp}(z,k_{F\perp})$ | × | |
| | $LT$ | $E'^{\perp}_{LT}(z,k_{F\perp}),\ G_{LT}(z,k_{F\perp}),\ G_{LT}^{\perp}(z,k_{F\perp})$ | ×, $G_{LT}(z)$ | |
| | $TT$ | $E'^{\perp}_{TT}(z,k_{F\perp}),\ G_{TT}^{\perp}(z,k_{F\perp}),\ G'^{\perp}_{TT}(z,k_{F\perp})$ | ×, ×, × | |
| $T$ | $U$ | $H(z,k_{F\perp})$ | $H(z)$ | Collins function |
| | $T(//)$ | $H_T^{\perp}(z,k_{F\perp})$ | × | spin transfer (transverse) |
| | $T(\perp)$ | $H'^{\perp}_T(z,k_{F\perp})$ | × | |
| | $L$ | $H_L(z,k_{F\perp})$ | $H_L(z)$ | |
| | $LL$ | $H_{LL}(z,k_{F\perp})$ | $H_{LL}(z)$ | |
| | $LT$ | $H_{LT}^{\perp}(z,k_{F\perp}),\ H'^{\perp}_{LT}(z,k_{F\perp})$ | ×, × | |
| | $TT$ | $H_{TT}^{\perp}(z,k_{F\perp}),\ H'^{\perp}_{TT}(z,k_{F\perp})$ | ×, × | |

# Twist-2 TMD FFs defined via quark-quark correlator (spin-1)

## Classified according to the polarization of the quark:

| | |
|---|---|
| Unpolaried quark | $D_1, D_{1T}^\perp, D_{1LL}, D_{1LT}^\perp, D_{1TT}^\perp$ |
| Longitudinally polarized quark | $G_{1L}, G_{1T}^\perp, G_{1LT}^\perp, G_{1TT}^\perp$ |
| Transversely polarized quark | $H_1^\perp, H_{1T}, H_{1T}^\perp, H_{1L}^\perp, H_{1LL}^\perp, H_{1LT}, H_{1LT}^\perp, H_{1TT}^\perp, H'^\perp_{1TT}$ |

## Classified according to the polarization of the hadron:

spin-1 hadron

spin-1/2 hadron

spin-0 hadron

$$\hat{\Xi}(z,k_{F\perp};p,S) = \hat{\Xi}(z,k_{F\perp};p\,|\,Upol) + \hat{\Xi}(z,k_{F\perp};p\,|\,Vpol) + \hat{\Xi}(z,k_{F\perp};p\,|\,Tpol)$$

| | | | | |
|---|---|---|---|---|
| number density: | | $D_1, H_1^\perp$ | | |
| induced polarization: | | | $D_{1T}^\perp$ | $D_{1LL}, D_{1LT}^\perp, D_{1TT}^\perp$ |
| spin transfer | "direct": | | $G_{1L}; H_{1T}, H_{1T}^\perp$ | |
| | "worm gear": | | $G_{1T}^\perp; H_{1L}^\perp$ | $G_{1LT}^\perp, G_{1TT}^\perp; H_{1LL}^\perp, H_{1LT}, H_{1LT}^\perp, H_{1TT}^\perp, H'^\perp_{1TT}$ |

Their contributions to the cross section are additive.

# Twist-3 TMD FFs defined via quark-gluon-quark correlator

## 36 twist-3 TMD FFs for spin-1 hadrons

QCD equation of motion $\gamma \cdot D(z)\psi(z) = 0$

⇨ totally 36 (real) equations.

chiral even: $$D_{dS}^{K}(z,k_{F\perp}) + G_{dS}^{K}(z,k_{F\perp}) = \frac{1}{z}\left[D_{S}^{K}(z,k_{F\perp}) + iG_{S}^{K}(z,k_{F\perp})\right]$$

$K = \text{null}, \quad S = T \text{ or } LT; \qquad K = {'\perp}, \quad S = TT;$

$K = \perp, \quad S = \text{null}, L, T, LL, LT, \text{ or } TT.$

chiral odd: $$H_{dS}^{K}(z,k_{F\perp}) + \frac{k_\perp^2}{2M^2} H_{dS}^{K'}(z,k_{F\perp}) = \frac{1}{2z}\left[H_{S}^{K}(z,k_{F\perp}) - iE_{S}^{K}(z,k_{F\perp})\right]$$

$(K, K') = (\text{null}, \perp), \quad S = \text{null}, L, \text{ or } LL;$

$(K, K') = (\perp, \perp') \text{ or } ('\perp, '\perp'), \quad S = T, \; LT \text{ or } TT.$

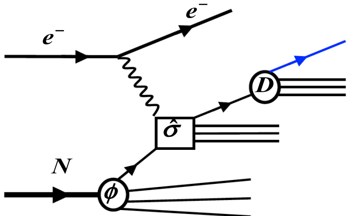
See e.g., K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD (2016); S.Y. Wei, K.B. Chen, Y.K. Song,& ZTL, PRD(2015).

# Contents

# Access TMDs via semi-inclusive high energy reactions

## Semi-inclusive high energy reactions

SiDIS:



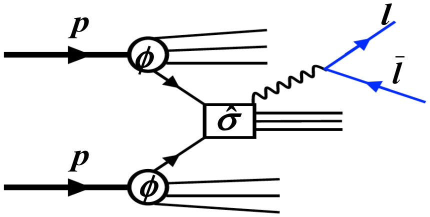
$e+N \to e+h+X$

DESY COMPASS Jefferson Lab EIC →

PDFs: $f_1, f_{1T}^{\perp}, g_{1L}, h_1, h_{1L}^{\perp}, h_{1T}^{\perp}\ldots$

FFs: $D_1, D_{1T}^{\perp}, G_{1L}, G_{1T}^{\perp}, H_1^{\perp}, \ldots$

Drell-Yan:

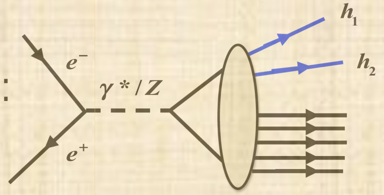


$p+p \to l+\bar{l}+X$

FNAL BNL JPARC GSI →

PDFs: $f_1, f_{1T}^{\perp}, g_{1L}, h_1, h_{1L}^{\perp}, h_{1T}^{\perp}\ldots$

$e^-e^+$ annihilation:



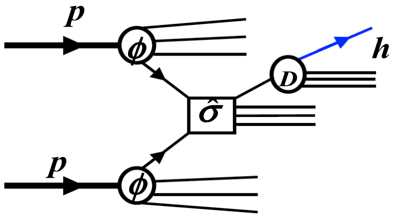
$e^-e^+ \to h_1h_2X$

BESIII BELLE CERN BABAR →

FFs: $D_1, D_{1T}^{\perp}, G_{1L}, G_{1T}^{\perp}, H_1^{\perp}, \ldots$

## Inclusive hadron production in hadron-hadron collisions $h_1+h_2 \to h+X$

$pp \to hX$



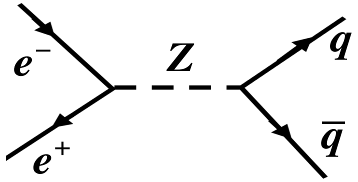
FNAL BNL JPARC GSI →

PDFs: $f_1, f_{1T}^{\perp}, g_{1L}, h_1, h_{1L}^{\perp}, h_{1T}^{\perp}\ldots$

FFs: $D_1, D_{1T}^{\perp}, G_{1L}, G_{1T}^{\perp}, H_1^{\perp}, \ldots$

# Quark polarization in $e^+e^- \to q\bar{q}$

## At the Z-pole: $e^+e^- \to Z \to q\bar{q}$



The cross section: $$\frac{d\hat{\sigma}^{ZZ}}{d\Omega} = \frac{\alpha^2}{4s}\chi\left[c_1^e c_1^q(1+\cos^2\theta) + 2c_3^e c_3^q \cos\theta\right]$$

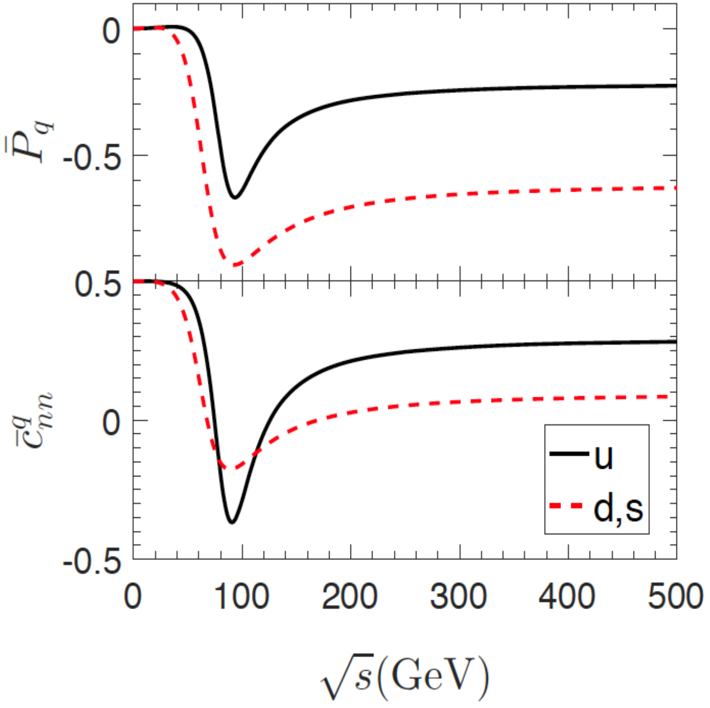
Longitudinal polarization of $q$ or $\bar{q}$ : $$\bar{P}_q^{ZZ} = -\frac{c_3^q}{c_1^q}$$

Correlation of transverse polarizations of $q$ and $\bar{q}$ : $$\bar{c}_{nn}^{qZZ} = \frac{c_2^q}{2c_1^q}$$

## At any energy: $e^+e^- \to \gamma^*/Z \to q\bar{q}$

$$\frac{d\hat{\sigma}}{d\Omega} = \frac{d\hat{\sigma}^{ZZ}}{d\Omega} + \frac{d\hat{\sigma}^{Z\gamma}}{d\Omega} + \frac{d\hat{\sigma}^{\gamma\gamma}}{d\Omega}$$

$$\bar{P}_q = -\frac{\chi c_1^e c_3^q + \chi_{\text{int}}^q c_V^e c_A^q}{e_q^2 + \chi c_1^e c_1^q + \chi_{\text{int}}^q c_V^e c_V^q}, \qquad \bar{c}_{nn}^q = \frac{e_q^2 + \chi c_1^e c_2^q + \chi_{int}^q c_V^e c_V^q}{2(e_q^2 + \chi c_1^e c_1^q + \chi_{\text{int}}^q c_V^e c_V^q)}$$



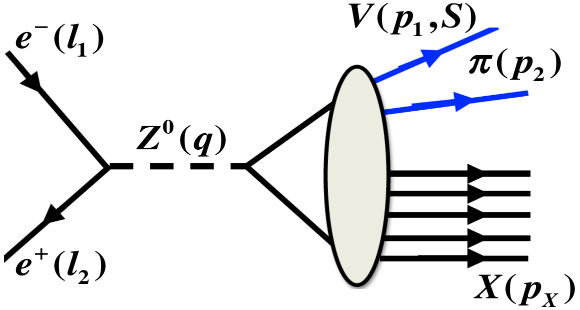
We can study FFs of unpolarized / longitudinally polarized quark in singly "polarized" process $e^-e^+ \to h_1(\uparrow) + h_2 + X$

transversely polarized quark in doubly "polarized" process $e^-e^+ \to h_1(\uparrow) + h_2(\uparrow) + X$

# Access polarization dependent FFs in singly polarized $e^-e^+ \to h_1\pi X$

$e^-e^+ \to Z \to V\pi X$ : the best place to study tensor polarization dependent FFs

**The general kinematic analysis**



$$\frac{2E_1E_2}{d^3p_1d^3p_2} = \frac{\alpha^2}{sQ^4}\chi L_{\mu\nu}(l_1,l_2)W^{\mu\nu}(q,p_1,S,p_2)$$

$$L_{\mu\nu}(l_1,l_2) = c_1^e\left[l_{1\mu}l_{2\nu} + l_{1\nu}l_{2\mu} - (l_1\cdot l_2)g_{\mu\nu}\right] + ic_3^e\varepsilon_{\mu\nu\rho\sigma}l_1^\rho l_2^\sigma$$

The hadronic tensor:

$W_{\mu\nu}(q,p_1,S,p_2) = W^{S\mu\nu}$ (the Symmetric part) $+iW^{A\mu\nu}$ (the Anti-symmetric part)

$$= \sum_{\sigma,i} W^S_{\sigma i}\ h^{S\mu\nu}_{\sigma i} + \sum_{\sigma,j} \tilde{W}^S_{\sigma j}\ \tilde{h}^{S\mu\nu}_{\sigma j} + i\sum_{\sigma,i} W^A_{\sigma i}\ h^{A\mu\nu}_{\sigma i} + i\sum_{\sigma,j} \tilde{W}^A_{\sigma j}\ \tilde{h}^{A\mu\nu}_{\sigma j}$$

$\sigma = U,V,S_{LL},S_{LT},S_{TT}$

polarization

the basic Lorentz tensors (BLTs):

$h^{S\mu\nu}_{\sigma i} = h^{S\nu\mu}_{\sigma i}$, $h^{A\mu\nu}_{\sigma i} = -h^{A\nu\mu}_{\sigma i}$ space reflection P-even: $\hat{\mathcal{P}}h^{\mu\nu} = h_{\mu\nu}$

$\tilde{h}^{S\mu\nu}_{\sigma i} = \tilde{h}^{S\nu\mu}_{\sigma i}$, $\tilde{h}^{A\mu\nu}_{\sigma i} = -\tilde{h}^{A\nu\mu}_{\sigma i}$ space reflection P-odd: $\hat{\mathcal{P}}\tilde{h}^{\mu\nu} = -\tilde{h}_{\mu\nu}$

Constraints：$W^{\mu\nu *} = W^{\nu\mu}$ (hermiticity), $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$ (current conservation)

See: D. Pitonyak, M. Schlegel, and A. Metz, PRD 89, 054032 (2014) (spin-1/2);
K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD95, 034003 (2016) (spin-1).

# General kinematic analysis for $e^+e^- \to V\pi X$

## The basic Lorentz tensors (BLTs) for the hadronic tensor

unpolarized part: 5+4=9

$$h_{Ui}^{S\mu\nu} = \left\{ g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2},\ p_{1q}^\mu p_{1q}^\nu,\ p_{2q}^\mu p_{2q}^\nu,\ p_{1q}^{\{\mu} p_{2q}^{\nu\}} \right\}$$

$$\tilde{h}_{Ui}^{S\mu\nu} = \left\{ \varepsilon^{\{\mu q p_1 p_2}(p_{1q}^{\nu\}},\ p_{2q}^{\nu\}}) \right\}$$

$$h_{U}^{A\mu\nu} = p_{1q}^{[\mu} p_{2q}^{\nu]}$$

$$\tilde{h}_{Ui}^{A\mu\nu} = \left\{ \varepsilon^{\mu\nu q p_1},\ \varepsilon^{\mu\nu q p_2} \right\}$$

$$\varepsilon^{\mu\nu\alpha p} \equiv \varepsilon^{\mu\nu\alpha\beta} p_\beta,\quad \varepsilon_\perp^{\mu\nu} \equiv \varepsilon^{\mu\nu\alpha\beta}\bar{n}_\alpha n_\beta$$

$$p_q \equiv p - \frac{p\cdot q}{q^2}q \qquad (p_q \cdot q = 0)$$

Vector polarization S-dependent part: 13+14=27

$$h_{Vi}^{S\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] \tilde{h}_{Ui}^{S\mu\nu},\ \ \varepsilon^{Sqp_1p_2} h_{Uj}^{S\mu\nu} \right\}$$

$$\tilde{h}_{Vi}^{S\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] h_{Ui}^{S\mu\nu},\ \ \varepsilon^{Sqp_1p_2} \tilde{h}_{Uj}^{S\mu\nu} \right\}$$

$$h_{Vi}^{A\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] \tilde{h}_{Ui}^{A\mu\nu},\ \ \varepsilon^{Sqp_1p_2} h_{U}^{A\mu\nu} \right\}$$

$$\tilde{h}_{Vi}^{A\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] h_{U}^{A\mu\nu},\ \ \varepsilon^{Sqp_1p_2} \tilde{h}_{Uj}^{A\mu\nu} \right\}$$

The regularity: (polarization dependent Lorentz tensor set) = (polarization dependent Lorentz (pseudo)scalar) × (the unpolarized set)

unpolarized

$$\begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix}$$

longitudinal polarization

$$\begin{pmatrix} h_{Li}^{S\mu\nu} \\ \tilde{h}_{Li}^{S\mu\nu} \\ h_{Li}^{A\mu\nu} \\ \tilde{h}_{Li}^{A\mu\nu} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \\ h_{U}^{A\mu\nu} \end{pmatrix}$$

transverse polarization

$$\begin{pmatrix} h_{Ti}^{S\mu\nu} \\ \tilde{h}_{Ti}^{S\mu\nu} \\ h_{Ti}^{A\mu\nu} \\ \tilde{h}_{Ti}^{A\mu\nu} \end{pmatrix} = \left\{ (p_2\cdot S) \begin{pmatrix} \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \\ h_{U}^{A\mu\nu} \end{pmatrix},\quad \varepsilon^{Sqp_1p_2} \begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{U}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix} \right\}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The basic Lorentz tensors (BLTs) for the hadronic tensor (continued)

$$S_{TT}^{p\beta} \equiv p_\alpha S_{TT}^{\alpha\beta}$$

$$\varepsilon^{abcd} \equiv \varepsilon^{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta$$

### $S_{LL}$-dependent part: 5+4=9

$$S_{LL}^{\mathcal{P}} = S_{LL}$$

$$\begin{pmatrix} h_{LLi}^{S\mu\nu} \\ \tilde{h}_{LLi}^{S\mu\nu} \\ h_{LLi}^{A\mu\nu} \\ \tilde{h}_{LLi}^{A\mu\nu} \end{pmatrix} = S_{LL} \begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{U}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix}$$

### $S_{LT}$-dependent part: 9+9=18

$$S_{LT} = (0, S_{LT}^x, S_{LT}^y, 0)$$

$$p_1 \cdot S_{LT} = 0, \quad q \cdot S_{LT} = 0 \qquad S_{LT\mu}^{\mathcal{P}} = S_{LT}^{\mu}$$

$$\begin{pmatrix} h_{LTi}^{S\mu\nu} \\ \tilde{h}_{LTi}^{S\mu\nu} \\ h_{LTi}^{A\mu\nu} \\ \tilde{h}_{LTi}^{A\mu\nu} \end{pmatrix} = \left\{ (p_2 \cdot S_{LT}) \begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{U}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix}, \quad \varepsilon^{S_{LT} q p_1 p_2} \begin{pmatrix} \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \\ h_{U}^{A\mu\nu} \end{pmatrix} \right\}$$

### $S_{TT}$-dependent part: 9+9=18

$$S_{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{TT}^{xx} & S_{TT}^{xy} & 0 \\ 0 & S_{TT}^{xy} & -S_{TT}^{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{TT\mu\nu}^{\mathcal{P}} = S_{TT}^{\mu\nu}$$

$$S_{TT}^{p_1\beta} = S_{TT}^{\alpha p_1} = 0$$

$$S_{TT}^{q\beta} = S_{TT}^{\alpha q} = 0$$

$$\begin{pmatrix} h_{TTi}^{S\mu\nu} \\ \tilde{h}_{TTi}^{S\mu\nu} \\ h_{TTi}^{A\mu\nu} \\ \tilde{h}_{TTi}^{A\mu\nu} \end{pmatrix} = \left\{ S_{TT}^{p_2 p_2} \begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{U}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix}, \quad \varepsilon^{S_{TT}^{p_2} q p_1 p_2} \begin{pmatrix} \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \\ h_{U}^{A\mu\nu} \end{pmatrix} \right\}$$

See K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD95, 034003 (2016).

# General kinematic analysis for $e^+e^- \to V\pi X$

## The cross section in terms of “structure functions” (the Lorentz invariant form)

The unpolarized part:

$$\frac{2E_1E_2 d\sigma^U}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi(\mathcal{F}_U + \tilde{\mathcal{F}}_U)$$

$$y_1 = \frac{2l_1 \cdot p_1}{Q^2}, \quad y_2 = \frac{2l_1 \cdot p_2}{Q^2}, \quad \tilde{y} = \frac{\varepsilon^{l_1 q p_1 p_2}}{Q^4}$$

$$\mathcal{F}_U = F_U^0 + y_1 F_U^1 + y_2 F_U^2 + y_1^2 F_U^{11} + y_2^2 F_U^{22} + y_1 y_2 F_U^{12}$$

$$\tilde{\mathcal{F}}_U = \tilde{y}(\tilde{F}_U^0 + y_1 \tilde{F}_U^1 + y_2 \tilde{F}_U^2)$$

$F_U^i$'s are combinations of $W_{Ui}$'s with coefficients such as $c_i^e$'s

All others have the same form:

$$\frac{2E_1E_2 d\sigma^V}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big[(q\cdot S)(\mathcal{F}_{V1} + \tilde{\mathcal{F}}_{V1}) + (p_2 \cdot S)(\mathcal{F}_{V2} + \tilde{\mathcal{F}}_{V2}) + \varepsilon^{S q p_1 p_2}(\mathcal{F}_{V3} + \tilde{\mathcal{F}}_{V3})\Big]$$

$$\frac{2E_1E_2 d\sigma^{LL}}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi S_{LL}(\mathcal{F}_{LL} + \tilde{\mathcal{F}}_{LL})$$

$$\frac{2E_1E_2 d\sigma^{LT}}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big[(p_2 \cdot S_{LT})(\mathcal{F}_{LT1} + \tilde{\mathcal{F}}_{LT1}) + \varepsilon^{S_{LT} q p_1 p_2}(\mathcal{F}_{LT2} + \tilde{\mathcal{F}}_{LT2})\Big]$$

$$\frac{2E_1E_2 d\sigma^{TT}}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big[S_{TT}^{p_2 p_2}(\mathcal{F}_{TT1} + \tilde{\mathcal{F}}_{TT1}) + \varepsilon^{S_{TT}^{p_2} q p_1 p_2}(\mathcal{F}_{TT2} + \tilde{\mathcal{F}}_{TT2})\Big]$$

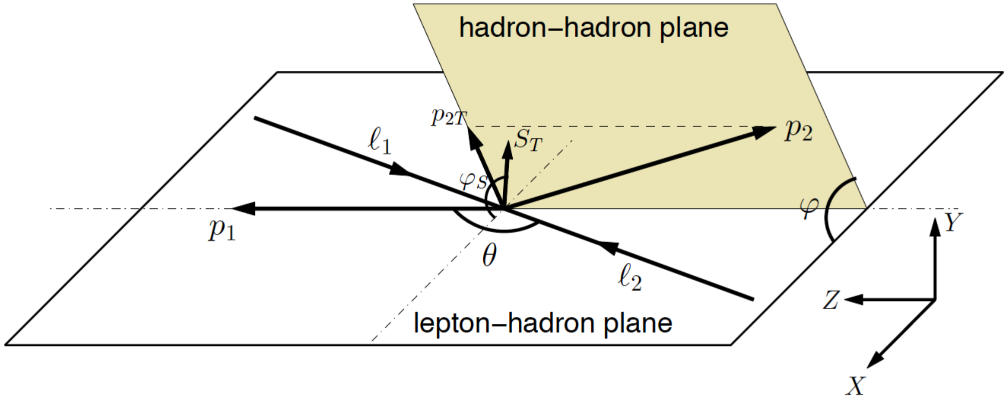
$$\mathcal{F}_U \leftrightarrow \mathcal{F}_\sigma$$

$$\tilde{\mathcal{F}}_U \leftrightarrow \tilde{\mathcal{F}}_\sigma$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The Helicity-Gottfried-Jackson (Helicity-GJ) frame

- c.m. frame of $e^+e^-$
- $p_1$ in $z$-direction
- lepton-hadron plane = oxz plane ($e^-$-$V$-plane)



$$V: \quad p_1 = (E_1,\ 0,\ 0,\ p_{1z})$$

$$\pi: \quad p_2 = (E_2,\ |\vec{p}_{2T}|\cos\varphi,\ |\vec{p}_{2T}|\sin\varphi,\ p_{2z})$$

$$e^-: \quad l_1 = Q(1,\ \sin\theta,\ 0,\ \cos\theta)/2$$

$$e^+: \quad l_2 = Q(1,\ \text{-}\sin\theta,\ 0,\ \text{-}\cos\theta)/2$$

$$Z: \quad q = l_1 + l_2 = Q(1,\ 0,\ 0,\ 0)$$

independent variables

$$s = q^2 = Q^2$$

$$\xi_1 = 2q \cdot p_1 / Q^2$$

$$\xi_2 = 2q \cdot p_2 / Q^2$$

$$\theta \quad \text{or} \quad y = 2l_2 \cdot p_1 / Q^2$$

$$p_{2T} \equiv |\vec{p}_{2T}|, \quad \varphi$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The cross section in Helicity-GJ-frame: unpolarized and longitudinally polarized parts

The structure functions: $F^{yy}_{jxx} = F^{yy}_{jxx}(s,\xi_1,\xi_2,p_{2T})$

$\tilde{F}^{yy}_{jxx} = \tilde{F}^{yy}_{jxx}(s,\xi_1,\xi_2,p_{2T})$

$$\frac{2E_1E_2\,d\sigma^U}{d^3p_1\,d^3p_2} = \frac{\alpha^2}{s^2}\chi(\mathcal{F}_U + \tilde{\mathcal{F}}_U)$$

$$\mathcal{F}_U = (1+\cos^2\theta)F_{1U} + \sin^2\theta F_{2U} + \cos\theta F_{3U}$$
$$+\cos\varphi[\sin\theta F^{\cos\varphi}_{1U} + \sin 2\theta F^{\cos\varphi}_{2U}]$$
$$+\cos 2\varphi \sin^2\theta F^{\cos 2\varphi}_U$$

$1$

$\cos\varphi$

$\cos 2\varphi$

$$\tilde{\mathcal{F}}_U = \sin\varphi[\sin\theta \tilde{F}^{\sin\varphi}_{1U} + \sin 2\theta \tilde{F}^{\sin\varphi}_{2U}]$$
$$+\sin 2\varphi \sin^2\theta \tilde{F}^{\sin 2\varphi}_U$$

$\sin\varphi$

$\sin 2\varphi$

---

$$\frac{2E_1E_2\,d\sigma^L}{d^3p_1\,d^3p_2} = \frac{\alpha^2}{s^2}\chi\lambda(\mathcal{F}_L + \tilde{\mathcal{F}}_L)$$

$$\mathcal{F}_L = \sin\varphi[\sin\theta F^{\sin\varphi}_{1L} + \sin 2\theta F^{\sin\varphi}_{2L}]$$
$$+\sin 2\varphi \sin^2\theta F^{\sin 2\varphi}_L$$

$\mathcal{F}_L \leftrightarrow \tilde{\mathcal{F}}_U, \quad \tilde{\mathcal{F}}_L \leftrightarrow \mathcal{F}_U$

$F^{xxx}_{jL} \leftrightarrow \tilde{F}^{xxx}_{jU}, \quad \tilde{F}^{xxx}_{jL} \leftrightarrow F^{xxx}_{jU}$

$$\tilde{\mathcal{F}}_L = (1+\cos^2\theta)\tilde{F}_{1L} + \sin^2\theta \tilde{F}_{2L} + \cos\theta \tilde{F}_{3L}$$
$$+\cos\varphi[\sin\theta \tilde{F}^{\cos\varphi}_{1L} + \sin 2\theta \tilde{F}^{\cos\varphi}_{2L}]$$
$$+\cos 2\varphi \sin^2\theta \tilde{F}^{\cos 2\varphi}_L$$

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$$\frac{2E_1E_2\,d\sigma^{LL}}{d^3p_1\,d^3p_2} = \frac{\alpha^2}{s^2}\chi S_{LL}(\mathcal{F}_{LL} + \tilde{\mathcal{F}}_{LL})$$

$$\mathcal{F}_{LL} = (1+\cos^2\theta)F_{1LL} + \sin^2\theta F_{2LL} + \cos\theta F_{3LL}$$
$$+\cos\varphi[\sin\theta F^{\cos\varphi}_{1LL} + \sin 2\theta F^{\cos\varphi}_{2LL}]$$
$$+\cos 2\varphi \sin^2\theta F^{\cos 2\varphi}_{LL}$$

$\mathcal{F}_{LL} \leftrightarrow \mathcal{F}_U, \quad \tilde{\mathcal{F}}_{LL} \leftrightarrow \tilde{\mathcal{F}}_U$

$F^{xxx}_{jLL} \leftrightarrow F^{xxx}_{jU}, \quad \tilde{F}^{xxx}_{jLL} \leftrightarrow \tilde{F}^{xxx}_{jU}$

$$\tilde{\mathcal{F}}_{LL} = \sin\varphi[\sin\theta \tilde{F}^{\sin\varphi}_{1LL} + \sin 2\theta \tilde{F}^{\sin\varphi}_{2LL}]$$
$$+\sin 2\varphi \sin^2\theta \tilde{F}^{\sin 2\varphi}_{LL}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The cross section in Helicity-GJ-frame: transverse polarization dependent parts

$$\frac{2E_1E_2\,d\sigma^T}{d^3p_1\,d^3p_2} = \frac{\alpha^2}{s^2}\chi\,|\vec{S}_T|\,(\mathcal{F}_T + \tilde{\mathcal{F}}_T)$$

$$|\vec{S}_T|^2 = (S_T^x)^2 + (S_T^y)^2$$

$$\tan\varphi_S = S_T^x / S_T^y$$

$$\begin{aligned}
\mathcal{F}_T = {} & \sin\varphi_S[\sin\theta F_{1T}^{\sin\varphi_S} + \sin 2\theta F_{2T}^{\sin\varphi_S}] && \qquad \sin\varphi_S \\
& + \sin(\varphi_S+\varphi)\sin^2\theta F_T^{\sin(\varphi_S+\varphi)} && \qquad \sin(\varphi_S+\varphi) \\
& + \sin(\varphi_S-\varphi)[(1+\cos^2\theta)F_{1T}^{\sin(\varphi_S-\varphi)} + \sin^2\theta F_{2T}^{\sin(\varphi_S-\varphi)} + \cos\theta F_{3T}^{\sin(\varphi_S-\varphi)}] && \qquad \sin(\varphi_S-\varphi) \\
& + \sin(\varphi_S-2\varphi)[\sin\theta F_{1T}^{\sin(\varphi_S-2\varphi)} + \sin 2\theta F_{2T}^{\sin(\varphi_S-2\varphi)}] && \qquad \sin(\varphi_S-2\varphi) \\
& + \sin(\varphi_S-3\varphi)\sin^2\theta F_T^{\sin(\varphi_S-3\varphi)} && \qquad \sin(\varphi_S-3\varphi)
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_T = {} & \cos\varphi_S[\sin\theta \tilde{F}_{1T}^{\cos\varphi_S} + \sin 2\theta \tilde{F}_{2T}^{\cos\varphi_S}] && \qquad \cos\varphi_S \\
& + \cos(\varphi_S+\varphi)\sin^2\theta \tilde{F}_T^{\cos(\varphi_S+\varphi)} && \qquad \cos(\varphi_S+\varphi) \\
& + \cos(\varphi_S-\varphi)[(1+\cos^2\theta)\tilde{F}_{1T}^{\cos(\varphi_S-\varphi)} + \sin^2\theta \tilde{F}_{2T}^{\cos(\varphi_S-\varphi)} + \cos\theta \tilde{F}_{3T}^{\cos(\varphi_S-\varphi)}] && \qquad \cos(\varphi_S-\varphi) \\
& + \cos(\varphi_S-2\varphi)[\sin\theta \tilde{F}_{1T}^{\cos(\varphi_S-2\varphi)} + \sin 2\theta \tilde{F}_{2T}^{\cos(\varphi_S-2\varphi)}] && \qquad \cos(\varphi_S-2\varphi) \\
& + \cos(\varphi_S-3\varphi)\sin^2\theta \tilde{F}_T^{\cos(\varphi_S-3\varphi)} && \qquad \cos(\varphi_S-3\varphi)
\end{aligned}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The cross section in Helicity-GJ-frame: transverse polarization dependent parts

$$\frac{2E_1E_2 d\sigma^T}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi|\vec{S}_T|(\mathcal{F}_T + \tilde{\mathcal{F}}_T)$$

$$|\vec{S}_T|^2 = (S_T^x)^2 + (S_T^y)^2,$$
$$\tan\varphi_S = S_T^x / S_T^y$$

$$\begin{aligned}
\mathcal{F}_T = {} & \sin\varphi_S[\sin\theta F_{1T}^{\sin\varphi_S} + \sin 2\theta F_{2T}^{\sin\varphi_S}] \\
& + \sin(\varphi_S+\varphi)\sin^2\theta F_T^{\sin(\varphi_S+\varphi)} \\
& + \sin(\varphi_S-\varphi)[(1+\cos^2\theta)F_{1T}^{\sin(\varphi_S-\varphi)} + \sin^2\theta F_{2T}^{\sin(\varphi_S-\varphi)} + \cos\theta F_{3T}^{\sin(\varphi_S-\varphi)}] \\
& + \sin(\varphi_S-2\varphi)[\sin\theta F_{1T}^{\sin(\varphi_S-2\varphi)} + \sin 2\theta F_{2T}^{\sin(\varphi_S-2\varphi)}] \\
& + \sin(\varphi_S-3\varphi)\sin^2\theta F_T^{\sin(\varphi_S-3\varphi)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_T = {} & \cos\varphi_S[\sin\theta \tilde{F}_{1T}^{\cos\varphi_S} + \sin 2\theta \tilde{F}_{2T}^{\cos\varphi_S}] \\
& + \cos(\varphi_S+\varphi)\sin^2\theta \tilde{F}_T^{\cos(\varphi_S+\varphi)} \\
& + \cos(\varphi_S-\varphi)[(1+\cos^2\theta)\tilde{F}_{1T}^{\cos(\varphi_S-\varphi)} + \sin^2\theta \tilde{F}_{2T}^{\cos(\varphi_S-\varphi)} + \cos\theta \tilde{F}_{3T}^{\cos(\varphi_S-\varphi)}] \\
& + \cos(\varphi_S-2\varphi)[\sin\theta \tilde{F}_{1T}^{\cos(\varphi_S-2\varphi)} + \sin 2\theta \tilde{F}_{2T}^{\cos(\varphi_S-2\varphi)}] \\
& + \cos(\varphi_S-3\varphi)\sin^2\theta \tilde{F}_T^{\cos(\varphi_S-3\varphi)}
\end{aligned}$$

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$$\frac{2E_1E_2 d\sigma^{LT}}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi|\vec{S}_{LT}|(\mathcal{F}_{LT} + \tilde{\mathcal{F}}_{LT})$$

$$|\vec{S}_{LT}|^2 = (S_{LT}^x)^2 + (S_{LT}^y)^2,$$
$$\tan\varphi_{LT} = S_{LT}^x / S_{LT}^y$$

$$\varphi_S \leftrightarrow \varphi_{LT}; \quad \mathcal{F}_T \leftrightarrow \tilde{\mathcal{F}}_{LT},\ \tilde{\mathcal{F}}_T \leftrightarrow \mathcal{F}_{LT}$$
$$F_{jT}^{xxx} \leftrightarrow \tilde{F}_{jLT}^{xxx},\ \tilde{F}_{jT}^{xxx} \leftrightarrow F_{jLT}^{xxx}$$

$$\begin{aligned}
\mathcal{F}_{LT} = {} & \cos\varphi_{LT}[\sin\theta F_{1LT}^{\cos\varphi_{LT}} + \sin 2\theta F_{2LT}^{\cos\varphi_{LT}}] \\
& + \cos(\varphi_{LT}+\varphi)\sin^2\theta F_{LT}^{\cos(\varphi_{LT}+\varphi)} \\
& + \cos(\varphi_{LT}-\varphi)[(1+\cos^2\theta)F_{1LT}^{\cos(\varphi_{LT}-\varphi)} + \sin^2\theta F_{2LT}^{\cos(\varphi_{LT}-\varphi)} + \cos\theta F_{3LT}^{\cos(\varphi_{LT}-\varphi)}] \\
& + \cos(\varphi_{LT}-2\varphi)[\sin\theta F_{1LT}^{\cos(\varphi_{LT}-2\varphi)} + \sin 2\theta F_{2LT}^{\cos(\varphi_{LT}-2\varphi)}] \\
& + \cos(\varphi_{LT}-3\varphi)\sin^2\theta F_{LT}^{\cos(\varphi_{LT}-3\varphi)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_{LT} = {} & \sin\varphi_{LT}[\sin\theta \tilde{F}_{1LT}^{\sin\varphi_{LT}} + \sin 2\theta \tilde{F}_{2LT}^{\sin\varphi_{LT}}] \\
& + \sin(\varphi_{LT}+\varphi)\sin^2\theta F_{LT}^{\sin(\varphi_{LT}+\varphi)} \\
& + \sin(\varphi_{LT}-\varphi)[(1+\cos^2\theta)\tilde{F}_{1LT}^{\sin(\varphi_{LT}-\varphi)} + \sin^2\theta \tilde{F}_{2LT}^{\sin(\varphi_{LT}-\varphi)} + \cos\theta \tilde{F}_{3LT}^{\sin(\varphi_{LT}-\varphi)}] \\
& + \sin(\varphi_{LT}-2\varphi)[\sin\theta \tilde{F}_{1LT}^{\sin(\varphi_{LT}-2\varphi)} + \sin 2\theta \tilde{F}_{2LT}^{\sin(\varphi_{LT}-2\varphi)}] \\
& + \sin(\varphi_{LT}-3\varphi)\sin^2\theta \tilde{F}_{LT}^{\sin(\varphi_{LT}-3\varphi)}
\end{aligned}$$

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$$\frac{2E_1E_2 d\sigma^{TT}}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi|\vec{S}_{TT}|(\mathcal{F}_{TT} + \tilde{\mathcal{F}}_{TT})$$

$$|\vec{S}_{TT}|^2 = (S_{TT}^{xx})^2 + (S_{TT}^{xy})^2,$$
$$\tan 2\varphi_{TT} = S_{TT}^{xx} / S_{TT}^{xy}$$

$$(2\varphi_{TT} - \varphi) \leftrightarrow \varphi_{LT}; \quad \mathcal{F}_{TT} \leftrightarrow \mathcal{F}_{LT},\ \tilde{\mathcal{F}}_{TT} \leftrightarrow \tilde{\mathcal{F}}_{LT}$$
$$F_{jTT}^{xxx} \leftrightarrow F_{jLT}^{xxx},\ F_{jTT}^{xxx} \leftrightarrow F_{jLT}^{xxx}$$

$$\begin{aligned}
\mathcal{F}_{TT} = {} & \cos 2\varphi_{TT}\sin^2\theta F_{TT}^{\cos 2\varphi_{TT}} \\
& + \cos(2\varphi_{TT}-\varphi)[\sin\theta F_{1TT}^{\cos(2\varphi_{TT}-\varphi)} + \sin 2\theta F_{2TT}^{\cos(2\varphi_{TT}-\varphi)}] \\
& + \cos(2\varphi_{TT}-2\varphi)[(1+\cos^2\theta)F_{1TT}^{\cos(2\varphi_{TT}-2\varphi)} + \sin^2\theta F_{2TT}^{\cos(2\varphi_{TT}-2\varphi)} + \cos\theta F_{3TT}^{\cos(2\varphi_{TT}-2\varphi)}] \\
& + \cos(2\varphi_{TT}-3\varphi)[\sin\theta F_{1TT}^{\cos(2\varphi_{TT}-3\varphi)} + \sin 2\theta F_{2TT}^{\cos(2\varphi_{TT}-3\varphi)}] \\
& + \cos(2\varphi_{TT}-4\varphi)\sin^2\theta F_{TT}^{\cos(2\varphi_{TT}-4\varphi)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_{TT} = {} & \sin 2\varphi_{TT}\sin^2\theta \tilde{F}_{TT}^{\sin 2\varphi_{TT}} \\
& + \sin(2\varphi_{TT}-\varphi)[\sin\theta \tilde{F}_{1TT}^{\sin(2\varphi_{TT}-\varphi)} + \sin 2\theta \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-\varphi)}] \\
& + \sin(2\varphi_{TT}-2\varphi)[(1+\cos^2\theta)\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-2\varphi)} + \sin^2\theta \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-2\varphi)} + \cos\theta \tilde{F}_{3TT}^{\sin(2\varphi_{TT}-2\varphi)}] \\
& + \sin(2\varphi_{TT}-3\varphi)[\sin\theta \tilde{F}_{1TT}^{\sin(2\varphi_{TT}-3\varphi)} + \sin 2\theta \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-3\varphi)}] \\
& + \sin(2\varphi_{TT}-4\varphi)\sin^2\theta \tilde{F}_{TT}^{\sin(2\varphi_{TT}-4\varphi)}
\end{aligned}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

| $\theta$-dep. | $1+\cos^2\theta$ | $\sin^2\theta$ | $\cos\theta$ | $\sin\theta$ | $\sin 2\theta$ | $\sin^2\theta$ | $\sin\theta$ | $\sin 2\theta$ | $\sin^2\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| $\varphi$-dep. | $1$ | | | $\cos\varphi$ | | $\cos 2\varphi$ | $\sin\varphi$ | | $\sin 2\varphi$ |
| U | $F_{1U}$ | $F_{2U}$ | $F_{3U}$ | $F_{1U}^{\cos\varphi}$ | $F_{2U}^{\cos\varphi}$ | $F_{U}^{\cos 2\varphi}$ | $\tilde{F}_{1U}^{\sin\varphi}$ | $\tilde{F}_{2U}^{\sin\varphi}$ | $\tilde{F}_{U}^{\sin 2\varphi}$ |
| L | $\tilde{F}_{1L}$ | $\tilde{F}_{2L}$ | $\tilde{F}_{3L}$ | $\tilde{F}_{1L}^{\cos\varphi}$ | $\tilde{F}_{2L}^{\cos\varphi}$ | $\tilde{F}_{L}^{\cos 2\varphi}$ | $F_{1L}^{\sin\varphi}$ | $F_{2L}^{\sin\varphi}$ | $F_{L}^{\sin 2\varphi}$ |
| LL | $F_{1LL}$ | $F_{2LL}$ | $F_{3LL}$ | $F_{1LL}^{\cos\varphi}$ | $F_{2LL}^{\cos\varphi}$ | $F_{LL}^{\cos 2\varphi}$ | $\tilde{F}_{1LL}^{\sin\varphi}$ | $\tilde{F}_{2LL}^{\sin\varphi}$ | $\tilde{F}_{LL}^{\sin 2\varphi}$ |
| T-PC | $F_{1T}^{\sin(\varphi_S-\varphi)}$ | $F_{2T}^{\sin(\varphi_S-\varphi)}$ | $F_{3T}^{\sin(\varphi_S-\varphi)}$ | $F_{1T}^{\sin(\varphi_S-2\varphi)}$ | $F_{2T}^{\sin(\varphi_S-2\varphi)}$ | $F_{T}^{\sin(\varphi_S-3\varphi)}$ | $F_{1T}^{\sin\varphi_S}$ | $F_{2T}^{\sin\varphi_S}$ | $F_{T}^{\sin(\varphi_S+\varphi)}$ |
| $\varphi$-dep. | $\sin(\varphi_S-\varphi)$ | | | $\sin(\varphi_S-2\varphi)$ | | $\sin(\varphi_S-3\varphi)$ | $\sin\varphi_S$ | | $\sin(\varphi_S+\varphi)$ |
| T-PV | $\tilde{F}_{1T}^{\cos(\varphi_S-\varphi)}$ | $\tilde{F}_{2T}^{\cos(\varphi_S-\varphi)}$ | $\tilde{F}_{3T}^{\cos(\varphi_S-\varphi)}$ | $\tilde{F}_{1T}^{\cos(\varphi_S-2\varphi)}$ | $\tilde{F}_{2T}^{\cos(\varphi_S-2\varphi)}$ | $\tilde{F}_{T}^{\cos(\varphi_S-3\varphi)}$ | $\tilde{F}_{1T}^{\cos\varphi_S}$ | $\tilde{F}_{2T}^{\cos\varphi_S}$ | $\tilde{F}_{T}^{\cos(\varphi_S+\varphi)}$ |
| $\varphi$-dep. | $\cos(\varphi_S-\varphi)$ | | | $\cos(\varphi_S-2\varphi)$ | | $\cos(\varphi_S-3\varphi)$ | $\cos\varphi_S$ | | $\cos(\varphi_S+\varphi)$ |
| LT-PC | $F_{1LT}^{\cos(\varphi_{LT}-\varphi)}$ | $F_{2LT}^{\cos(\varphi_{LT}-\varphi)}$ | $F_{3LT}^{\cos(\varphi_{LT}-\varphi)}$ | $F_{1LT}^{\cos(\varphi_{LT}-2\varphi)}$ | $F_{2LT}^{\cos(\varphi_{LT}-2\varphi)}$ | $F_{LT}^{\cos(\varphi_{LT}-3\varphi)}$ | $F_{1LT}^{\cos\varphi_{LT}}$ | $F_{2LT}^{\cos\varphi_{LT}}$ | $F_{LT}^{\cos(\varphi_{LT}+\varphi)}$ |
| $\varphi$-dep. | $\cos(\varphi_{LT}-\varphi)$ | | | $\cos(\varphi_{LT}-2\varphi)$ | | $\cos(\varphi_{LT}-3\varphi)$ | $\cos\varphi_{LT}$ | | $\cos(\varphi_{LT}+\varphi)$ |
| LT-PV | $\tilde{F}_{1LT}^{\sin(\varphi_{LT}-\varphi)}$ | $\tilde{F}_{2LT}^{\sin(\varphi_{LT}-\varphi)}$ | $\tilde{F}_{3LT}^{\sin(\varphi_{LT}-\varphi)}$ | $\tilde{F}_{1LT}^{\sin(\varphi_{LT}-2\varphi)}$ | $\tilde{F}_{2LT}^{\sin(\varphi_{LT}-2\varphi)}$ | $\tilde{F}_{LT}^{\sin(\varphi_{LT}-3\varphi)}$ | $\tilde{F}_{1LT}^{\sin\varphi_{LT}}$ | $\tilde{F}_{2LT}^{\sin\varphi_{LT}}$ | $\tilde{F}_{LT}^{\sin(\varphi_{LT}+\varphi)}$ |
| $\varphi$-dep. | $\sin(\varphi_{LT}-\varphi)$ | | | $\sin(\varphi_{LT}-2\varphi)$ | | $\sin(\varphi_{LT}-3\varphi)$ | $\sin\varphi_{LT}$ | | $\sin(\varphi_{LT}+\varphi)$ |
| TT-PC | $F_{1TT}^{\cos(2\varphi_{TT}-2\varphi)}$ | $F_{2TT}^{\cos(2\varphi_{TT}-2\varphi)}$ | $F_{3TT}^{\cos(2\varphi_{TT}-2\varphi)}$ | $F_{1TT}^{\cos(2\varphi_{TT}-3\varphi)}$ | $F_{2TT}^{\cos(2\varphi_{TT}-3\varphi)}$ | $F_{TT}^{\cos(2\varphi_{TT}-4\varphi)}$ | $F_{1TT}^{\cos(2\varphi_{TT}-\varphi)}$ | $F_{2TT}^{\cos(2\varphi_{TT}-\varphi)}$ | $F_{TT}^{\cos 2\varphi_{TT}}$ |
| $\varphi$-dep. | $\cos(2\varphi_{TT}-2\varphi)$ | | | $\cos(2\varphi_{TT}-3\varphi)$ | | $\cos(2\varphi_{TT}-4\varphi)$ | $\cos(2\varphi_{TT}-\varphi)$ | | $\cos 2\varphi_{TT}$ |
| TT-PV | $\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-2\varphi)}$ | $\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-2\varphi)}$ | $\tilde{F}_{3TT}^{\sin(2\varphi_{TT}-2\varphi)}$ | $\tilde{F}_{TT}^{\sin(2\varphi_{TT}-3\varphi)}$ | $\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-3\varphi)}$ | $\tilde{F}_{TT}^{\sin(2\varphi_{TT}-4\varphi)}$ | $\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-\varphi)}$ | $\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-\varphi)}$ | $\tilde{F}_{TT}^{\sin 2\varphi_{TT}}$ |
| $\varphi$-dep. | $\sin(2\varphi_{TT}-2\varphi)$ | | | $\sin(2\varphi_{TT}-3\varphi)$ | | $\sin(2\varphi_{TT}-4\varphi)$ | $\sin(2\varphi_{TT}-\varphi)$ | | $\sin 2\varphi_{TT}$ |

# General kinematic analysis for $e^+e^- \to V\pi X$

## Azimuthal asymmetries in the unpolarized case

parity conserved:

$$\langle\cos\varphi\rangle_U = [\sin\theta F_{1U}^{\cos\varphi} + \sin 2\theta F_{2U}^{\cos\varphi}] / 2F_{Ut}$$

$$\langle\cos 2\varphi\rangle_U = \sin^2\theta F_U^{\cos 2\varphi} / 2F_{Ut}$$

parity violated:

$$\langle\sin\varphi\rangle_U = [\sin\theta \tilde{F}_{1U}^{\sin\varphi} + \sin 2\theta \tilde{F}_{2U}^{\sin\varphi}] / 2F_{Ut}$$

$$\langle\sin 2\varphi\rangle_U = \sin^2\theta \tilde{F}_U^{\sin 2\varphi} / 2F_{Ut}$$

$$F_{Ut} = \int \frac{d\varphi}{2\pi}(\mathcal{F}_U + \tilde{\mathcal{F}}_U) = (1+\cos^2\theta)F_{1U} + \sin^2\theta F_{2U} + \cos\theta F_{3U}$$

## Hadron polarizations

E.g.:

$$\lambda_{ave} = \frac{\mathcal{F}_L + \tilde{\mathcal{F}}_L}{\mathcal{F}_U + \tilde{\mathcal{F}}_U} \qquad S_{LL,ave} = \frac{1}{2}\frac{\mathcal{F}_{LL} + \tilde{\mathcal{F}}_{LL}}{\mathcal{F}_U + \tilde{\mathcal{F}}_U} \qquad S_{LT,ave}^i = \frac{2}{3}\frac{\mathcal{F}_{LT}^i + \tilde{\mathcal{F}}_{LT}^i}{\mathcal{F}_U + \tilde{\mathcal{F}}_U}$$

In practice, often integrated over the azimuthal angle φ ⇨

# General kinematic analysis for $e^+e^- \to V\pi X$

## Integrated over the azimuthal angle $\varphi$ — inclusive $e^+e^- \to VX$

$$\int\frac{d\varphi}{2\pi}\frac{2E_1E_2d^6\sigma}{d^3p_1d^3p_2}=\frac{\alpha^2}{s^2}\chi\Big\{\big(\langle\mathcal{F}_U\rangle+\langle\tilde{\mathcal{F}}_U\rangle\big)+\lambda\big(\langle\mathcal{F}_L\rangle+\langle\tilde{\mathcal{F}}_L\rangle\big)+S_{LL}\big(\langle\mathcal{F}_{LL}\rangle+\langle\tilde{\mathcal{F}}_{LL}\rangle\big)+|\vec{S}_T|\big(\langle\mathcal{F}_T\rangle+\langle\tilde{\mathcal{F}}_T\rangle\big)+|\vec{S}_{LT}|\big(\langle\mathcal{F}_{LT}\rangle+\langle\tilde{\mathcal{F}}_{LT}\rangle\big)+|\vec{S}_{TT}|\big(\langle\mathcal{F}_{TT}\rangle+\langle\tilde{\mathcal{F}}_{TT}\rangle\big)\Big\}$$

$$\frac{2E_1d^3\sigma_{in}}{d^3p_1}=\frac{\alpha^2}{s^2}\chi\Big\{\big(\mathcal{F}_{U,in}+\tilde{\mathcal{F}}_{U,in}\big)+\lambda\big(\mathcal{F}_{L,in}+\tilde{\mathcal{F}}_{L,in}\big)+S_{LL}\big(\mathcal{F}_{LL,in}+\tilde{\mathcal{F}}_{LL,in}\big)+|\vec{S}_T|\big(\mathcal{F}_{T,in}+\tilde{\mathcal{F}}_{T,in}\big)+|\vec{S}_{LT}|\big(\mathcal{F}_{LT,in}+\tilde{\mathcal{F}}_{LT,in}\big)+|\vec{S}_{TT}|\big(\mathcal{F}_{TT,in}+\tilde{\mathcal{F}}_{TT,in}\big)\Big\}$$

| | | |
|---|---|---|
| $\langle\mathcal{F}_U\rangle=(1+\cos^2\theta)F_{1U}+\sin^2\theta F_{2U}+\cos\theta F_{3U}$ | P-even, T-even | $\mathcal{F}_{U,in}=(1+\cos^2\theta)F_{1U,in}+\sin^2\theta F_{2U,in}+\cos\theta F_{3U,in}$ |
| $\langle\tilde{\mathcal{F}}_U\rangle=0$ | P-odd, T-odd | $\tilde{\mathcal{F}}_{U,in}=0$ |
| $\langle\mathcal{F}_L\rangle=0$ | P-even, T-odd | $\mathcal{F}_{L,in}=0$ |
| $\langle\tilde{\mathcal{F}}_L\rangle=(1+\cos^2\theta)\tilde{F}_{1L}+\sin^2\theta\tilde{F}_{2L}+\cos\theta\tilde{F}_{3L}$ | P-odd, T-even | $\tilde{\mathcal{F}}_{L,in}=(1+\cos^2\theta)\tilde{F}_{1L,in}+\sin^2\theta\tilde{F}_{2L,in}+\cos\theta\tilde{F}_{3L,in}$ |
| $\langle\mathcal{F}_{LL}\rangle=(1+\cos^2\theta)F_{1LL}+\sin^2\theta F_{2LL}+\cos\theta F_{3LL}$ | P-even, T-even | $\mathcal{F}_{LL,in}=(1+\cos^2\theta)F_{1LL,in}+\sin^2\theta F_{2LL,in}+\cos\theta F_{3LL,in}$ |
| $\langle\tilde{\mathcal{F}}_{LL}\rangle=0$ | P-odd, T-odd | $\tilde{\mathcal{F}}_{LL,in}=0$ |
| $\langle\mathcal{F}_T\rangle=\sin\varphi_S(\sin\theta F_{1T}^{\sin\varphi_S}+\sin2\theta F_{2T}^{\sin\varphi_S})$ | P-even, T-odd | $\mathcal{F}_{T,in}=\sin\varphi_S(\sin\theta F_{1T,in}^{\sin\varphi_S}+\sin2\theta F_{2T,in}^{\sin\varphi_S})$ |
| $\langle\tilde{\mathcal{F}}_T\rangle=\cos\varphi_S(\sin\theta\tilde{F}_{1T}^{\cos\varphi_S}+\sin2\theta\tilde{F}_{2T}^{\cos\varphi_S})$ | P-odd, T-even | $\tilde{\mathcal{F}}_{T,in}=\cos\varphi_S(\sin\theta\tilde{F}_{1T,in}^{\cos\varphi_S}+\sin2\theta\tilde{F}_{2T,in}^{\cos\varphi_S})$ |
| $\langle\mathcal{F}_{LT}\rangle=\cos\varphi_{LT}(\sin\theta F_{1LT}^{\cos\varphi_{LT}}+\sin2\theta F_{2LT}^{\cos\varphi_{LT}})$ | P-even, T-even | $\mathcal{F}_{LT,in}=\cos\varphi_{LT}(\sin\theta F_{1LT,in}^{\cos\varphi_{LT}}+\sin2\theta F_{2LT,in}^{\cos\varphi_{LT}})$ |
| $\langle\tilde{\mathcal{F}}_{LT}\rangle=\sin\varphi_{LT}(\sin\theta\tilde{F}_{1LT}^{\sin\varphi_{LT}}+\sin2\theta\tilde{F}_{2LT}^{\sin\varphi_{LT}})$ | P-odd, T-odd | $\tilde{\mathcal{F}}_{LT,in}=\sin\varphi_{LT}(\sin\theta\tilde{F}_{1LT,in}^{\sin\varphi_{LT}}+\sin2\theta\tilde{F}_{2LT,in}^{\sin\varphi_{LT}})$ |
| $\langle\mathcal{F}_{TT}\rangle=\cos2\varphi_{TT}\sin^2\theta F_{TT}^{\cos2\varphi_{TT}}$ | P-even, T-even | $\mathcal{F}_{TT,in}=\cos2\varphi_{TT}\sin^2\theta F_{TT,in}^{\cos2\varphi_{TT}}$ |
| $\langle\tilde{\mathcal{F}}_{TT}\rangle=\sin2\varphi_{TT}\sin^2\theta\tilde{F}_{TT}^{\sin2\varphi_{TT}}$ | P-odd, T-odd | $\tilde{\mathcal{F}}_{TT,in}=\sin2\varphi_{TT}\sin^2\theta\tilde{F}_{TT,in}^{\sin2\varphi_{TT}}$ |

19 “structure functions” left, 11 parity conserved, 8 parity violated.

$$F_{xxx,in}^{yy}=\int\frac{dp_{2T}^2dp_{2z}}{2E_2}F_{xxx}^{yy}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## Hadron polarizations averaged over the azimuthal angle $\varphi$

Longitudinal components

$$\langle\lambda\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)\tilde{F}_{1L} + \sin^2\theta\tilde{F}_{2L} + \cos\theta\tilde{F}_{3L}]$$

parity violated

$$\langle S_{LL}\rangle = \frac{1}{2F_{Ut}}[(1+\cos^2\theta)F_{1LL} + \sin^2\theta F_{2LL} + \cos\theta F_{3LL}]$$

parity conserved

Transversal components

w.r.t. the hadron-hadron plane

$$\langle S_T^n\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)F_{1T}^{\sin(\varphi_S-\varphi)} + \sin^2\theta F_{2T}^{\sin(\varphi_S-\varphi)} + \cos\theta F_{3T}^{\sin(\varphi_S-\varphi)}]$$

$$\langle S_T^t\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)\tilde{F}_{1T}^{\cos(\varphi_S-\varphi)} + \sin^2\theta\tilde{F}_{2T}^{\cos(\varphi_S-\varphi)} + \cos\theta\tilde{F}_{3T}^{\cos(\varphi_S-\varphi)}]$$

$$\langle S_{LT}^n\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)\tilde{F}_{1LT}^{\sin(\varphi_{LT}-\varphi)} + \sin^2\theta\tilde{F}_{2LT}^{\sin(\varphi_{LT}-\varphi)} + \cos\theta\tilde{F}_{3LT}^{\sin(\varphi_{LT}-\varphi)}]$$

$$\langle S_{LT}^t\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)F_{1LT}^{\cos(\varphi_{LT}-\varphi)} + \sin^2\theta F_{2LT}^{\cos(\varphi_{LT}-\varphi)} + \cos\theta F_{3LT}^{\cos(\varphi_{LT}-\varphi)}]$$

$$\langle S_{TT}^{nn}\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)F_{1TT}^{\cos(2\varphi_{TT}-2\varphi)} + \sin^2\theta F_{2TT}^{\cos(2\varphi_{TT}-2\varphi)} + \cos\theta F_{3TT}^{\cos(2\varphi_{TT}-2\varphi)}]$$

$$\langle S_{TT}^{nt}\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-2\varphi)} + \sin^2\theta\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-2\varphi)} + \cos\theta\tilde{F}_{3TT}^{\sin(2\varphi_{TT}-2\varphi)}]$$

w.r.t. the lepton-hadron plane

$$\langle S_T^x\rangle = \frac{2}{3F_{Ut}}[\sin\theta\tilde{F}_{1T}^{\cos\varphi_S} + \sin 2\theta\tilde{F}_{2T}^{\cos\varphi_S}]$$

$$\langle S_T^y\rangle = \frac{2}{3F_{Ut}}[\sin\theta F_{1T}^{\sin\varphi_S} + \sin 2\theta F_{2T}^{\sin\varphi_S}]$$

$$\langle S_{LT}^x\rangle = \frac{2}{3F_{Ut}}[\sin\theta F_{1LT}^{\cos\varphi_{LT}} + \sin 2\theta F_{2LT}^{\cos\varphi_{LT}}]$$

$$\langle S_{LT}^y\rangle = \frac{2}{3F_{Ut}}[\sin\theta\tilde{F}_{1LT}^{\sin\varphi_{LT}} + \sin 2\theta\tilde{F}_{2LT}^{\sin\varphi_{LT}}]$$

$$\langle S_{TT}^{xx}\rangle = \frac{2}{3F_{Ut}}\sin^2\theta F_{1TT}^{\cos 2\varphi_{TT}}$$

$$\langle S_{TT}^{xy}\rangle = \frac{2}{3F_{Ut}}\sin^2\theta\tilde{F}_{1TT}^{\sin 2\varphi_{TT}}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## Hadron polarizations averaged over the azimuthal angle $\varphi$

### For the semi-inclusive process $e^+e^- \to V\pi X$

**Longitudinal components**

$$\langle\lambda\rangle = \frac{2}{3F_{Ut}}[(1+\cos^2\theta)\tilde{F}_{1L} + \sin^2\theta\tilde{F}_{2L} + \cos\theta\tilde{F}_{3L}]$$

$$\langle S_{LL}\rangle = \frac{1}{2F_{Ut}}[(1+\cos^2\theta)F_{1LL} + \sin^2\theta F_{2LL} + \cos\theta F_{3LL}]$$

**Transversal components**

w.r.t. the lepton-hadron plane

$$\langle S_T^x\rangle = \frac{2}{3F_{Ut}}[\sin\theta\tilde{F}_{1T}^{\cos\varphi_S} + \sin 2\theta\tilde{F}_{2T}^{\cos\varphi_S}]$$

$$\langle S_T^y\rangle = \frac{2}{3F_{Ut}}[\sin\theta F_{1T}^{\sin\varphi_S} + \sin 2\theta F_{2T}^{\sin\varphi_S}]$$

$$\langle S_{LT}^x\rangle = \frac{2}{3F_{Ut}}[\sin\theta F_{1LT}^{\cos\varphi_{LT}} + \sin 2\theta F_{2LT}^{\cos\varphi_{LT}}]$$

$$\langle S_{LT}^y\rangle = \frac{2}{3F_{Ut}}[\sin\theta\tilde{F}_{1LT}^{\sin\varphi_{LT}} + \sin 2\theta\tilde{F}_{2LT}^{\sin\varphi_{LT}}]$$

$$\langle S_{TT}^{xx}\rangle = \frac{2}{3F_{Ut}}\sin^2\theta F_{1TT}^{\cos 2\varphi_{TT}}$$

$$\langle S_{TT}^{xy}\rangle = \frac{2}{3F_{Ut}}\sin^2\theta\tilde{F}_{1TT}^{\sin 2\varphi_{TT}}$$

### For the inclusive process $e^+e^- \to VX$

**Longitudinal components**

$$\langle\lambda\rangle_{in} = \frac{2}{3F_{Ut,in}}[(1+\cos^2\theta)\tilde{F}_{1L,in} + \sin^2\theta\tilde{F}_{2L,in} + \cos\theta\tilde{F}_{3L,in}]$$

$$\langle S_{LL}\rangle_{in} = \frac{1}{2F_{Ut,in}}[(1+\cos^2\theta)F_{1LL,in} + \sin^2\theta F_{2LL,in} + \cos\theta F_{3LL,in}]$$

**Transversal components**

w.r.t. the lepton-hadron plane

$$\langle S_T^x\rangle_{in} = \frac{2}{3F_{Ut,in}}[\sin\theta\tilde{F}_{1T,in}^{\cos\varphi_S} + \sin 2\theta\tilde{F}_{2T,in}^{\cos\varphi_S}]$$

$$\langle S_T^y\rangle_{in} = \frac{2}{3F_{Ut,in}}[\sin\theta F_{1T,in}^{\sin\varphi_S} + \sin 2\theta F_{2T,in}^{\sin\varphi_S}]$$

$$\langle S_{LT}^x\rangle_{in} = \frac{2}{3F_{Ut,in}}[\sin\theta F_{1LT,in}^{\cos\varphi_{LT}} + \sin 2\theta F_{2LT,in}^{\cos\varphi_{LT}}]$$

$$\langle S_{LT}^y\rangle_{in} = \frac{2}{3F_{Ut,in}}[\sin\theta\tilde{F}_{1LT,in}^{\sin\varphi_{LT}} + \sin 2\theta\tilde{F}_{2LT,in}^{\sin\varphi_{LT}}]$$

$$\langle S_{TT}^{xx}\rangle_{in} = \frac{2}{3F_{Ut,in}}\sin^2\theta F_{1TT,in}^{\cos 2\varphi_{TT}}$$

$$\langle S_{TT}^{xy}\rangle_{in} = \frac{2}{3F_{Ut,in}}\sin^2\theta\tilde{F}_{1TT,in}^{\sin 2\varphi_{TT}}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

Further integrated over the polar angle $\theta$ inclusive $e^+e^- \to VX$

$$\int \frac{d\Omega}{4\pi}\frac{2E_1E_2d^6\sigma}{d^3p_1d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big\{\left(\bar{\mathcal{F}}_U + \bar{\tilde{\mathcal{F}}}_U\right) + \lambda\left(\bar{\mathcal{F}}_L + \bar{\tilde{\mathcal{F}}}_L\right) + S_{LL}\left(\bar{\mathcal{F}}_{LL} + \bar{\tilde{\mathcal{F}}}_{LL}\right) + |\vec{S}_T|\left(\bar{\mathcal{F}}_T + \bar{\tilde{\mathcal{F}}}_T\right) + |\vec{S}_{LT}|\left(\bar{\mathcal{F}}_{LT} + \bar{\tilde{\mathcal{F}}}_{LT}\right) + |\vec{S}_{TT}|\left(\bar{\mathcal{F}}_{TT} + \bar{\tilde{\mathcal{F}}}_{TT}\right)\Big\}$$

$$\frac{2E_1d^3\bar{\sigma}_{in}}{d^3p_1} = \frac{\alpha^2}{s^2}\chi\Big\{\left(\bar{\mathcal{F}}_{U,in} + \bar{\tilde{\mathcal{F}}}_{U,in}\right) + \lambda\left(\bar{\mathcal{F}}_{L,in} + \bar{\tilde{\mathcal{F}}}_{L,in}\right) + S_{LL}\left(\bar{\mathcal{F}}_{LL,in} + \bar{\tilde{\mathcal{F}}}_{LL,in}\right) + |\vec{S}_T|\left(\bar{\mathcal{F}}_{T,in} + \bar{\tilde{\mathcal{F}}}_{T,in}\right) + |\vec{S}_{LT}|\left(\bar{\mathcal{F}}_{LT,in} + \bar{\tilde{\mathcal{F}}}_{LT,in}\right) + |\vec{S}_{TT}|\left(\bar{\mathcal{F}}_{TT,in} + \bar{\tilde{\mathcal{F}}}_{TT,in}\right)\Big\}$$

| | | |
|---|---|---|
| $\bar{\mathcal{F}}_U = (4F_{1U} + 2F_{2U})/3$ | P-even, T-even | $\bar{\mathcal{F}}_{U,in} = (4F_{1U,in} + 2F_{2U,in})/3$ |
| $\bar{\tilde{\mathcal{F}}}_U = 0$ | P-odd, T-odd | $\bar{\tilde{\mathcal{F}}}_{U,in} = 0$ |
| $\bar{\mathcal{F}}_L = 0$ | P-even, T-odd | $\bar{\mathcal{F}}_{L,in} = 0$ |
| $\bar{\tilde{\mathcal{F}}}_L = (4\tilde{F}_{1L} + 2\tilde{F}_{2L})/3$ | P-odd, T-even | $\bar{\tilde{\mathcal{F}}}_{L,in} = (4\tilde{F}_{1L,in} + 2\tilde{F}_{2L,in})/3$ |
| $\bar{\mathcal{F}}_{LL} = (4F_{1LL} + 2F_{2LL})/3$ | P-even, T-even | $\bar{\mathcal{F}}_{LL,in} = (4F_{1LL,in} + 2F_{2LL,in})/3$ |
| $\bar{\tilde{\mathcal{F}}}_{LL} = 0$ | P-odd, T-odd | $\bar{\tilde{\mathcal{F}}}_{LL,in} = 0$ |
| $\bar{\mathcal{F}}_T = \pi\sin\varphi_S F_{1T}^{\sin\varphi_S}/4$ | P-even, T-odd | $\bar{\mathcal{F}}_{T,in} = \pi\sin\varphi_S F_{1T,in}^{\sin\varphi_S}/4$ |
| $\bar{\tilde{\mathcal{F}}}_T = \pi\cos\varphi_S \tilde{F}_{1T}^{\cos\varphi_S}/4$ | P-odd, T-even | $\bar{\tilde{\mathcal{F}}}_{T,in} = \pi\cos\varphi_S \tilde{F}_{1T,in}^{\cos\varphi_S}/4$ |
| $\bar{\mathcal{F}}_{LT} = \pi\cos\varphi_{LT} F_{1LT}^{\cos\varphi_{LT}}/4$ | P-even, T-even | $\bar{\mathcal{F}}_{LT,in} = \pi\cos\varphi_{LT} F_{1LT,in}^{\cos\varphi_{LT}}/4$ |
| $\bar{\tilde{\mathcal{F}}}_{LT} = \pi\sin\varphi_{LT} \tilde{F}_{1LT}^{\sin\varphi_{LT}}/4$ | P-odd, T-odd | $\bar{\tilde{\mathcal{F}}}_{LT,in} = \pi\sin\varphi_{LT} \tilde{F}_{1LT,in}^{\sin\varphi_{LT}}/4$ |
| $\bar{\mathcal{F}}_{TT} = 2\cos 2\varphi_{TT} F_{TT}^{\cos 2\varphi_{TT}}/3$ | P-even, T-even | $\bar{\mathcal{F}}_{TT,in} = 2\cos 2\varphi_{TT} F_{TT,in}^{\cos 2\varphi_{TT}}/3$ |
| $\bar{\tilde{\mathcal{F}}}_{TT} = 2\sin 2\varphi_{TT} \tilde{F}_{TT}^{\sin 2\varphi_{TT}}/3$ | P-odd, T-odd | $\bar{\tilde{\mathcal{F}}}_{TT,in} = 2\sin 2\varphi_{TT} \tilde{F}_{TT,in}^{\sin 2\varphi_{TT}}/3$ |

12 “structure functions” left, 7 parity conserved, 5 parity violated.

# General kinematic analysis for $e^+e^- \to V\pi X$

## Hadron polarizations averaged over the azimuthal angle $\varphi$ and the polar angle $\theta$

$$e^+e^- \to V\pi X$$

Longitudinal components

$$\bar{\lambda} = 4(2\tilde{F}_{1L} + \tilde{F}_{2L}) / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_{LL} = (2F_{1LL} + F_{2LL}) / 3\bar{\mathcal{F}}_U$$

Transversal components

w.r.t. the hadron-hadron plane

$$\bar{S}_T^n = 4(2F_{1T}^{\sin(\varphi_S-\varphi)} + F_{2T}^{\sin(\varphi_S-\varphi)}) / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_T^t = 4(2\tilde{F}_{1T}^{\cos(\varphi_S-\varphi)} + \tilde{F}_{2T}^{\cos(\varphi_S-\varphi)}) / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_{LT}^n = 4(2\tilde{F}_{1LT}^{\sin(\varphi_{LT}-\varphi)} + \tilde{F}_{2LT}^{\sin(\varphi_{LT}-\varphi)}) / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_{LT}^t = 4(2F_{1LT}^{\cos(\varphi_{LT}-\varphi)} + F_{2LT}^{\cos(\varphi_{LT}-\varphi)}) / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_{TT}^{nn} = 4(2F_{1TT}^{\cos(2\varphi_{TT}-2\varphi)} + F_{2TT}^{\cos(2\varphi_{TT}-2\varphi)}) / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_{TT}^{nt} = 4(2\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-2\varphi)} + \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-2\varphi)}) / 9\bar{\mathcal{F}}_U$$

w.r.t. the lepton-hadron plane

$$\bar{S}_T^x = \pi\tilde{F}_{1T}^{\cos\varphi_S} / 6\bar{\mathcal{F}}_U$$

$$\bar{S}_T^y = \pi F_{1T}^{\sin\varphi_S} / 6\bar{\mathcal{F}}_U$$

$$\bar{S}_{LT}^x = \pi F_{1LT}^{\cos\varphi_{LT}} / 6\bar{\mathcal{F}}_U$$

$$\bar{S}_{LT}^y = \pi\tilde{F}_{1LT}^{\sin\varphi_{LT}} / 6\bar{\mathcal{F}}_U$$

$$\bar{S}_{TT}^{xx} = 4F_{1TT}^{\cos 2\varphi_{TT}} / 9\bar{\mathcal{F}}_U$$

$$\bar{S}_{TT}^{xy} = 4\tilde{F}_{1TT}^{\sin 2\varphi_{TT}} / 9\bar{\mathcal{F}}_U$$

$$e^+e^- \to VX$$

$$\bar{\lambda}_{in} = 4(2\tilde{F}_{1L,in} + \tilde{F}_{2L,in}) / 9\bar{\mathcal{F}}_{U,in}$$

$$\bar{S}_{LL,in} = (2F_{1LL,in} + F_{2LL,in}) / 3\bar{\mathcal{F}}_{U,in}$$

w.r.t. the lepton-hadron plane

$$\bar{S}_{T,in}^x = \pi\tilde{F}_{1T,in}^{\cos\varphi_S} / 6\bar{\mathcal{F}}_{U,in}$$

$$\bar{S}_{T,in}^y = \pi F_{1T,in}^{\sin\varphi_S} / 6\bar{\mathcal{F}}_{U,in}$$

$$\bar{S}_{LT,in}^x = \pi F_{1LT,in}^{\cos\varphi_{LT}} / 6\bar{\mathcal{F}}_{U,in}$$

$$\bar{S}_{LT,in}^y = \pi\tilde{F}_{1LT,in}^{\sin\varphi_{LT}} / 6\bar{\mathcal{F}}_{U,in}$$

$$\bar{S}_{TT,in}^{xx} = 4F_{1TT,in}^{\cos 2\varphi_{TT}} / 9\bar{\mathcal{F}}_{U,in}$$

$$\bar{S}_{TT,in}^{xy} = 4\tilde{F}_{1TT,in}^{\sin 2\varphi_{TT}} / 9\bar{\mathcal{F}}_{U,in}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## Number of independent structure functions

$$\frac{2E_1E_2d^6\sigma}{d^3p_1d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big\{\left(\mathcal{F}_U+\tilde{\mathcal{F}}_U\right)+\lambda\left(\mathcal{F}_L+\tilde{\mathcal{F}}_L\right) + S_{LL}\left(\mathcal{F}_{LL}+\tilde{\mathcal{F}}_{LL}\right)+|\vec{S}_T|\left(\mathcal{F}_T+\tilde{\mathcal{F}}_T\right) + |\vec{S}_{LT}|\left(\mathcal{F}_{LT}+\tilde{\mathcal{F}}_{LT}\right)+|\vec{S}_{TT}|\left(\mathcal{F}_{TT}+\tilde{\mathcal{F}}_{TT}\right)\Big\}$$

$$\int\frac{d\varphi}{2\pi}\frac{2E_1E_2d^6\sigma}{d^3p_1d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big\{\left(\langle\mathcal{F}_U\rangle+\langle\tilde{\mathcal{F}}_U\rangle\right)+\lambda\left(\langle\mathcal{F}_L\rangle+\langle\tilde{\mathcal{F}}_L\rangle\right) + S_{LL}\left(\langle\mathcal{F}_{LL}\rangle+\langle\tilde{\mathcal{F}}_{LL}\rangle\right)+|\vec{S}_T|\left(\langle\mathcal{F}_T\rangle+\langle\tilde{\mathcal{F}}_T\rangle\right) + |\vec{S}_{LT}|\left(\langle\mathcal{F}_{LT}\rangle+\langle\tilde{\mathcal{F}}_{LT}\rangle\right)+|\vec{S}_{TT}|\left(\langle\mathcal{F}_{TT}\rangle+\langle\tilde{\mathcal{F}}_{TT}\rangle\right)\Big\}$$

$$\int\frac{d\Omega}{4\pi}\frac{2E_1E_2d^6\sigma}{d^3p_1d^3p_2} = \frac{\alpha^2}{s^2}\chi\Big\{\left(\bar{\mathcal{F}}_U+\bar{\tilde{\mathcal{F}}}_U\right)+\lambda\left(\bar{\mathcal{F}}_L+\bar{\tilde{\mathcal{F}}}_L\right) + S_{LL}\left(\bar{\mathcal{F}}_{LL}+\bar{\tilde{\mathcal{F}}}_{LL}\right)+|\vec{S}_T|\left(\bar{\mathcal{F}}_T+\bar{\tilde{\mathcal{F}}}_T\right) + |\vec{S}_{LT}|\left(\bar{\mathcal{F}}_{LT}+\bar{\tilde{\mathcal{F}}}_{LT}\right)+|\vec{S}_{TT}|\left(\bar{\mathcal{F}}_{TT}+\bar{\tilde{\mathcal{F}}}_{TT}\right)\Big\}$$

81 (41,40) $\xrightarrow[\text{the azimuthal angle }\varphi]{\text{integrated over}}$ 19 (11,8) $\xrightarrow[\text{the polar angle }\theta]{\text{further integrated over}}$ 12 (7,5)

# Contents

## I. Introduction

Transverse momentum dependent fragmentation functions (TMD FFs) defined via quark-quark correlator

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- The hadronic tensor and structure functions up to twist-3
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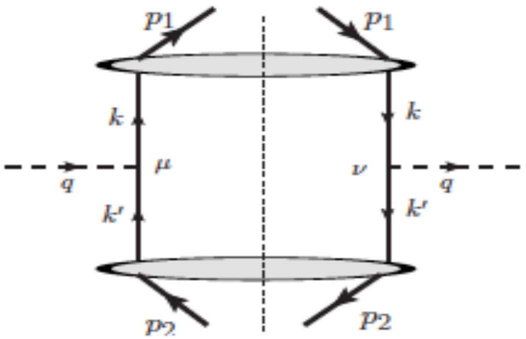
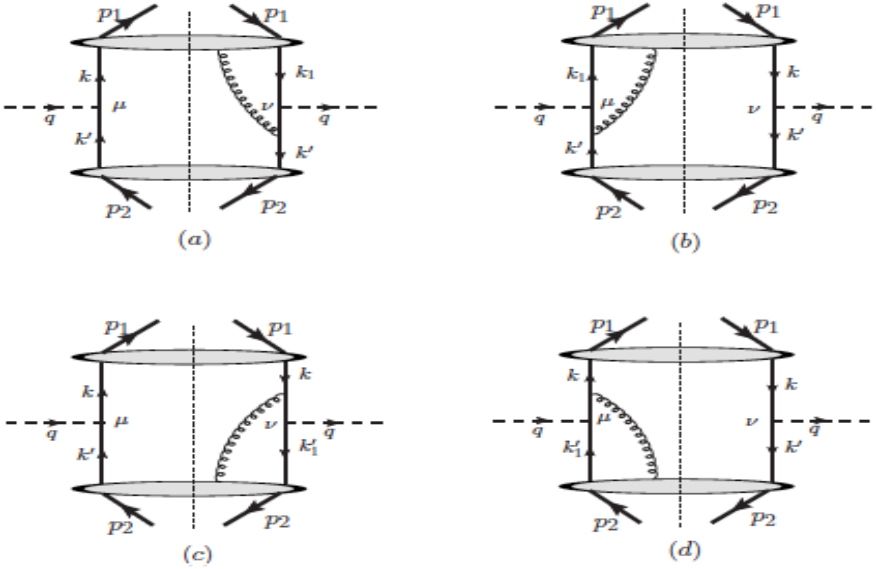
## IV. Summary and outlook

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Up to twist-3 (LO pQCD):





$$W_{\mu\nu}^{(0)} = \frac{1}{p_1^+ p_2^-}\int \frac{d^2k_\perp}{(2\pi)^2}\frac{d^2k_\perp^{'}}{(2\pi)^2}\delta^2(k_\perp + k_\perp^{'} - q_\perp)\mathrm{Tr}[\Xi^{(0)}(z_1,k_\perp,p_1,S)\Gamma_\mu \bar{\Xi}^{(0)}(z_2,k_\perp^{'},p_2)\Gamma_\nu]$$

$$W_{\mu\nu}^{(1L)} = \frac{-1}{\sqrt{2Qp_1^+ p_2^-}}\int \frac{d^2k_\perp}{(2\pi)^2}\frac{d^2k_\perp^{'}}{(2\pi)^2}\delta^2(k_\perp + k_\perp^{'} - q_\perp)\mathrm{Tr}[\gamma_\rho \not{n}\Gamma_\nu \Xi^{(1)\rho}(z_1,k_\perp,p_1,S)\Gamma_\mu \bar{\Xi}^{(0)}(z_2,k_\perp^{'},p_2)]$$

See D. Boer, R. Jakob, and P. J. Mulders, Nucl. Phys. B504, 345 (1997) (spin-1/2).
K.B. Chen, S.Y. Wei, W.H. Yang, & ZTL, PRD94, 034003 (2016) (spin-1).

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Structure functions at twist-2:

only $q \to VX,\ \bar{q} \to \pi X$ part

(hadron) unpolarized

$$F_{1U1} = 2c_1^e c_1^q \mathcal{C}[D_1 \bar{D}_1]$$

$$F_{3U1} = 4c_3^e c_3^q \mathcal{C}[D_1 \bar{D}_1]$$

$$F_{U1}^{\cos 2\varphi} = -8c_1^e c_2^q \mathcal{C}[w_{hh} H_1^\perp \bar{H}_1^\perp]$$

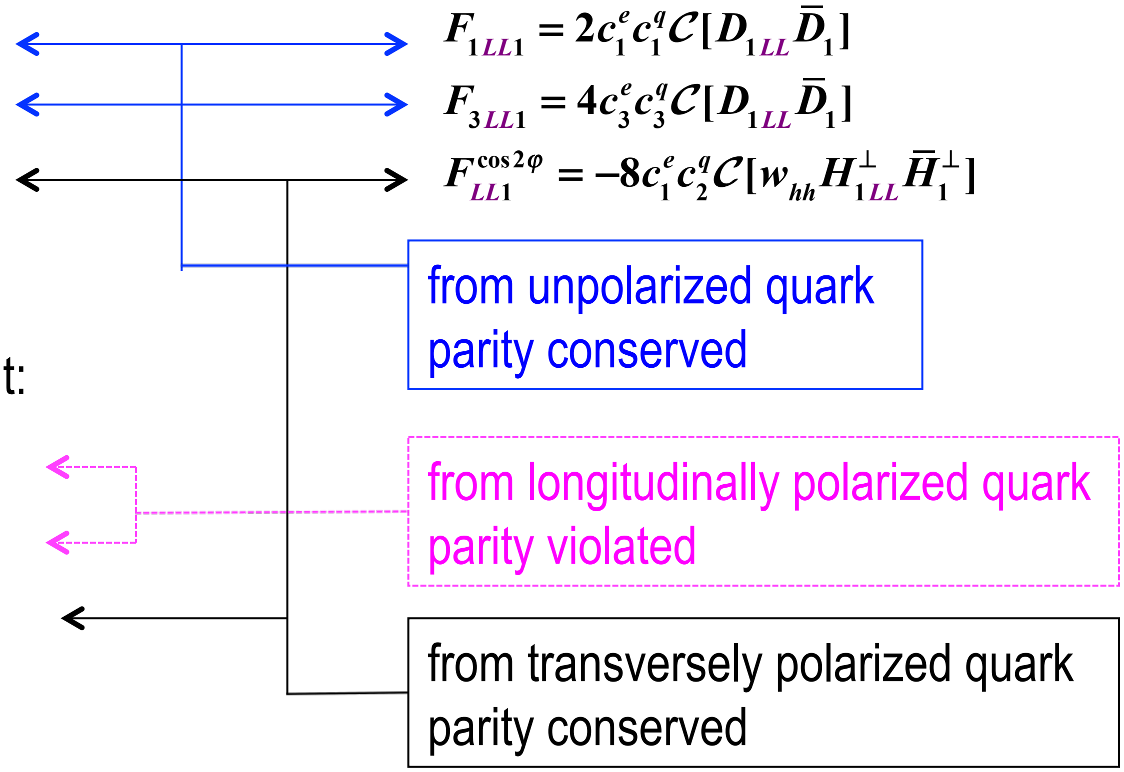
longitudinal polarization dependent:

$$\tilde{F}_{1L1} = -2c_1^e c_3^q \mathcal{C}[G_{1L} \bar{D}_1]$$

$$\tilde{F}_{3L1} = -4c_3^e c_1^q \mathcal{C}[G_{1L} \bar{D}_1]$$

$$F_{L1}^{\sin 2\varphi} = -8c_1^e c_2^q \mathcal{C}[w_{hh} H_{1L}^\perp \bar{H}_1^\perp]$$

totally 27 non-zeros at twist-2



$S_{LL}$-dependent:

$$F_{1LL1} = 2c_1^e c_1^q \mathcal{C}[D_{1LL} \bar{D}_1]$$

$$F_{3LL1} = 4c_3^e c_3^q \mathcal{C}[D_{1LL} \bar{D}_1]$$

$$F_{LL1}^{\cos 2\varphi} = -8c_1^e c_2^q \mathcal{C}[w_{hh} H_{1LL}^\perp \bar{H}_1^\perp]$$

from unpolarized quark
parity conserved

from longitudinally polarized quark
parity violated

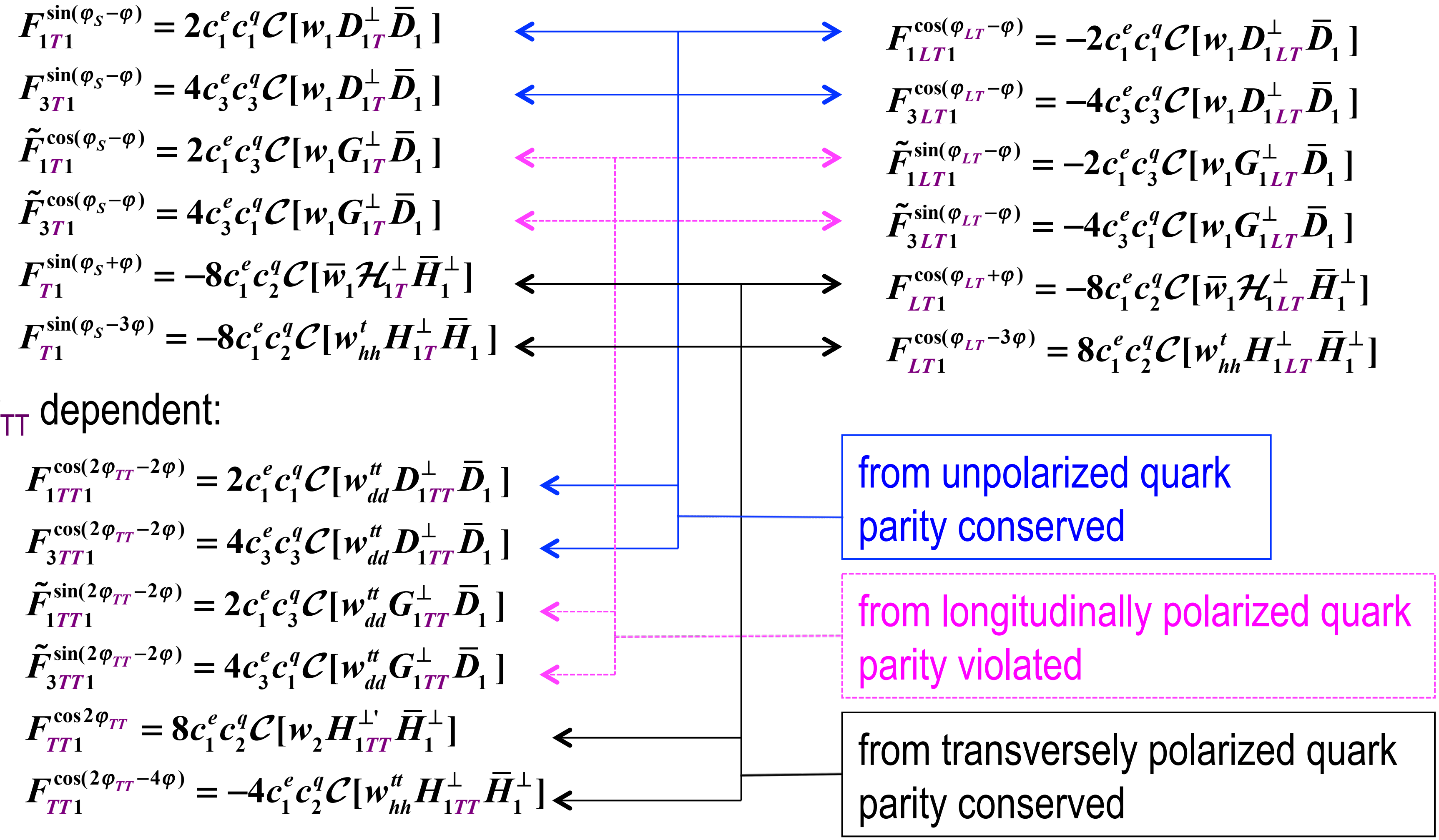
from transversely polarized quark
parity conserved

$$\mathcal{C}[wD_1 \bar{D}_1] \equiv \frac{1}{z_1 z_2} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{d^2k'_\perp}{(2\pi)^2} \delta^2(k_\perp + k'_\perp - q_\perp) w(k_\perp, k'_\perp) D_1(z_1, k_\perp) \bar{D}_1(z_2, k'_\perp)$$

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Structure functions at twist-2:

only $q \to VX,\ \bar{q} \to \pi X$ part



Transverse polarization $S_T$ dependent:

$$F_{1T1}^{\sin(\varphi_S-\varphi)} = 2c_1^e c_1^q \mathcal{C}[w_1 D_{1T}^\perp \bar{D}_1]$$

$$F_{3T1}^{\sin(\varphi_S-\varphi)} = 4c_3^e c_3^q \mathcal{C}[w_1 D_{1T}^\perp \bar{D}_1]$$

$$\tilde{F}_{1T1}^{\cos(\varphi_S-\varphi)} = 2c_1^e c_3^q \mathcal{C}[w_1 G_{1T}^\perp \bar{D}_1]$$

$$\tilde{F}_{3T1}^{\cos(\varphi_S-\varphi)} = 4c_3^e c_1^q \mathcal{C}[w_1 G_{1T}^\perp \bar{D}_1]$$

$$F_{T1}^{\sin(\varphi_S+\varphi)} = -8c_1^e c_2^q \mathcal{C}[\bar{w}_1 \mathcal{H}_{1T}^\perp \bar{H}_1^\perp]$$

$$F_{T1}^{\sin(\varphi_S-3\varphi)} = -8c_1^e c_2^q \mathcal{C}[w_{hh}^t H_{1T}^\perp \bar{H}_1]$$

$S_{LT}$ dependent:

$$F_{1LT1}^{\cos(\varphi_{LT}-\varphi)} = -2c_1^e c_1^q \mathcal{C}[w_1 D_{1LT}^\perp \bar{D}_1]$$

$$F_{3LT1}^{\cos(\varphi_{LT}-\varphi)} = -4c_3^e c_3^q \mathcal{C}[w_1 D_{1LT}^\perp \bar{D}_1]$$

$$\tilde{F}_{1LT1}^{\sin(\varphi_{LT}-\varphi)} = -2c_1^e c_3^q \mathcal{C}[w_1 G_{1LT}^\perp \bar{D}_1]$$

$$\tilde{F}_{3LT1}^{\sin(\varphi_{LT}-\varphi)} = -4c_3^e c_1^q \mathcal{C}[w_1 G_{1LT}^\perp \bar{D}_1]$$

$$F_{LT1}^{\cos(\varphi_{LT}+\varphi)} = -8c_1^e c_2^q \mathcal{C}[\bar{w}_1 \mathcal{H}_{1LT}^\perp \bar{H}_1^\perp]$$

$$F_{LT1}^{\cos(\varphi_{LT}-3\varphi)} = 8c_1^e c_2^q \mathcal{C}[w_{hh}^t H_{1LT}^\perp \bar{H}_1^\perp]$$

$S_{TT}$ dependent:

$$F_{1TT1}^{\cos(2\varphi_{TT}-2\varphi)} = 2c_1^e c_1^q \mathcal{C}[w_{dd}^{tt} D_{1TT}^\perp \bar{D}_1]$$

$$F_{3TT1}^{\cos(2\varphi_{TT}-2\varphi)} = 4c_3^e c_3^q \mathcal{C}[w_{dd}^{tt} D_{1TT}^\perp \bar{D}_1]$$

$$\tilde{F}_{1TT1}^{\sin(2\varphi_{TT}-2\varphi)} = 2c_1^e c_3^q \mathcal{C}[w_{dd}^{tt} G_{1TT}^\perp \bar{D}_1]$$

$$\tilde{F}_{3TT1}^{\sin(2\varphi_{TT}-2\varphi)} = 4c_3^e c_1^q \mathcal{C}[w_{dd}^{tt} G_{1TT}^\perp \bar{D}_1]$$

$$F_{TT1}^{\cos 2\varphi_{TT}} = 8c_1^e c_2^q \mathcal{C}[w_2 H_{1TT}^{\perp'} \bar{H}_1^\perp]$$

$$F_{TT1}^{\cos(2\varphi_{TT}-4\varphi)} = -4c_1^e c_2^q \mathcal{C}[w_{hh}^{tt} H_{1TT}^\perp \bar{H}_1^\perp]$$

from unpolarized quark
parity conserved

from longitudinally polarized quark
parity violated

from transversely polarized quark
parity conserved

| $\theta$-dep. | $1+\cos^2\theta$ | $\sin^2\theta$ | $\cos\theta$ | $\sin\theta$ | $\sin 2\theta$ | $\sin^2\theta$ | $\sin\theta$ | $\sin 2\theta$ | $\sin^2\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| $\varphi$-dep. | $1$ | | | $\cos\varphi$ | | $\cos 2\varphi$ | $\sin\varphi$ | | $\sin 2\varphi$ |
| U | $F_{1U}$ | $F_{2U}$ | $F_{3U}$ | $F_{1U}^{\cos\varphi}$ | $F_{2U}^{\cos\varphi}$ | $F_{U}^{\cos 2\varphi}$ | $\tilde{F}_{1U}^{\sin\varphi}$ | $\tilde{F}_{2U}^{\sin\varphi}$ | $\tilde{F}_{U}^{\sin 2\varphi}$ |
| twist-2 | $C[D_1\bar{D}_1]$ | $0$ | $C[D_1\bar{D}_1]$ | $0$ | $0$ | $C[w_{hh}H_1^\perp\bar{H}_1^\perp]$ | $0$ | $0$ | $0$ |
| L | $\tilde{F}_{1L}$ | $\tilde{F}_{2L}$ | $\tilde{F}_{3L}$ | $\tilde{F}_{1L}^{\cos\varphi}$ | $\tilde{F}_{2L}^{\cos\varphi}$ | $\tilde{F}_{L}^{\cos 2\varphi}$ | $F_{1L}^{\sin\varphi}$ | $F_{2L}^{\sin\varphi}$ | $F_{L}^{\sin 2\varphi}$ |
| twist-2 | $C[G_{1L}\bar{D}_1]$ | $0$ | $C[G_{1L}\bar{D}_1]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $C[w_{hh}H_{1L}^\perp\bar{H}_1^\perp]$ |
| LL | $F_{1LL}$ | $F_{2LL}$ | $F_{3LL}$ | $F_{1LL}^{\cos\varphi}$ | $F_{2LL}^{\cos\varphi}$ | $F_{LL}^{\cos 2\varphi}$ | $\tilde{F}_{1LL}^{\sin\varphi}$ | $\tilde{F}_{2LL}^{\sin\varphi}$ | $\tilde{F}_{LL}^{\sin 2\varphi}$ |
| twist-2 | $C[D_{1LL}\bar{D}_1]$ | $0$ | $C[D_{1LL}\bar{D}_1]$ | $0$ | $0$ | $C[w_{hh}H_{1LL}^\perp\bar{H}_1^\perp]$ | $0$ | $0$ | $0$ |
| T-PC | $F_{1T}^{\sin(\varphi_S-\varphi)}$ | $F_{2T}^{\sin(\varphi_S-\varphi)}$ | $F_{3T}^{\sin(\varphi_S-\varphi)}$ | $F_{1T}^{\sin(\varphi_S-2\varphi)}$ | $F_{2T}^{\sin(\varphi_S-2\varphi)}$ | $F_{T}^{\sin(\varphi_S-3\varphi)}$ | $F_{1T}^{\sin\varphi_S}$ | $F_{2T}^{\sin\varphi_S}$ | $F_{T}^{\sin(\varphi_S+\varphi)}$ |
| $\varphi$-dep. | $\sin(\varphi_S-\varphi)$ | | | $\sin(\varphi_S-2\varphi)$ | | $\sin(\varphi_S-3\varphi)$ | $\sin\varphi_S$ | | $\sin(\varphi_S+\varphi)$ |
| twist-2 | $C[w_1D_{1T}^\perp\bar{D}_1]$ | $0$ | $C[w_1D_{1T}^\perp\bar{D}_1]$ | $0$ | $0$ | $C[w_{hh}^tH_{1T}^\perp\bar{H}_1^\perp]$ | $0$ | $0$ | $C[\bar{w}_1\mathcal{H}_{1T}^\perp\bar{H}_1^\perp]$ |
| T-PV | $\tilde{F}_{1T}^{\cos(\varphi_S-\varphi)}$ | $\tilde{F}_{2T}^{\cos(\varphi_S-\varphi)}$ | $\tilde{F}_{3T}^{\cos(\varphi_S-\varphi)}$ | $\tilde{F}_{1T}^{\cos(\varphi_S-2\varphi)}$ | $\tilde{F}_{2T}^{\cos(\varphi_S-2\varphi)}$ | $\tilde{F}_{T}^{\cos(\varphi_S-3\varphi)}$ | $\tilde{F}_{1T}^{\cos\varphi_S}$ | $\tilde{F}_{2T}^{\cos\varphi_S}$ | $\tilde{F}_{T}^{\cos(\varphi_S+\varphi)}$ |
| $\varphi$-dep. | $\cos(\varphi_S-\varphi)$ | | | $\cos(\varphi_S-2\varphi)$ | | $\cos(\varphi_S-3\varphi)$ | $\cos\varphi_S$ | | $\cos(\varphi_S+\varphi)$ |
| twist-2 | $C[w_1G_{1T}^\perp\bar{D}_1]$ | $0$ | $C[w_1G_{1T}^\perp\bar{D}_1]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| LT-PC | $F_{1LT}^{\cos(\varphi_{LT}-\varphi)}$ | $F_{2LT}^{\cos(\varphi_{LT}-\varphi)}$ | $F_{3LT}^{\cos(\varphi_{LT}-\varphi)}$ | $F_{1LT}^{\cos(\varphi_{LT}-2\varphi)}$ | $F_{2LT}^{\cos(\varphi_{LT}-2\varphi)}$ | $F_{LT}^{\cos(\varphi_{LT}-3\varphi)}$ | $F_{1LT}^{\cos\varphi_{LT}}$ | $F_{2LT}^{\cos\varphi_{LT}}$ | $F_{LT}^{\cos(\varphi_{LT}+\varphi)}$ |
| $\varphi$-dep. | $\cos(\varphi_{LT}-\varphi)$ | | | $\cos(\varphi_{LT}-2\varphi)$ | | $\cos(\varphi_{LT}-3\varphi)$ | $\cos\varphi_{LT}$ | | $\cos(\varphi_{LT}+\varphi)$ |
| twist-2 | $C[w_1D_{1LT}^\perp\bar{D}_1]$ | $0$ | $C[w_1D_{1LT}^\perp\bar{D}_1]$ | $0$ | $0$ | $C[w_{hh}^tH_{1LT}^\perp\bar{H}_1^\perp]$ | $0$ | $0$ | $C[\bar{w}_1\mathcal{H}_{1LT}^\perp\bar{H}_1^\perp]$ |
| LT-PV | $\tilde{F}_{1LT}^{\sin(\varphi_{LT}-\varphi)}$ | $\tilde{F}_{2LT}^{\sin(\varphi_{LT}-\varphi)}$ | $\tilde{F}_{3LT}^{\sin(\varphi_{LT}-\varphi)}$ | $\tilde{F}_{1LT}^{\sin(\varphi_{LT}-2\varphi)}$ | $\tilde{F}_{2LT}^{\sin(\varphi_{LT}-2\varphi)}$ | $\tilde{F}_{LT}^{\sin(\varphi_{LT}-3\varphi)}$ | $\tilde{F}_{1LT}^{\sin\varphi_{LT}}$ | $\tilde{F}_{2LT}^{\sin\varphi_{LT}}$ | $\tilde{F}_{LT}^{\sin(\varphi_{LT}+\varphi)}$ |
| $\varphi$-dep. | $\sin(\varphi_{LT}-\varphi)$ | | | $\sin(\varphi_{LT}-2\varphi)$ | | $\sin(\varphi_{LT}-3\varphi)$ | $\sin\varphi_{LT}$ | | $\sin(\varphi_{LT}+\varphi)$ |
| twist-2 | $C[w_1G_{1LT}^\perp\bar{D}_1]$ | $0$ | $C[w_1G_{1LT}^\perp\bar{D}_1]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| TT-PC | $F_{1TT}^{\cos(2\varphi_{TT}-2\varphi)}$ | $F_{2TT}^{\cos(2\varphi_{TT}-2\varphi)}$ | $F_{3TT}^{\cos(2\varphi_{TT}-2\varphi)}$ | $F_{1TT}^{\cos(2\varphi_{TT}-3\varphi)}$ | $F_{2TT}^{\cos(2\varphi_{TT}-3\varphi)}$ | $F_{TT}^{\cos(2\varphi_{TT}-4\varphi)}$ | $F_{1TT}^{\cos(2\varphi_{TT}-\varphi)}$ | $F_{2TT}^{\cos(2\varphi_{TT}-\varphi)}$ | $F_{TT}^{\cos 2\varphi_{TT}}$ |
| $\varphi$-dep. | $\cos(2\varphi_{TT}-2\varphi)$ | | | $\cos(2\varphi_{TT}-3\varphi)$ | | $\cos(2\varphi_{TT}-4\varphi)$ | $\cos(2\varphi_{TT}-\varphi)$ | | $\cos 2\varphi_{TT}$ |
| twist-2 | $C[w_{dd}^{tt}D_{1TT}^\perp\bar{D}_1]$ | $0$ | $C[w_{dd}^{tt}D_{1TT}^\perp\bar{D}_1]$ | $0$ | $0$ | $C[w_{hh}^{tt}H_{1TT}^\perp\bar{H}_1^\perp]$ | $0$ | $0$ | $C[w_2\mathcal{H}_{1TT}^\perp\bar{H}_1^\perp]$ |
| TT-PV | $\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-2\varphi)}$ | $\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-2\varphi)}$ | $\tilde{F}_{3TT}^{\sin(2\varphi_{TT}-2\varphi)}$ | $\tilde{F}_{TT}^{\sin(2\varphi_{TT}-3\varphi)}$ | $\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-3\varphi)}$ | $\tilde{F}_{TT}^{\sin(2\varphi_{TT}-4\varphi)}$ | $\tilde{F}_{1TT}^{\sin(2\varphi_{TT}-\varphi)}$ | $\tilde{F}_{2TT}^{\sin(2\varphi_{TT}-\varphi)}$ | $\tilde{F}_{TT}^{\sin 2\varphi_{TT}}$ |
| $\varphi$-dep. | $\sin(2\varphi_{TT}-2\varphi)$ | | | $\sin(2\varphi_{TT}-3\varphi)$ | | $\sin(2\varphi_{TT}-4\varphi)$ | $\sin(2\varphi_{TT}-\varphi)$ | | $\sin 2\varphi_{TT}$ |
| twist-2 | $C[w_{dd}^{tt}G_{1TT}^\perp\bar{D}_1]$ | $0$ | $C[w_{dd}^{tt}G_{1TT}^\perp\bar{D}_1]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Understanding the twist-2 results:

only $q \to VX$, $\bar{q} \to \pi X$ part

$e^+e^- \to Z \to q\bar{q}$

$e^+e^- \to Z \to q\bar{q} \to V\pi X$

$$\frac{d\hat{\sigma}}{d\Omega} = \frac{\alpha^2}{4s}\chi\left[c_1^e c_1^q (1+\cos^2\theta) + 2c_3^e c_3^q \cos\theta\right]$$

unpolarized quark

$$F_{1U1} = 2c_1^e c_1^q \mathcal{C}[D_1 \bar{D}_1] \qquad F_{1LL1} = 2c_1^e c_1^q \mathcal{C}[D_{1LL} \bar{D}_1] \qquad F_{1T1}^{\sin(\varphi_S-\varphi)} = 2c_1^e c_1^q \mathcal{C}[w_1 D_{1T}^\perp \bar{D}_1]$$

$$F_{3U1} = 4c_3^e c_3^q \mathcal{C}[D_1 \bar{D}_1] \qquad F_{3LL1} = 4c_3^e c_3^q \mathcal{C}[D_{1LL} \bar{D}_1] \qquad F_{3T1}^{\sin(\varphi_S-\varphi)} = 4c_3^e c_3^q \mathcal{C}[w_1 D_{1T}^\perp \bar{D}_1]$$

$$F_{1LT1}^{\cos(\varphi_{LT}-\varphi)} = -2c_1^e c_1^q \mathcal{C}[w_1 D_{1LT}^\perp \bar{D}_1] \qquad F_{1TT1}^{\cos(2\varphi_{TT}-2\varphi)} = 2c_1^e c_1^q \mathcal{C}[w_{dd}^{tt} D_{1TT}^\perp \bar{D}_1]$$

$$F_{3LT1}^{\cos(\varphi_{LT}-\varphi)} = -4c_3^e c_3^q \mathcal{C}[w_1 D_{1LT}^\perp \bar{D}_1] \qquad F_{3TT1}^{\cos(2\varphi_{TT}-2\varphi)} = 4c_3^e c_3^q \mathcal{C}[w_{dd}^{tt} D_{1TT}^\perp \bar{D}_1]$$

---

$$P_q(\theta) = -\frac{c_1^e c_3^q (1+\cos^2\theta) + 2c_3^e c_1^q \cos\theta}{c_1^e c_1^q (1+\cos^2\theta) + 2c_3^e c_3^q \cos\theta}$$

longitudinally polarized quark

$$\tilde{F}_{1L1} = -2c_1^e c_3^q \mathcal{C}[G_{1L} \bar{D}_1] \qquad \tilde{F}_{1T1}^{\cos(\varphi_S-\varphi)} = 2c_1^e c_3^q \mathcal{C}[w_1 G_{1T}^\perp \bar{D}_1]$$

$$\tilde{F}_{3L1} = -4c_3^e c_1^q \mathcal{C}[G_{1L} \bar{D}_1] \qquad \tilde{F}_{3T1}^{\cos(\varphi_S-\varphi)} = 4c_3^e c_1^q \mathcal{C}[w_1 G_{1T}^\perp \bar{D}_1]$$

$$\tilde{F}_{1LT1}^{\sin(\varphi_{LT}-\varphi)} = -2c_1^e c_3^q \mathcal{C}[w_1 G_{1LT}^\perp \bar{D}_1] \qquad \tilde{F}_{1TT1}^{\sin(2\varphi_{TT}-2\varphi)} = 2c_1^e c_3^q \mathcal{C}[w_{dd}^{tt} G_{1TT}^\perp \bar{D}_1]$$

$$\tilde{F}_{3LT1}^{\sin(\varphi_{LT}-\varphi)} = -4c_3^e c_1^q \mathcal{C}[w_1 G_{1LT}^\perp \bar{D}_1] \qquad \tilde{F}_{3TT1}^{\sin(2\varphi_{TT}-2\varphi)} = 4c_3^e c_1^q \mathcal{C}[w_{dd}^{tt} G_{1TT}^\perp \bar{D}_1]$$

---

$$c_{nn}^q(\theta) = \frac{c_1^e c_2^q \sin^2\theta}{c_1^e c_1^q (1+\cos^2\theta) + 2c_3^e c_3^q \cos\theta}$$

transversally polarized quark

$$F_{U1}^{\cos 2\varphi} = -8c_1^e c_2^q \mathcal{C}[w_{hh} H_1^\perp \bar{H}_1^\perp] \qquad F_{LL1}^{\cos 2\varphi} = -8c_1^e c_2^q \mathcal{C}[w_{hh} H_{1LL}^\perp \bar{H}_1^\perp]$$

$$F_{L1}^{\sin 2\varphi} = -8c_1^e c_2^q \mathcal{C}[w_{hh} H_{1L}^\perp \bar{H}_1^\perp]$$

$$F_{LT1}^{\cos(\varphi_{LT}+\varphi)} = -8c_1^e c_2^q \mathcal{C}[\bar{w}_1 \mathcal{H}_{1LT}^\perp \bar{H}_1^\perp]$$

$$F_{LT1}^{\cos(\varphi_{LT}-3\varphi)} = 8c_1^e c_2^q \mathcal{C}[w_{hh}^t H_{1LT}^\perp \bar{H}_1^\perp]$$

$$F_{T1}^{\sin(\varphi_S+\varphi)} = -8c_1^e c_2^q \mathcal{C}[\bar{w}_1 \mathcal{H}_{1T}^\perp \bar{H}_1^\perp] \qquad F_{TT1}^{\cos 2\varphi_{TT}} = 8c_1^e c_2^q \mathcal{C}[w_2 H_{1TT}^{\perp'} \bar{H}_1^\perp]$$

$$F_{T1}^{\sin(\varphi_S-3\varphi)} = -8c_1^e c_2^q \mathcal{C}[w_{hh}^t H_{1T}^\perp \bar{H}_1] \qquad F_{TT1}^{\cos(2\varphi_{TT}-4\varphi)} = -4c_1^e c_2^q \mathcal{C}[w_{hh}^{tt} H_{1TT}^\perp \bar{H}_1^\perp]$$

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Azimuthal asymmetries up to twist-3:

unpolarized, twist-2:

$$\langle \cos 2\varphi \rangle_U^{(0)} = -\frac{C(y)\sum_q c_1^e c_1^q \mathcal{C}[w_{hh} H_1^\perp \bar{H}_1^\perp]}{\sum_q T_0^q(y)\mathcal{C}[D_1 \bar{D}_1]}$$

Collins asymmetry

---

unpolarized, twist-3:

$$\langle \cos \varphi \rangle_U^{(1)} = -\frac{2D(y)}{z_1 z_2 Q} \frac{M_1 \sum_q T_2^q(y)\mathcal{C}[w_1 D^\perp z_2 \bar{D}_1] + T_4^q(y)\mathcal{C}[\bar{w}_1 H z_2 \bar{H}_1^\perp] + \ldots\ldots}{\sum_q T_0^q(y)\mathcal{C}[D_1 \bar{D}_1]}$$

similar to Cahn effect in SiDIS

$$\langle \sin \varphi \rangle_U^{(1)} = \frac{2D(y)}{z_1 z_2 Q} \frac{M_1 \sum_q T_3^q(y)\mathcal{C}[w_1 G^\perp z_2 \bar{D}_1] + 2c_3^e c_2^q \mathcal{C}[\bar{w}_1 E z_2 \bar{H}_1^\perp] + \ldots\ldots}{\sum_q T_0^q(y)\mathcal{C}[D_1 \bar{D}_1]}$$

parity violating

---

polarized, twist-2:
(academic)

$$\langle \sin 2\varphi \rangle_L^{(0)} = -\frac{\lambda C(y)\sum_q c_1^e c_2^q \mathcal{C}[w_{hh} H_{1L}^\perp \bar{H}_1^\perp]}{\sum_q T_0^q(y)\mathcal{C}[(D_1 - \lambda G_{1L})\bar{D}_1]}$$

$$\langle \cos 2\varphi \rangle_{LL}^{(0)} = -\frac{C(y)\sum_q c_1^e c_2^q \mathcal{C}[w_{hh}(H_1^\perp + S_{LL} H_{1LL}^\perp)\bar{H}_1^\perp]}{\sum_q T_0^q(y)\mathcal{C}[(D_1 + S_{LL} D_{1LL})\bar{D}_1]}$$

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Hadron polarizations at twist-2 (averaged over $\varphi$):

### Longitudinal components

$$\langle \lambda \rangle^{(0)} = \frac{2}{3}\frac{\sum_q P_q(y)T_0^q(y)\mathcal{C}[G_{1L}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

spin transfer, parity violated

$$\langle S_{LL} \rangle^{(0)} = \frac{1}{2}\frac{\sum_q T_0^q(y)\mathcal{C}[D_{1LL}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

induced polarization, parity conserved

### Transverse components w.r.t. hadron-hadron plane

$$\langle S_T^t \rangle^{(0)} = -\frac{2}{3}\frac{\sum_q P_q(y)T_0^q(y)\mathcal{C}[w_1 G_{1T}^{\perp}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{LT}^n \rangle^{(0)} = \frac{2}{3}\frac{\sum_q P_q(y)T_0^q(y)\mathcal{C}[w_1 G_{1LT}^{\perp}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{TT}^{nt} \rangle^{(0)} = -\frac{2}{3}\frac{\sum_q P_q(y)T_0^q(y)\mathcal{C}[w_{dd}^{tt} G_{1TT}^{\perp}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

"worm-gear" effects, parity violated

$$\langle S_T^n \rangle^{(0)} = \frac{2}{3}\frac{\sum_q T_0^q(y)\mathcal{C}[w_1 D_{1T}^{\perp}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{LT}^t \rangle^{(0)} = -\frac{2}{3}\frac{\sum_q T_0^q(y)\mathcal{C}[w_1 D_{1LT}^{\perp}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{TT}^{nn} \rangle^{(0)} = -\frac{2}{3}\frac{\sum_q T_0^q(y)\mathcal{C}[w_{dd}^{tt} D_{1TT}^{\perp}\bar{D}_1]}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

induced polarization, parity conserved

strong energy dependence

very different

weak energy dependence

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Twist-3 contributions (averaged over $\varphi$):

"single-spin asymmetries"

### Transverse components

w.r.t. lepton-hadron plane exist at twist-3

$$\langle S_T^x\rangle^{(1)} = -\frac{8}{3z_1Q}\frac{M_1\sum_q \tilde{T}_3^q(y)\mathcal{C}[\mathcal{G}_T^\perp \bar{D}_1]+\ldots\ldots}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_T^y\rangle^{(1)} = \frac{8}{3z_1Q}\frac{M_1\sum_q \tilde{T}_2^q(y)\mathcal{C}[\mathcal{D}_T^\perp \bar{D}_1]+\ldots\ldots}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{LT}^x\rangle^{(1)} = -\frac{8}{3z_1Q}\frac{M_1\sum_q \tilde{T}_2^q(y)\mathcal{C}[\mathcal{D}_{LT}^\perp \bar{D}_1]+\ldots\ldots}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{LT}^y\rangle^{(1)} = \frac{8}{3z_1Q}\frac{M_1\sum_q \tilde{T}_3^q(y)\mathcal{C}[\mathcal{G}_{LT}^\perp \bar{D}_1]+\ldots\ldots}{\sum_q T_0^q(y)\mathcal{C}[D_1\bar{D}_1]}$$

For $e^+e^- \to \gamma^* \to V\pi X$

$$\langle S_T^y\rangle^{(1,em)} = \frac{8M_1\tilde{B}(y)}{3z_1QA(y)}\frac{\sum_q e_q^2\mathcal{C}[\mathcal{D}_T^\perp \bar{D}_1]+\ldots\ldots}{\sum_q e_q^2\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_{LT}^x\rangle^{(1,em)} = -\frac{8M_1\tilde{B}(y)}{3z_1QA(y)}\frac{\sum_q e_q^2\mathcal{C}[\mathcal{D}_{LT}^\perp \bar{D}_1]+\ldots\ldots}{\sum_q e_q^2\mathcal{C}[D_1\bar{D}_1]}$$

$$\langle S_T^x\rangle^{(1,em)} = \langle S_{LT}^y\rangle^{(1,em)} = 0$$

For the inclusive $e^+e^- \to Z \to VX$

$$\langle S_T^x\rangle_{in}^{(1)} = -\frac{8M_1D(y)}{3z_1Q}\frac{\sum_q T_3^q(y)G_T(z_1)}{\sum_q T_0^q(y)D_1(z_1)}$$

$$\langle S_T^y\rangle_{in}^{(1)} = \frac{8M_1D(y)}{3z_1Q}\frac{\sum_q T_2^q(y)D_T(z_1)}{\sum_q T_0^q(y)D_1(z_1)}$$

$$\langle S_{LT}^x\rangle_{in}^{(1)} = -\frac{8M_1D(y)}{3z_1Q}\frac{\sum_q T_2^q(y)D_{LT}(z_1)}{\sum_q T_0^q(y)D_1(z_1)}$$

$$\langle S_{LT}^y\rangle_{in}^{(1)} = \frac{8M_1D(y)}{3z_1Q}\frac{\sum_q T_3^q(y)G_{LT}(z_1)}{\sum_q T_0^q(y)D_1(z_1)}$$

$e^+e^- \to \gamma^* \to VX$

$$\langle S_T^y\rangle_{in}^{(1,em)} = \frac{8M_1\tilde{B}(y)}{3z_1QA(y)}\frac{\sum_q e_q^2 D_T(z_1)}{\sum_q e_q^2 D_1(z_1)}$$

$$\langle S_{LT}^x\rangle_{in}^{(1,em)} = -\frac{8M_1\tilde{B}(y)}{3z_1QA(y)}\frac{\sum_q e_q^2 D_{LT}(z_1)}{\sum_q e_q^2 D_1(z_1)}$$

$$\langle S_T^x\rangle_{in}^{(1,em)} = \langle S_{LT}^y\rangle_{in}^{(1,em)} = 0$$

# Parton model results for $e^+e^- \to Z \to V\pi X$

Hadron polarizations at twist-2 (averaged over $\theta$ and $\varphi$):

## Longitudinal components

$$\bar{\lambda}^{(0)} = \frac{2}{3}\frac{\sum_q \bar{P}_q c_1^q \mathcal{C}[G_{1L}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

spin transfer, parity violated

$$\bar{S}_{LL}^{(0)} = \frac{1}{2}\frac{\sum_q c_1^q \mathcal{C}[D_{1LL}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

induced polarization, parity conserved

## Transverse components w.r.t. hadron-hadron plane

$$\bar{S}_T^{t(0)} = -\frac{2}{3}\frac{\sum_q \bar{P}_q c_1^q \mathcal{C}[w_1 G_{1T}^{\perp}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

$$\bar{S}_{LT}^{n(0)} = \frac{2}{3}\frac{\sum_q \bar{P}_q c_1^q \mathcal{C}[w_1 G_{1LT}^{\perp}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

$$\bar{S}_{TT}^{nt(0)} = -\frac{2}{3}\frac{\sum_q \bar{P}_q c_1^q \mathcal{C}[w_{dd}^{tt} G_{1TT}^{\perp}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

"worm-gear" effects, parity violated

$$\bar{S}_T^{n(0)} = \frac{2}{3}\frac{\sum_q c_1^q \mathcal{C}[w_1 D_{1T}^{\perp}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

$$\bar{S}_{LT}^{t(0)} = -\frac{2}{3}\frac{\sum_q c_1^q \mathcal{C}[w_1 D_{1LT}^{\perp}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

$$\bar{S}_{TT}^{nn(0)} = -\frac{2}{3}\frac{\sum_q c_1^q \mathcal{C}[w_{dd}^{tt} D_{1TT}^{\perp}\bar{D}_1]}{\sum_q c_1^q \mathcal{C}[D_1\bar{D}_1]}$$

induced polarization, parity conserved

strong energy dependence

very different

weak energy dependence

# Parton model results for $e^+e^- \to Z \to V\pi X$

## Twist-3 contributions (averaged over $\theta$ and $\varphi$): “single-spin asymmetries”

### Transverse components

w.r.t. lepton-hadron plane exist at twist-3

$$\bar{S}_T^{x(1)} = -\frac{\pi}{2z_1Q}\frac{M_1\sum_q c_3^e c_1^q \mathcal{C}[\mathcal{G}_T^\perp \bar{D}_1]+......}{\sum_q c_1^e c_1^q \mathcal{C}[D_1 \bar{D}_1]}$$

$$\bar{S}_T^{y(1)} = -\frac{\pi}{2z_1Q}\frac{M_1\sum_q c_3^e c_3^q \mathcal{C}[\mathcal{D}_T^\perp \bar{D}_1]+......}{\sum_q c_1^e c_1^q \mathcal{C}[D_1 \bar{D}_1]}$$

$$\bar{S}_{LT}^{x(1)} = \frac{\pi}{2z_1Q}\frac{M_1\sum_q c_3^e c_3^q \mathcal{C}[\mathcal{D}_{LT}^\perp \bar{D}_1]+......}{\sum_q c_1^e c_1^q \mathcal{C}[D_1 \bar{D}_1]}$$

$$\bar{S}_{LT}^{y(1)} = \frac{\pi}{2z_1Q}\frac{M_1\sum_q c_3^e c_1^q \mathcal{C}[\mathcal{G}_{LT}^\perp \bar{D}_1]+......}{\sum_q c_1^e c_1^q \mathcal{C}[D_1 \bar{D}_1]}$$

For $e^+e^- \to \gamma^* \to V\pi X$

$$\bar{S}_T^{x(1,em)} = \bar{S}_T^{y(1,em)} = 0$$

$$\bar{S}_{LT}^{x(1,em)} = \bar{S}_{LT}^{y(1,em)} = 0$$

For the inclusive $e^+e^- \to Z \to VX$

$$\bar{S}_{T,in}^{x(1)} = -\frac{\pi M_1}{2z_1Q}\frac{\sum_q c_3^e c_1^q G_T(z_1)}{\sum_q c_1^e c_1^q D_1(z_1)}$$

$$\bar{S}_{T,in}^{y(1)} = -\frac{\pi M_1}{2z_1Q}\frac{\sum_q c_3^e c_3^q D_T(z_1)}{\sum_q c_1^e c_1^q D_1(z_1)}$$

$$\bar{S}_{LT,in}^{x(1)} = \frac{\pi M_1}{2z_1Q}\frac{\sum_q c_3^e c_3^q D_{LT}(z_1)}{\sum_q c_1^e c_1^q D_1(z_1)}$$

$$\bar{S}_{LT,in}^{y(1)} = \frac{\pi M_1}{2z_1Q}\frac{\sum_q c_3^e c_1^q G_{LT}(z_1)}{\sum_q c_1^e c_1^q D_1(z_1)}$$

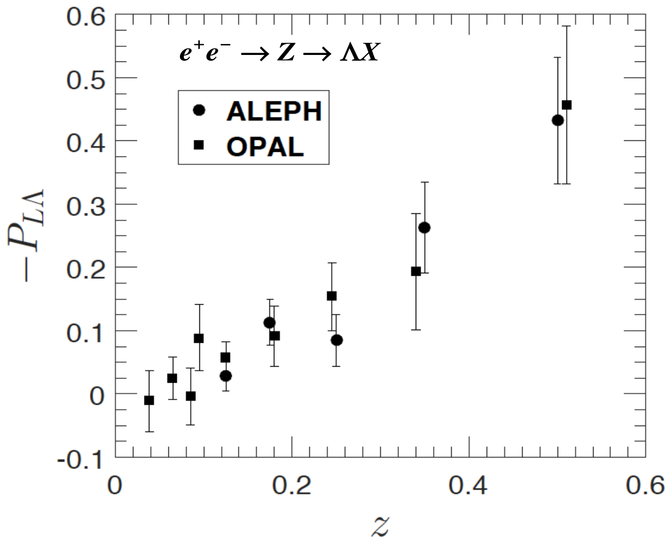
$e^+e^- \to \gamma^* \to VX$

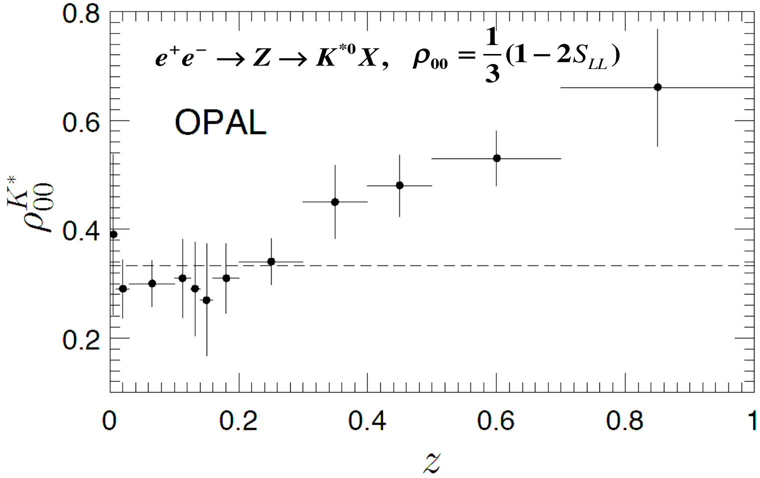
$$\bar{S}_{T,in}^{x(1,em)} = \bar{S}_{T,in}^{y(1,em)} = 0$$

$$\bar{S}_{LT,in}^{x(1,em)} = \bar{S}_{LT,in}^{y(1,em)} = 0$$

# Numerical estimations for inclusive processes

We have some data from LEP on $e^+e^- \to Z \to hX$





For $e^+e^- \to \gamma^*/Z \to hX$ at any given energy

$$\bar{P}_{L\Lambda}(z_1,Q)=\bar{\lambda}^{(0)}(z_1,Q)=\frac{2}{3}\frac{\sum_q \bar{P}_q(Q)W_q(Q)G_{1L}(z_1,Q)}{\sum_q W_q(Q)D_1(z_1,Q)}$$

spin transfer, parity violated,
strong energy dependence

$$\bar{S}_{LL}^{(0)}(z_1,Q)=\frac{1}{2}\frac{\sum_q W_q(Q)D_{1LL}(z_1,Q)}{\sum_q W_q(Q)D_1(z_1,Q)}$$

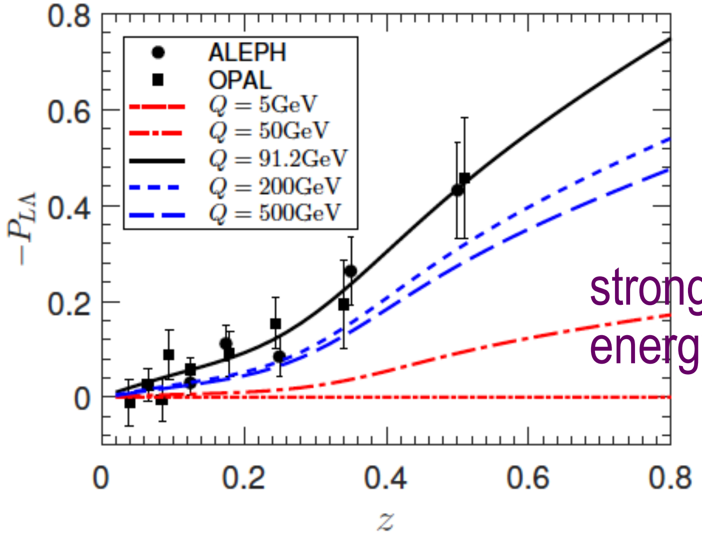
induced polarization, parity conserved
weak energy dependence

We can initialize a phenomenological analysis at leading twist with pQCD evolutions of FFs.

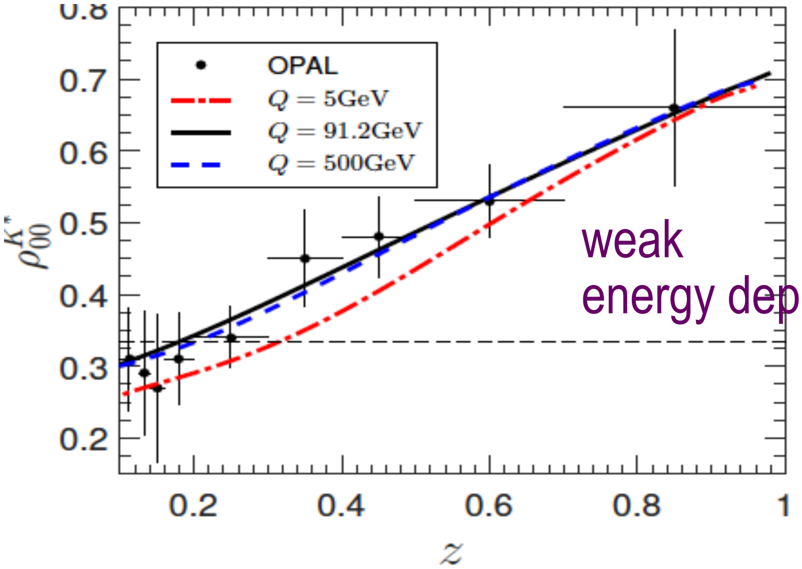
See K.B. Chen, W.H. Yang, Y.J. Zhou, & ZTL, arXiv:1609.07001 [hep-ph] (2016).

# Numerical estimations for inclusive processes

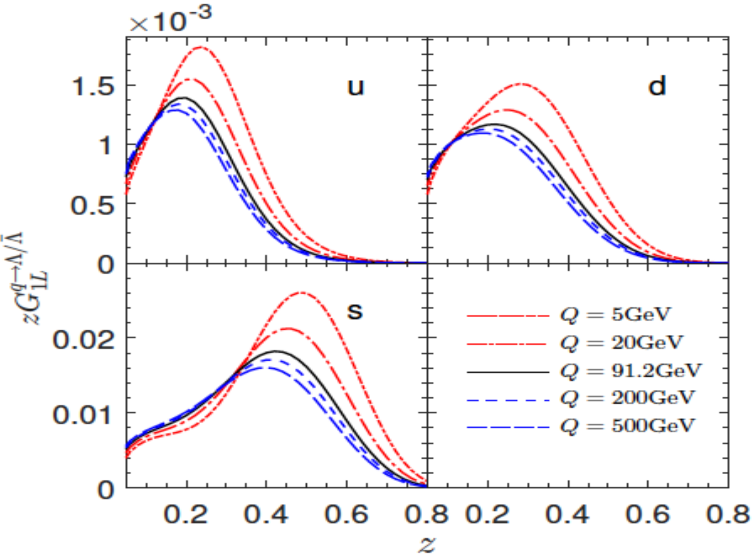
Leading twist and leading order pQCD evolution

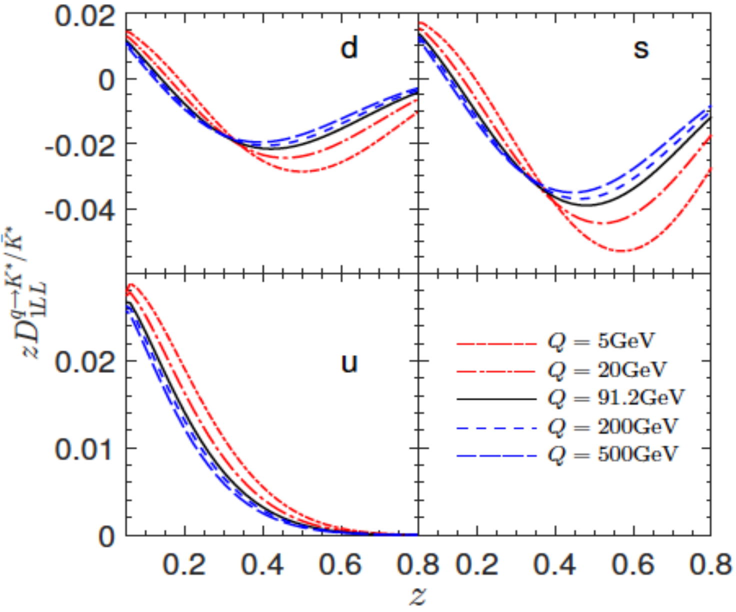


strong
energy dependence



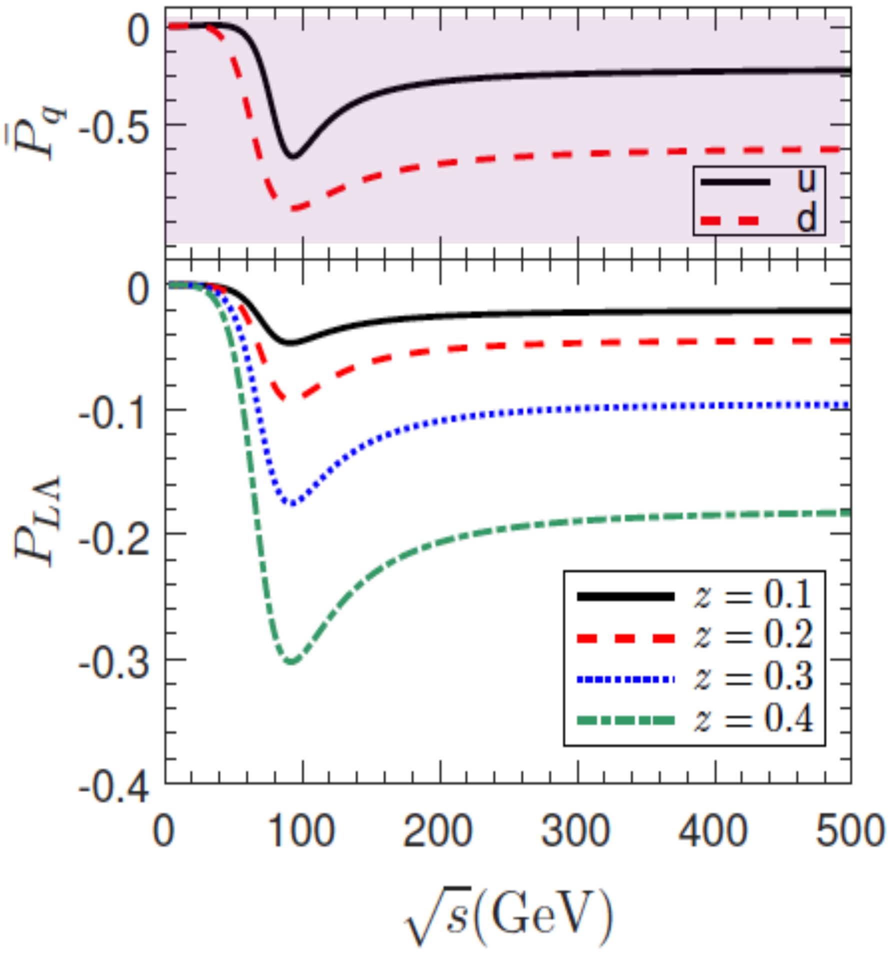
weak
energy dependence

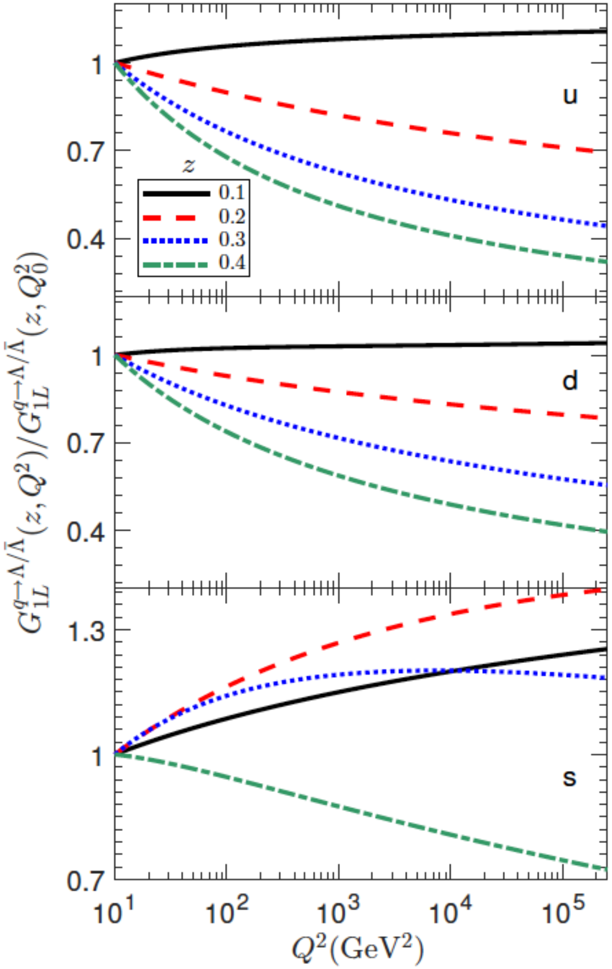




# Numerical estimations for inclusive processes

Leading twist and leading order pQCD evolution





Extend to other processes such as $pp \to hX$ and study at RHIC ............

# Contents

# Summary and Outlook

✪ A systematic study of $e^+e^- \to V\pi X$, a good place to study spin dependent FFs

★ A general and complete kinematic analysis:

- There are in total 81 structure functions or basic Lorentz tensors (BLT):

  $$\begin{pmatrix}\text{polarization dependent}\\ \text{Lorentz tensor set(s)}\end{pmatrix} = \begin{pmatrix}\text{polarization dependent}\\ \text{Lorentz scalar(s)}\end{pmatrix} \times \begin{pmatrix}\text{the unpolarized set}\end{pmatrix}$$

- 4 azimuthal asymmetries in the unpolarized case: $\langle\cos\varphi\rangle, \langle\cos 2\varphi\rangle, \langle\sin\varphi\rangle, \langle\sin 2\varphi\rangle$
  8 polarization components: 2 "longitudinal"; 6 "transverse" that can be measured w.r.t. the lepton-hadron or hadron-hadron plane, conveniently averaged over φ.

★ Complete parton model results up to twist-3 at LO pQCD:

- 27 non-vanishing structure functions at twist-2, 36 twist-3.
- Azimuthal asymmetries in the unpolarized case: a twist-2 $\langle\cos 2\varphi\rangle_U^{(0)}$ (Collins asymmetry), a twist-3 $\langle\cos\varphi\rangle_U^{(1)}$ (similar to "Cahn effect") and a parity violating $\langle\sin\varphi\rangle_U^{(1)}$.
- Hadron polarizations (averaged over azimuthal angle φ):
  twist-2: 2 "longitudinal" and 6 transverse components w.r.t. the hadron-hadron plane;
  twist-3: 6 transverse components w.r.t. the lepton-hadron plane, also for inclusive reaction.

★ A rough estimation for $P_{L\Lambda}$ and $\rho_{00}^{K*}$ at leading twist with LO pQCD evolution:

While $P_{L\Lambda}$ has a very strong $\sqrt{s}$ dependence and vanishes at low $\sqrt{s}$, $\rho_{00}^{K*}$ depends weakly on $\sqrt{s}$.

**Thank you for your attention!**

# General kinematic analysis for $e^+e^- \to V\pi X$

## The basic Lorentz tensors (BLTs) for the hadronic tensor

unpolarized part: 5+4=9

$$h_{Ui}^{S\mu\nu} = \left\{ g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2},\ p_{1q}^\mu p_{1q}^\nu,\ p_{2q}^\mu p_{2q}^\nu,\ p_{1q}^{\{\mu} p_{2q}^{\nu\}} \right\}$$

$$\tilde{h}_{Ui}^{S\mu\nu} = \left\{ \varepsilon^{\{\mu q p_1 p_2}(p_{1q}^{\nu\}},\ p_{2q}^{\nu\}}) \right\}$$

$$h_{U}^{A\mu\nu} = p_{1q}^{[\mu} p_{2q}^{\nu]}$$

$$\tilde{h}_{Ui}^{A\mu\nu} = \left\{ \varepsilon^{\mu\nu q p_1},\ \varepsilon^{\mu\nu q p_2} \right\}$$

$$A^{\{\mu} B^{\nu\}} \equiv A^\mu B^\nu + A^\nu B^\mu$$

$$A^{[\mu} B^{\nu]} \equiv A^\mu B^\nu - A^\nu B^\mu$$

$$\varepsilon^{\mu\nu\alpha p} \equiv \varepsilon^{\mu\nu\alpha\beta} p_\beta,\quad \varepsilon_\perp^{\mu\nu} \equiv \varepsilon^{\mu\nu\alpha\beta} \bar{n}_\alpha n_\beta$$

$$p_q \equiv p - \frac{p\cdot q}{q^2} q \qquad (p_q \cdot q = 0)$$

Vector polarization $S$-dependent part: 13+14=27

$$h_{Vi}^{S\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] \tilde{h}_{Ui}^{S\mu\nu},\ \ \varepsilon^{Sqp_1p_2} h_{Uj}^{S\mu\nu} \right\}$$

$$\tilde{h}_{Vi}^{S\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] h_{Ui}^{S\mu\nu},\ \ \varepsilon^{Sqp_1p_2} \tilde{h}_{Uj}^{S\mu\nu} \right\}$$

$$h_{Vi}^{A\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] \tilde{h}_{Ui}^{A\mu\nu},\ \ \varepsilon^{Sqp_1p_2} h_{U}^{A\mu\nu} \right\}$$

$$\tilde{h}_{Vi}^{A\mu\nu} = \left\{ \left[ (q\cdot S),\ (p_2\cdot S) \right] h_{U}^{A\mu\nu},\ \ \varepsilon^{Sqp_1p_2} \tilde{h}_{Uj}^{A\mu\nu} \right\}$$

The regularity: (polarization dependent Lorentz tensor set(s)) = (polarization dependent Lorentz scalar(s)) × (the unpolarized set)

unpolarized

$$\begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix}$$

longitudinal polarization

$$\begin{pmatrix} h_{Li}^{S\mu\nu} \\ \tilde{h}_{Li}^{S\mu\nu} \\ h_{Li}^{A\mu\nu} \\ \tilde{h}_{Li}^{A\mu\nu} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \\ h_{Ui}^{A\mu\nu} \end{pmatrix}$$

transverse polarization

$$\begin{pmatrix} h_{Ti}^{S\mu\nu} \\ \tilde{h}_{Ti}^{S\mu\nu} \\ h_{Ti}^{A\mu\nu} \\ \tilde{h}_{Ti}^{A\mu\nu} \end{pmatrix} = \left\{ (p_2\cdot S) \begin{pmatrix} \tilde{h}_{Ui}^{S\mu\nu} \\ h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \\ h_{Ui}^{A\mu\nu} \end{pmatrix},\quad \varepsilon^{Sqp_1p_2} \begin{pmatrix} h_{Ui}^{S\mu\nu} \\ \tilde{h}_{Ui}^{S\mu\nu} \\ h_{U}^{A\mu\nu} \\ \tilde{h}_{Ui}^{A\mu\nu} \end{pmatrix} \right\}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The cross section in Helicity-GJ-frame: $S_{TT}$-dependent part

$$\frac{2E_1E_2 d\sigma^{LT}}{d^3p_1 d^3p_2} = \frac{\alpha^2}{s^2}\chi\,|\vec{S}_{LT}|(\mathcal{F}_{LT} + \tilde{\mathcal{F}}_{LT})$$

$$|\vec{S}_{LT}|^2 = (S_{LT}^x)^2 + (S_{LT}^y)^2$$

$$\tan\varphi_{LT} = S_{LT}^x / S_{LT}^y$$

$$\varphi_S \leftrightarrow \varphi_{LT}$$

$$\mathcal{F}_T \leftrightarrow \tilde{\mathcal{F}}_{LT},\ \tilde{\mathcal{F}}_T \leftrightarrow \mathcal{F}_{LT}$$

$$F_{jT}^{xxx} \leftrightarrow \tilde{F}_{jLT}^{xxx},\ \tilde{F}_{jT}^{xxx} \leftrightarrow F_{jLT}^{xxx}$$

$$\begin{aligned}\mathcal{F}_{LT} = {} & \cos\varphi_{LT}[\sin\theta F_{1LT}^{\cos\varphi_{LT}} + \sin 2\theta F_{2LT}^{\cos\varphi_{LT}}] \\ & + \cos(\varphi_{LT}+\varphi)\sin^2\theta F_{LT}^{\cos(\varphi_{LT}+\varphi)} \\ & + \cos(\varphi_{LT}-\varphi)[(1+\cos^2\theta)F_{1LT}^{\cos(\varphi_{LT}-\varphi)} + \sin^2\theta F_{2LT}^{\cos(\varphi_{LT}-\varphi)} + \cos\theta F_{3LT}^{\cos(\varphi_{LT}-\varphi)}] \\ & + \cos(\varphi_{LT}-2\varphi)[\sin\theta F_{1LT}^{\cos(\varphi_{LT}-2\varphi)} + \sin 2\theta F_{2LT}^{\cos(\varphi_{LT}-2\varphi)}] \\ & + \cos(\varphi_{LT}-3\varphi)\sin^2\theta F_{LT}^{\cos(\varphi_{LT}-3\varphi)}\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{F}}_{LT} = {} & \sin\varphi_{LT}[\sin\theta \tilde{F}_{1LT}^{\sin\varphi_{LT}} + \sin 2\theta \tilde{F}_{2LT}^{\sin\varphi_{LT}}] \\ & + \sin(\varphi_{LT}+\varphi)\sin^2\theta F_{LT}^{\sin(\varphi_{LT}+\varphi)} \\ & + \sin(\varphi_{LT}-\varphi)[(1+\cos^2\theta)\tilde{F}_{1LT}^{\sin(\varphi_{LT}-\varphi)} + \sin^2\theta \tilde{F}_{2LT}^{\sin(\varphi_{LT}-\varphi)} + \cos\theta \tilde{F}_{3LT}^{\sin(\varphi_{LT}-\varphi)}] \\ & + \sin(\varphi_{LT}-2\varphi)[\sin\theta \tilde{F}_{1LT}^{\sin(\varphi_{LT}-2\varphi)} + \sin 2\theta \tilde{F}_{2LT}^{\sin(\varphi_{LT}-2\varphi)}] \\ & + \sin(\varphi_{LT}-3\varphi)\sin^2\theta \tilde{F}_{LT}^{\sin(\varphi_{LT}-3\varphi)}\end{aligned}$$

# General kinematic analysis for $e^+e^- \to V\pi X$

## The cross section in Helicity-GJ-frame: $S_{TT}$-dependent part

$$\frac{2E_1E_2\,d\sigma^{TT}}{d^3p_1\,d^3p_2} = \frac{\alpha^2}{s^2}\chi\,|\vec{S}_{TT}|\,(\mathcal{F}_{TT} + \tilde{\mathcal{F}}_{TT})$$

$$|\vec{S}_{TT}|^2 = (S_{TT}^{xx})^2 + (S_{TT}^{xy})^2$$

$$\tan 2\varphi_{TT} = S_{TT}^{xx} \,/\, S_{TT}^{xy}$$

$$(2\varphi_{TT} - \varphi) \leftrightarrow \varphi_{LT}$$

$$\mathcal{F}_{TT} \leftrightarrow \mathcal{F}_{LT},\ \tilde{\mathcal{F}}_{TT} \leftrightarrow \tilde{\mathcal{F}}_{LT}$$

$$F_{jTT}^{xxx} \leftrightarrow F_{jLT}^{xxx},\ F_{jTT}^{xxx} \leftrightarrow F_{jLT}^{xxx}$$

$$\begin{aligned}
\mathcal{F}_{TT} &= \cos 2\varphi_{TT} \sin^2\theta F_{TT}^{\cos 2\varphi_{TT}} \\
&+ \cos(2\varphi_{TT} - \varphi)[\sin\theta F_{1TT}^{\cos(2\varphi_{TT}-\varphi)} + \sin 2\theta F_{2TT}^{\cos(2\varphi_{TT}-\varphi)}] \\
&+ \cos(2\varphi_{TT} - 2\varphi)[(1+\cos^2\theta) F_{1TT}^{\cos(2\varphi_{TT}-2\varphi)} + \sin^2\theta F_{2TT}^{\cos(2\varphi_{TT}-2\varphi)} + \cos\theta F_{3TT}^{\cos(2\varphi_{TT}-2\varphi)}] \\
&+ \cos(2\varphi_{TT} - 3\varphi)[\sin\theta F_{1TT}^{\cos(2\varphi_{TT}-3\varphi)} + \sin 2\theta F_{2TT}^{\cos(2\varphi_{TT}-3\varphi)}] \\
&+ \cos(2\varphi_{TT} - 4\varphi)\sin^2\theta F_{TT}^{\cos(2\varphi_{TT}-4\varphi)}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_{TT} &= \sin 2\varphi_{TT} \sin^2\theta \tilde{F}_{TT}^{\sin 2\varphi_{TT}} \\
&+ \sin(2\varphi_{TT} - \varphi)[\sin\theta \tilde{F}_{1TT}^{\sin(2\varphi_{TT}-\varphi)} + \sin 2\theta \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-\varphi)}] \\
&+ \sin(2\varphi_{TT} - 2\varphi)[(1+\cos^2\theta) \tilde{F}_{1TT}^{\sin(2\varphi_{TT}-2\varphi)} + \sin^2\theta \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-2\varphi)} + \cos\theta \tilde{F}_{3TT}^{\sin(2\varphi_{TT}-2\varphi)}] \\
&+ \sin(2\varphi_{TT} - 3\varphi)[\sin\theta \tilde{F}_{1TT}^{\sin(2\varphi_{TT}-3\varphi)} + \sin 2\theta \tilde{F}_{2TT}^{\sin(2\varphi_{TT}-3\varphi)}] \\
&+ \sin(2\varphi_{TT} - 4\varphi)\sin^2\theta \tilde{F}_{TT}^{\sin(2\varphi_{TT}-4\varphi)}
\end{aligned}$$

# TMD PDFs defined via quark-quark correlator

**The Lorentz decomposition**

totally 8(twist 2)+16(twist 3)+8(twist 4) components

$$\tilde{\Phi}^{(0)}(x,k_\perp;p,S)=\lambda M e_L(x,k_\perp)+(k_\perp\cdot S_T)e_T(x,k_\perp)$$

twist-3

$$\tilde{\Phi}^{(0)}_\alpha(x,k_\perp;p,S)=\frac{p^+}{M}\bar{n}_\alpha\left[\lambda M g_{1L}(x,k_\perp)+(k_\perp\cdot S_T)g_{1T}^\perp(x,k_\perp)\right]$$

twist-2

$$-MS_{T\alpha}g_T(x,k_\perp)-\frac{k_{\perp\alpha}}{M}\left[\lambda M g_L^\perp(x,k_\perp)+(k_\perp\cdot S_T)g_T^\perp(x,k_\perp)\right]+\tilde{k}_{\perp\alpha}g^\perp(x,k_\perp)$$

$$+\frac{M}{p^+}n_\alpha\left[\lambda M g_{3L}(x,k_\perp)+(k_\perp\cdot S_T)g_{3T}^\perp(x,k_\perp)\right]$$

twist-4

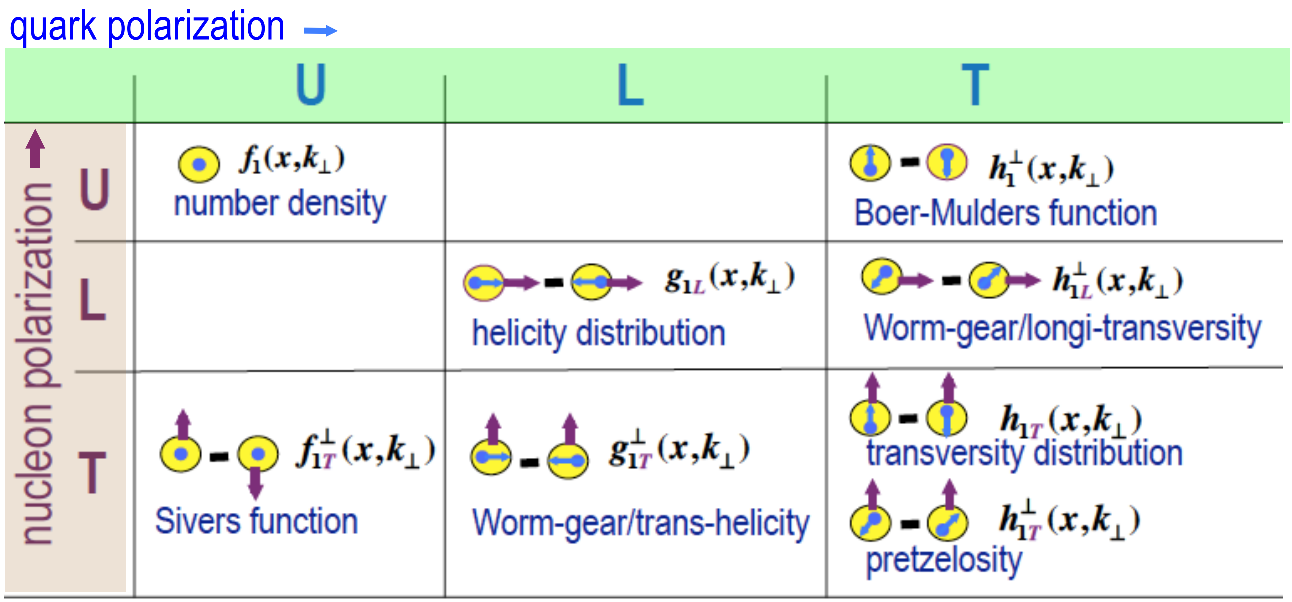
$$\Phi^{(0)}_{\rho\alpha}(x,k_\perp;p,S)=\frac{p^+}{M}\bar{n}_{[\rho}\left\{MS_{T\alpha]}h_{1T}(x,k_\perp)+\frac{k_{\perp\alpha]}}{M}\left[\lambda M h_{1L}^\perp(x,k_\perp)+(k_\perp\cdot S_T)h_{1T}^\perp(x,k_\perp)\right]+\tilde{k}_{\perp\alpha]}h_1^\perp(x,k_\perp)\right\}$$

$$+\ S_{T[\rho}k_{\perp\alpha]}h_T^{\perp'}(x,k_\perp)+M\varepsilon_{\perp\rho\alpha}h(x,k_\perp)-\bar{n}_{[\rho}n_{\alpha]}\left[\lambda M h_L(x,k_\perp)-(k_\perp\cdot S_T)h_T^\perp(x,k_\perp)\right]$$

$$+\frac{M}{p^+}n_{[\rho}\left\{MS_{T\alpha]}h_{3T}(x,k_\perp)+k_{\perp\alpha]}\left[\lambda M h_{3L}^\perp(x,k_\perp)+(k_\perp\cdot S_T)h_{3T}^\perp(x,k_\perp)\right]+\tilde{k}_{\perp\alpha]}h_3^\perp(x,k_\perp)\right\}$$

# TMD PDFs defined via quark-quark correlator

Twist-2 TMD PDFs

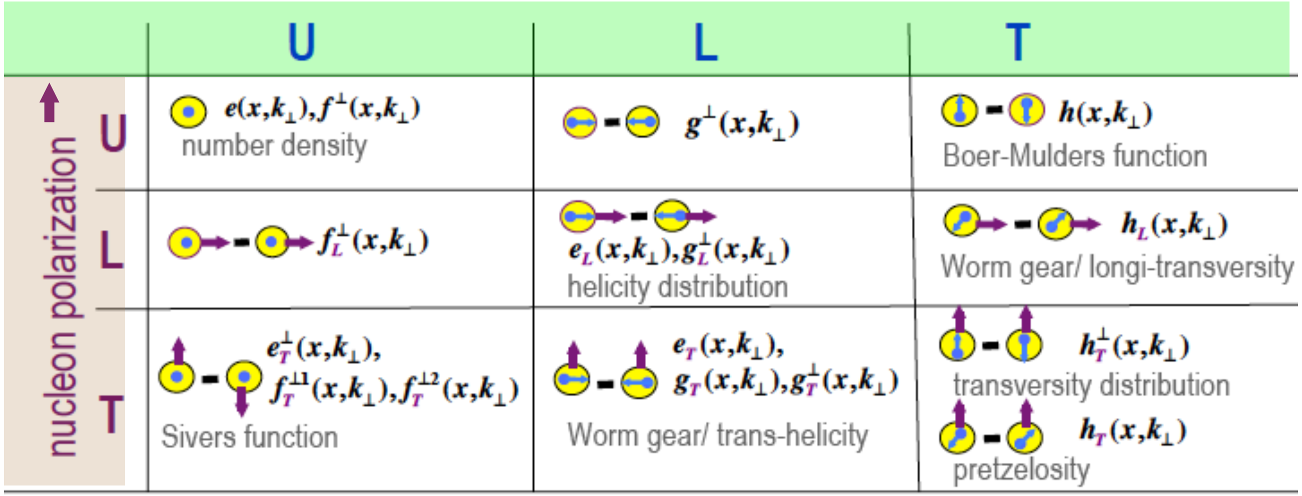


quark polarization →

nucleon polarization ↑

| | U | L | T |
|---|---|---|---|
| U | $f_1(x,k_\perp)$<br>number density | | $h_1^\perp(x,k_\perp)$<br>Boer-Mulders function |
| L | | $g_{1L}(x,k_\perp)$<br>helicity distribution | $h_{1L}^\perp(x,k_\perp)$<br>Worm-gear/longi-transversity |
| T | $f_{1T}^\perp(x,k_\perp)$<br>Sivers function | $g_{1T}^\perp(x,k_\perp)$<br>Worm-gear/trans-helicity | $h_{1T}(x,k_\perp)$<br>transversity distribution<br>$h_{1T}^\perp(x,k_\perp)$<br>pretzelosity |

Twist-3 TMD PDFs



nucleon polarization ↑

| | U | L | T |
|---|---|---|---|
| U | $e(x,k_\perp),f^\perp(x,k_\perp)$<br>number density | $g^\perp(x,k_\perp)$ | $h(x,k_\perp)$<br>Boer-Mulders function |
| L | $f_L^\perp(x,k_\perp)$ | $e_L(x,k_\perp),g_L^\perp(x,k_\perp)$<br>helicity distribution | $h_L(x,k_\perp)$<br>Worm gear/ longi-transversity |
| T | $e_T^\perp(x,k_\perp),$<br>$f_T^{\perp 1}(x,k_\perp),f_T^{\perp 2}(x,k_\perp)$<br>Sivers function | $e_T(x,k_\perp),$<br>$g_T(x,k_\perp),g_T^\perp(x,k_\perp)$<br>Worm gear/ trans-helicity | $h_T^\perp(x,k_\perp)$<br>transversity distribution<br>$h_T(x,k_\perp)$<br>pretzelosity |

# TMD FFs defined via quark-quark correlator (for spin-1/2 hadrons)

**The Lorentz decomposition**

totally 8(twist 2)+16(twist 3)+8(twist 4) components

$$z\tilde{\Xi}^{(0)}(z,k_{F\perp};p,S) = \lambda M E_L(z,k_{F\perp}) + (k_{F\perp}\cdot S_T)E_T(z,k_{F\perp})$$

twist-3

$$z\tilde{\Xi}^{(0)}_{\alpha}(z,k_{F\perp};p,S) = \frac{p^+}{M}\bar{n}_\alpha\Big[\lambda M G_{1L}(z,k_{F\perp}) + (k_{F\perp}\cdot S_T)G_{1T}^{\perp}(z,k_{F\perp})\Big]$$
$$- M S_{T\alpha}G_T(z,k_{F\perp}) - \frac{k_{F\perp\alpha}}{M}\Big[\lambda M G_L^{\perp}(z,k_{F\perp}) + (k_{F\perp}\cdot S_T)G_T^{\perp}(z,k_{F\perp})\Big] + \tilde{k}_{F\perp\alpha}G^{\perp}(z,k_{F\perp})$$
$$+ \frac{M}{p^+}n_\alpha\Big[\lambda M G_{3L}(z,k_{F\perp}) + (k_{F\perp}\cdot S_T)G_{3T}^{\perp}(z,k_{F\perp})\Big]$$

twist-2

twist-4

$$z\Xi^{(0)}_{\rho\alpha}(z,k_{F\perp};p,S) = \frac{p^+}{M}\bar{n}_{[\rho}\Big\{M S_{T\alpha]}H_{1T}(z,k_{F\perp}) + \frac{k_{F\perp\alpha]}}{M}\Big[\lambda M H_{1L}^{\perp}(z,k_{F\perp}) + (k_{F\perp}\cdot S_T)H_{1T}^{\perp}(z,k_{F\perp})\Big] + \tilde{k}_{F\perp\alpha]}H_1^{\perp}(z,k_{F\perp})\Big\}$$
$$+ S_{T[\rho}k_{F\perp\alpha]}H_T^{\perp'}(z,k_{F\perp}) + M\varepsilon_{\perp\rho\alpha}H(z,k_{F\perp}) - \bar{n}_{[\rho}n_{\alpha]}\Big[\lambda M H_L(z,k_{F\perp}) - (k_{F\perp}\cdot S_T)H_T^{\perp}(z,k_{F\perp})\Big]$$
$$+ \frac{M}{p^+}n_{[\rho}\Big\{M S_{T\alpha]}H_{3T}(z,k_{F\perp}) + \frac{k_{F\perp\alpha]}}{M}\Big[\lambda M H_{3L}^{\perp}(z,k_{F\perp}) + (k_{F\perp}\cdot S_T)H_{3T}^{\perp}(z,k_{F\perp})\Big] + \tilde{k}_{F\perp\alpha]}H_3^{\perp}(z,k_{F\perp})\Big\}$$

# TMD FFs defined via quark-quark correlator (T-dep. part)

## The Lorentz decomposition

totally 10(twist-2)+20(twist-3)+10(twist-4) components

The tensor polarization dependent part

$$z\tilde{\Xi}_S^{T(0)}(z,k_{F\perp};p,S) = (\tilde{k}_{F\perp}\cdot S_{LT})E'^{\perp}_{LT}(z,k_{F\perp}) + \frac{S_{TT}^{\tilde{k}_F k_F}}{M}E'^{\perp}_{TT}(z,k_{F\perp})$$

$$z\tilde{\Xi}_\alpha^{T(0)}(z,k_{F\perp};p,S) = \frac{p^+}{M}\bar{n}_\alpha\left[(\tilde{k}_{F\perp}\cdot S_{LT})G^{\perp}_{1LT}(z,k_{F\perp}) + \frac{S_{TT}^{\tilde{k}_F k_F}}{M}G^{\perp}_{1TT}(z,k_{F\perp})\right]$$

$$- M\tilde{S}_{LT\alpha}G_{LT}(z,k_{F\perp}) - \tilde{S}^{k_F}_{TT\alpha}G'^{\perp}_{TT}(z,k_{F\perp}) - \tilde{k}_{F\perp\alpha}\left[S_{LL}G^{\perp}_{LL}(z,k_{F\perp}) + \frac{k_{F\perp}\cdot S_{LT}}{M}G^{\perp}_{LT}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M^2}G^{\perp}_{TT}(z,k_{F\perp})\right]$$

$$+ \frac{M}{p^+}n_\alpha\left[(\tilde{k}_{F\perp}\cdot S_{LT})G^{\perp}_{3LT}(z,k_{F\perp}) + \frac{S_{TT}^{\tilde{k}_F k_F}}{M}G^{\perp}_{3TT}(z,k_{F\perp})\right]$$

$$z\Xi_{\rho\alpha}^{T(0)}(z,k_{F\perp};p,S) = -\frac{p^+}{M}\bar{n}_{[\rho}\left\{M\tilde{S}_{LT\alpha]}H_{1LT}(z,k_{F\perp}) + \tilde{S}^{k_F}_{TT\alpha]}H'^{\perp}_{1TT}(z,k_{F\perp}) + \tilde{k}_{F\alpha]}\left[S_{LL}H^{\perp}_{1LL}(z,k_{F\perp}) + \frac{k_{F\perp}\cdot S_{LT}}{M}H^{\perp}_{1LT}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M}H^{\perp}_{1TT}(z,k_{F\perp})\right]\right]\right\}$$

$$+ \varepsilon_{\perp\rho\alpha}\left[MS_{LL}H_{LL}(z,k_{F\perp}) + (k_{F\perp}\cdot S_{LT})H^{\perp}_{LT}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M}H^{\perp}_{TT}(z,k_{F\perp})\right] + \bar{n}_{[\rho}n_{\alpha]}\left[(\tilde{k}_{F\perp}\cdot S_{LT})H'^{\perp}_{LT}(z,k_{F\perp}) + \frac{S_{TT}^{\tilde{k}_F k_F}}{M}H'^{\perp}_{TT}(z,k_{F\perp})\right]$$

$$- \frac{M}{p^+}n_{[\rho}\left\{\tilde{k}_{F\perp\alpha]}\left[S_{LL}H^{\perp}_{3LL}(z,k_{F\perp}) + \frac{k_{F\perp}\cdot S_{LT}}{M}H^{\perp}_{3LT}(z,k_{F\perp}) + \frac{S_{TT}^{k_F k_F}}{M^2}H^{\perp}_{3TT}(z,k_{F\perp})\right] + M\tilde{S}_{LT\alpha]}H_{3LT}(z,k_{F\perp}) + \tilde{S}^{k_F}_{TT\alpha]}H'^{\perp}_{3TT}(z,k_{F\perp})\right\}$$

twist-3

twist-2

twist-4