Beyond traditional parton showers

Jeff Forshaw

Overview

1. All-order calculations in perturbative QCD

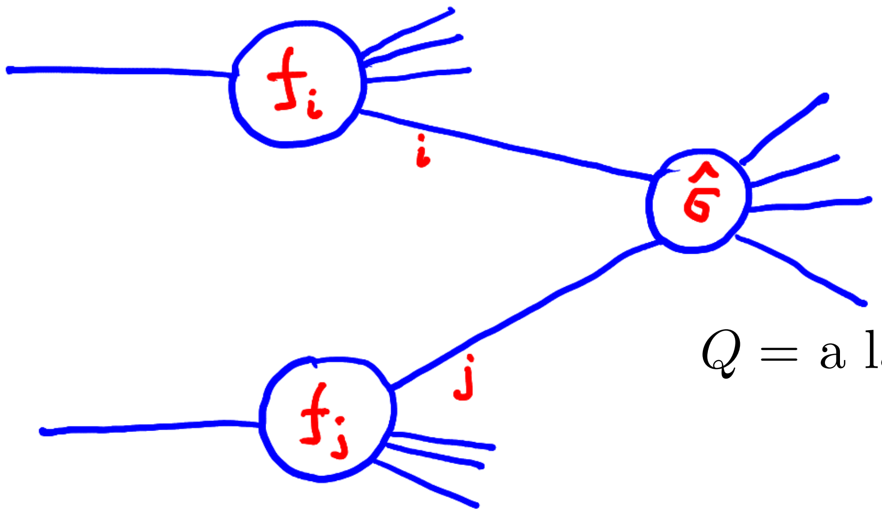
2. A case study: “Jet Vetoing”

3. A general algorithm

Factorization allows us to make predictions

$$d\sigma = \sum_{i,j} \int dx_1 dx_2 \; f_i(x_1,\mu_F) f_j(x_2,\mu_F) \; d\hat{\sigma}_{ij}(Q,\mu_R,\mu_F)$$

f_i is a universal function

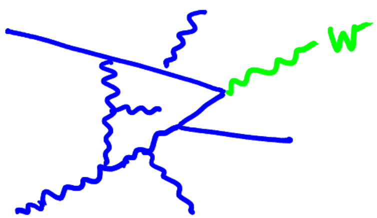


Q = a large invariant mass.

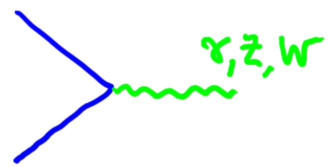
Collins, Soper, Sterman
Nucl. Phys B308 (1988) 833.

$$\hat{\sigma} \sim \alpha_s^n \left(\hat{\sigma}_{LO} + \alpha_s \hat{\sigma}_{NLO} + \alpha_s^2 \hat{\sigma}_{NNLO} + \alpha_s^3 \hat{\sigma}_{NNNLO} + \dots \right)$$

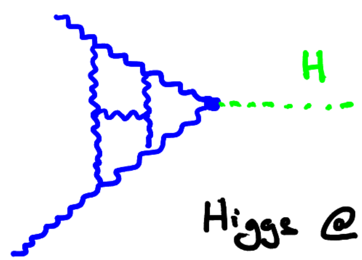
- $\hat{\sigma}_{LO}$: order of magnitude
- $\hat{\sigma}_{NLO}$: automation
- $\hat{\sigma}_{NNLO}$: state of art
- $\hat{\sigma}_{NNNLO}$: world record in QCD



W + 4 jets @ NLO
[2010]
$n = 4$



Drell-Yan @ LO
$n = 0$

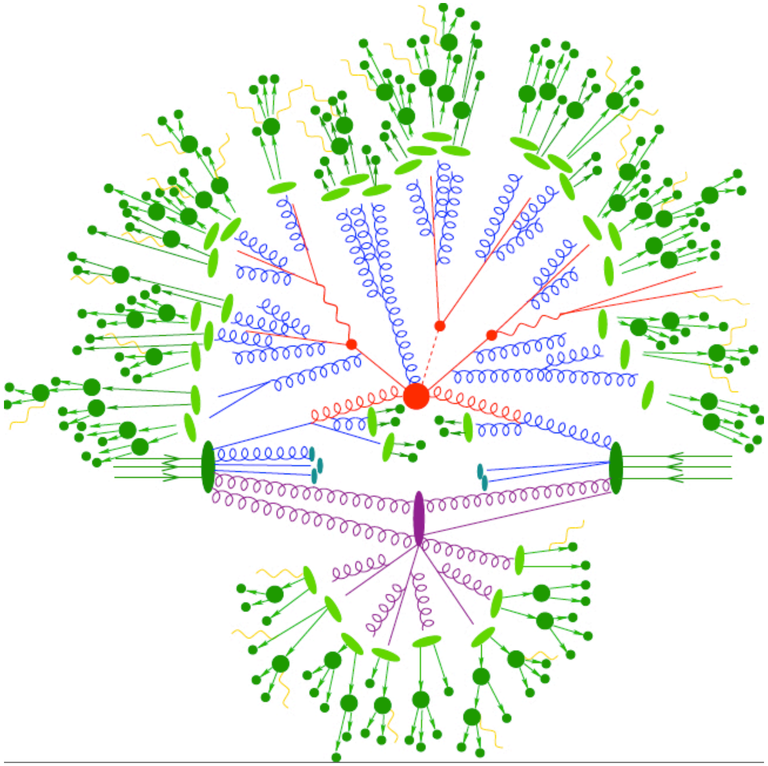


Higgs @ N3LO
$\sim 10^5$ Feynman diagrams
$\sim$ ½ billion integrals

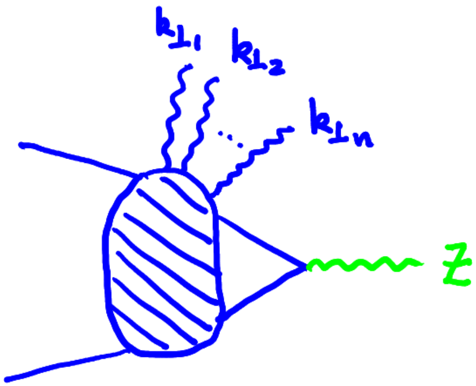
Many observables are (often by construction) sensitive to multi-particle production in the final state.

The production of many particles is typically not rare: the rate is enhanced by large logarithmic terms in the perturbation series. **Fixed order calculations not always sufficient.**

QCD calculations implemented into “parton shower” Monte Carlo codes.



A classic example: Drell-Yan



$$\frac{\mathrm{d}\sigma}{\mathrm{d}p_T} = \frac{\mathrm{d}}{\mathrm{d}p_T}\left(\sigma_0 \exp\left(-\frac{\alpha_s C_F}{2\pi} L^2\right)\right)$$

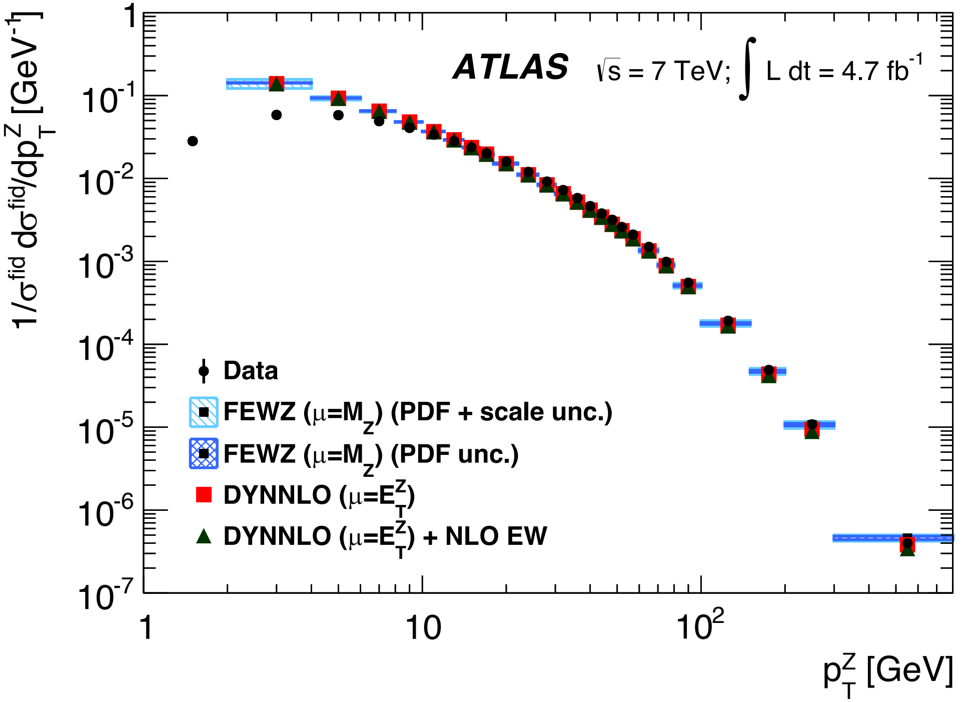
p_T = transverse momentum of the Z particle

$$L = \ln(Q^2/p_T^2)$$

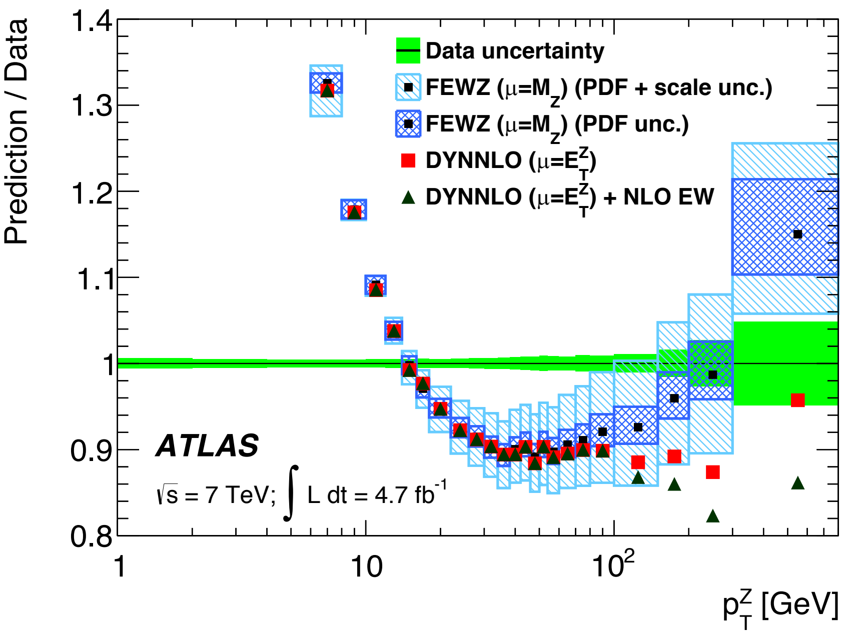
- Fixed order in α_s fails if $L \gg 1$.
- Accounts for all terms $\sim \alpha_s^n L^{2n}$.
- Simple interpretation as a non-emission probability.

$$\int_0^{p_T} \frac{\mathrm{d}\sigma}{\mathrm{d}p'_T}\mathrm{d}p'_T = \sigma_0 \exp\left(-\frac{\alpha_s C_F}{2\pi} L^2\right)$$

= rate of producing Z particles with transverse momentum below p_T



NNLO fails at low p_T



Beyond the simplest approximation: systematic resummation of the large logarithms is possible.

$$\mathcal{O} = 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \cdots$$

For many ("global") observables:

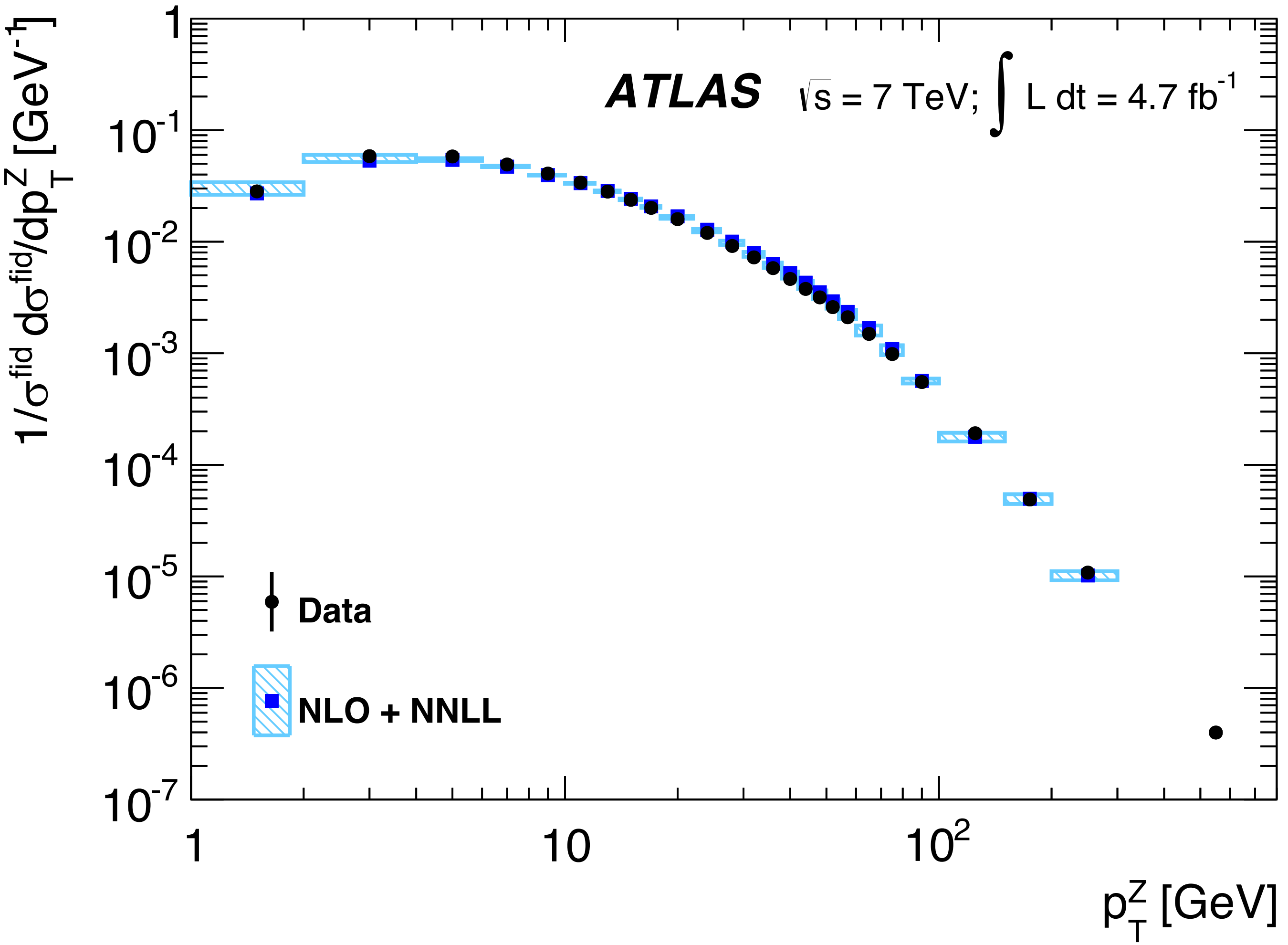
$$\mathcal{O} = C(\alpha_s)\ \exp\left[L g_{LL}(\alpha_s L) + g_{NLL}(\alpha_s L) + \alpha_s g_{NNLL}(\alpha_s L) + \cdots\right]$$

$$C(\alpha_s) = 1 + \alpha_s C_{NNLL} + \cdots$$

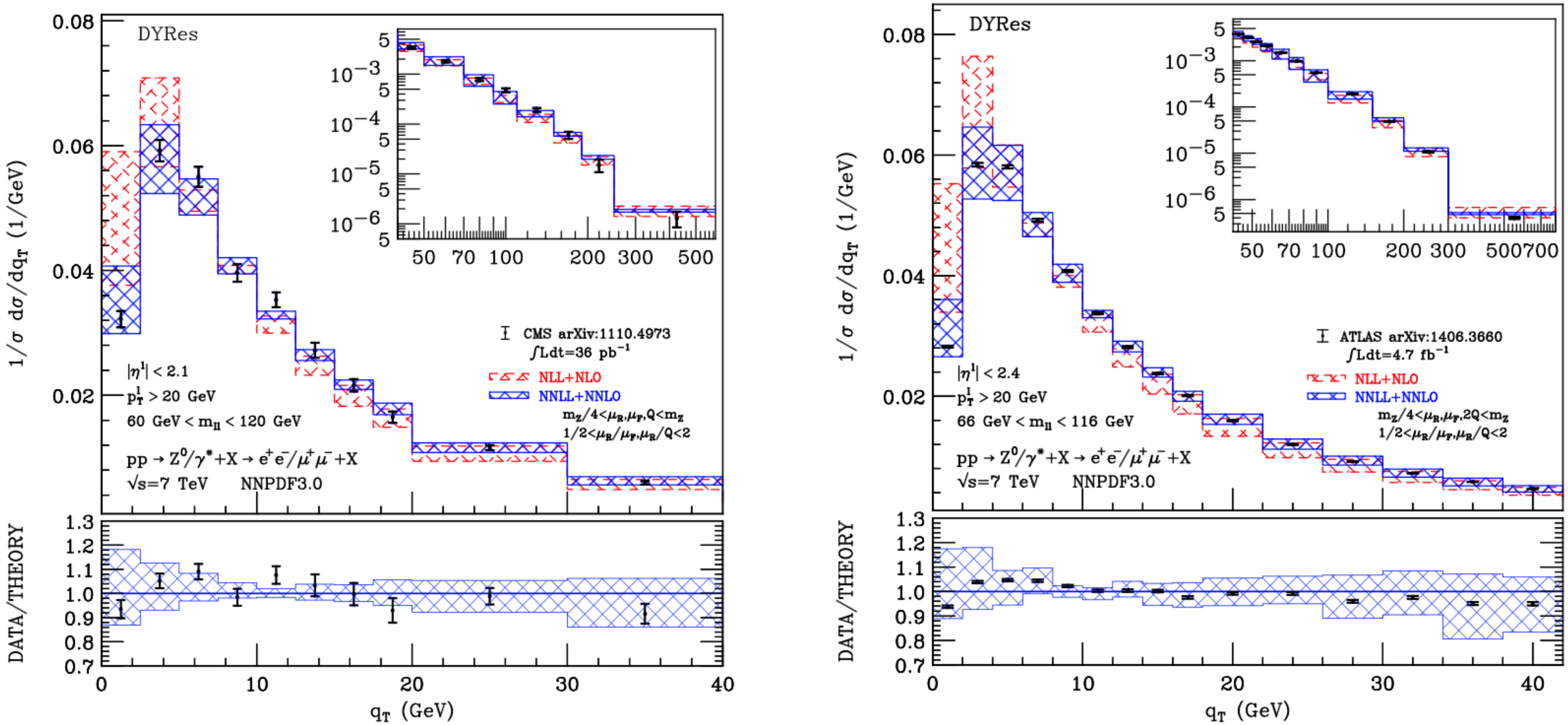
Can match the resummed result with the fixed order result.

Progress in automated resummation
e^+e^- at NLL: Banfi, Salam, Zanderighi hep-ph//0407286
e^+e^- at NNLL: Banfi, McAslan, Monni, Zanderighi arXiv:1412.2126
Gerwick, Schumann, Höche, Marzani arXiv: 1411.7325

All orders (resummed) works well



Banfi, Dasgupta, Marzani and Tomlinson, arXiv:1205.4760



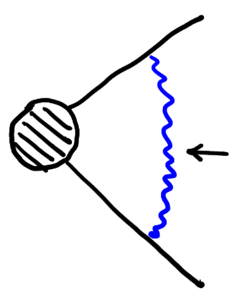
NNLO + NNLL

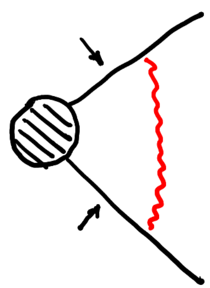
Catani, de Florian, Ferrera, Grazzini, arXiv:1507.06937

The large logarithms of infra-red origin are a result of:

1. **Soft** gluon emission

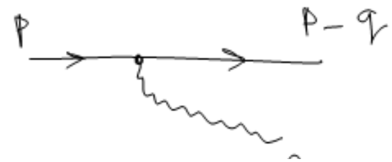
2. **Collinear** parton branching



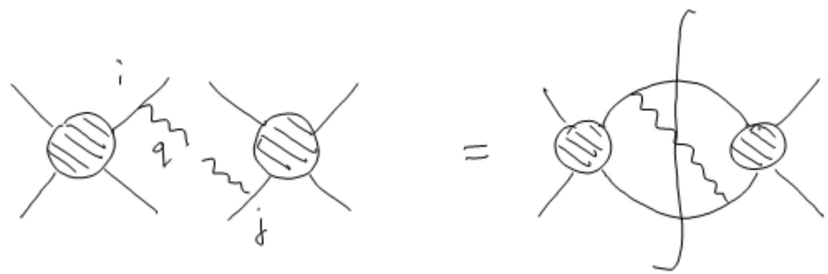


Goal is to systematically sum up all enhanced terms, either on a case-by-case basis or more generally.

SOFT GLUONS



$$= 2gp_\mu \delta_{\lambda\lambda'} T^a_{ij}$$

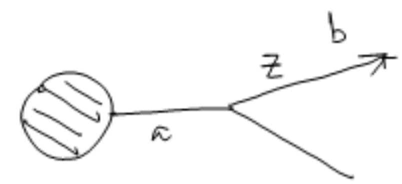


$$d\sigma_{n+1} = d\sigma_n \frac{\alpha_s}{2\pi} \frac{dE}{E} \frac{d\Omega}{2\pi} \sum_{ij} C_{ij} E^2 \, \frac{p_i \cdot p_j}{p_i \cdot q \; p_j \cdot q}$$

- Only have to consider soft gluons off the external legs of a hard subprocess.
- <u>Colour factor is the “problem”.</u>

COLLINEAR EMISSIONS

Colour structure is easier. It is as if emission is off the parton to which it is collinear.



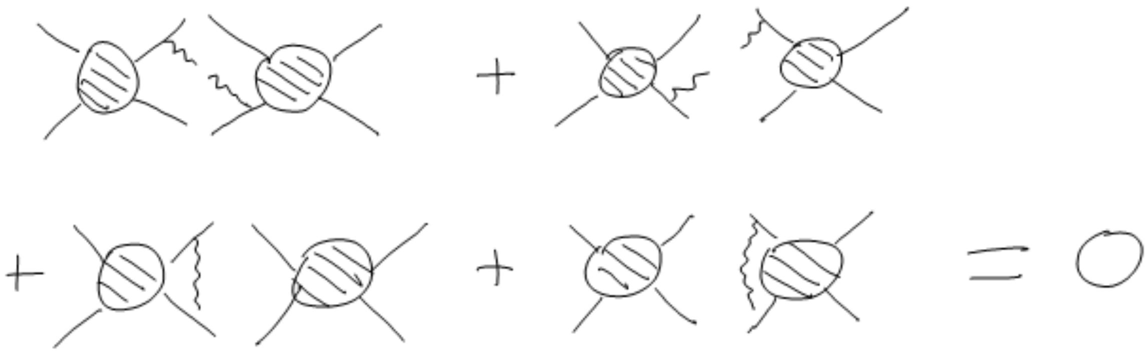
$$d\sigma_{n+1} = d\sigma_n \frac{\alpha_s}{2\pi}\frac{dq^2}{q^2}dz\ P_{ba}(z)$$

Simulation codes exploit the fact that in the "large N_c" approximation both wide-angle soft and collinear emissions can be included via a classical branching algorithm, i.e. quantum interference included by clever re-arrangement of the interference terms = the HERWIG project.

This beautiful physics is also a serious drawback.

Not all observables are affected by soft and/or collinear enhancements

Intuitive: inclusive observables do not care that the outgoing partons may subsequently radiate additional soft and/or collinear particles.



Bloch-Nordsieck Theorem

Real & virtual graphs cancel exactly in soft approximation if the real emissions are integrated over without restriction.

Weighting the real emissions induces a miscancellation and a logarithm,

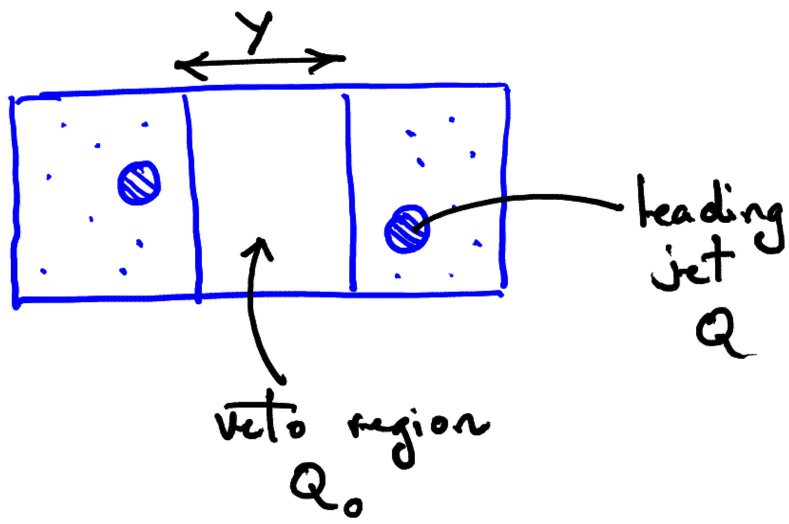
e.g.

$$\alpha_s \int_0^Q \frac{\mathrm{d}k_T}{k_T} - \alpha_s \int_0^\mu \frac{\mathrm{d}k_T}{k_T} = \alpha_s \ln \frac{Q}{\mu}$$

Virtual loop

Real emission phase-space if emissions forbidden above μ

Case study: JET VETOING & "Gaps between jets"



Jets produced with $p_T = Q \gg Q_0$

Observable restricts emission only in the gap region therefore expect

$$\alpha_s^n \log^n(Q/Q_0)$$

i.e. do not expect collinear enhancement since we sum inclusively over the collinear regions of the incoming and outgoing partons.

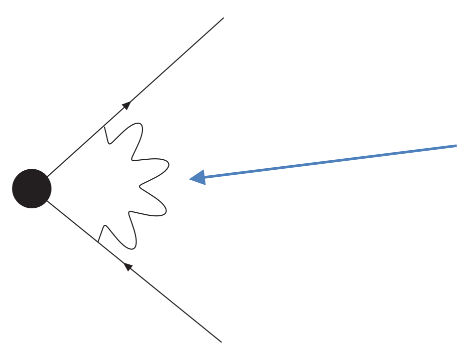
Real emissions are forbidden in the phase-space region

$$-Y/2 < y < Y/2$$
$$k_T > Q_0$$

"*By Bloch-Nordsieck, all other real emissions cancel and we therefore only need to compute the virtual soft gluon corrections to the primary hard scattering.*"

$e^+e^- \to q\bar{q}$ case is very simple:

$$\sigma_{\text{gap}} = \sigma_0 \ \exp\left(-C_F \frac{\alpha_s}{\pi} Y \ln\left(\frac{Q}{Q_0}\right)\right)$$



The virtual gluon is integrated over "in gap" momenta, i.e. the region where real emissions are forbidden.

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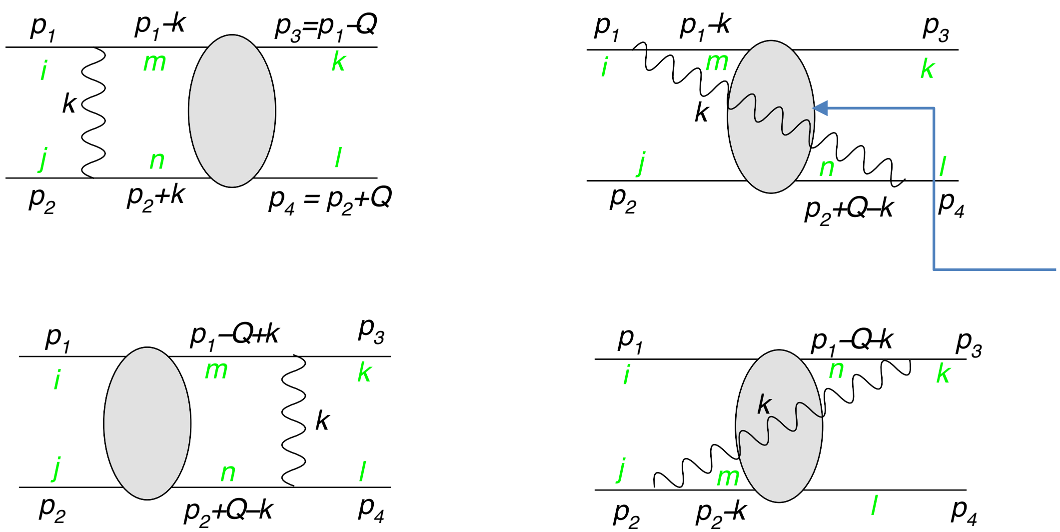


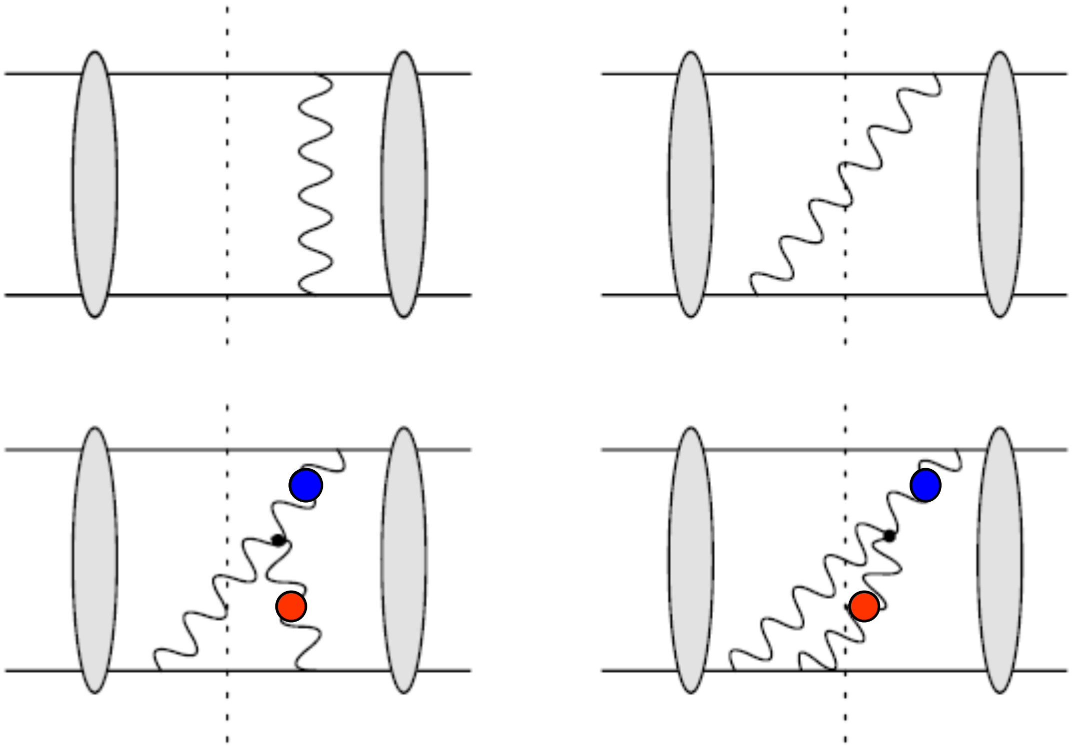
The virtual gluon is integrated over "in gap" momenta, i.e. the region where real emissions are forbidden.

But this is over an over simplification: we cannot get rid of the real emissions so easily….

Dasgupta & Salam 2001

hep-ph/0104277

This observable is **non-global**



Real & virtual corrections cancel for "out of gap" gluons.

But these do not if the gluon marked with a red blob is in the forbidden region: the 2nd cut is not allowed.

The observable is sensitive to "out of gap" gluons.

The real emissions no longer cancel once we start to evolve emissions (such as those denoted by the blue blob in the above) which lie *outside of the gap* region and which have $k_T > Q_0$.

If $k_T < Q_0$ then subsequent evolution also has $k_T < Q_0$ and cancellation works.

- The observable is sensitive to gluon emissions outside of the gap, even though it sums inclusively over that region.

- Not a surprise: emissions outside of the gap can radiate back into the gap.

- Must include *any number of emissions* outside of the gap and their subsequent evolution.

- <u>Colour structure now becomes much more complicated.</u>

The amplitude can be projected onto a colour basis, e.g. for quark-quark scattering:

Kidonakis, Oderda & Sterman
hep-ph/9803241

$$(M)_{ij}^{kl} = M^{(1)} C_{ijkl}^{(1)} + M^{(8)} C_{ijkl}^{(8)}$$

$$C_{ijkl}^{(8)} = (T^a)_{ik}(T^a)_{jl}$$
$$C_{ijkl}^{(1)} = \delta_{ik}\delta_{jl}.$$

i.e. $\mathbf{M} = \begin{pmatrix} M^{(1)} \\ M^{(8)} \end{pmatrix}$ and $\sigma = \mathbf{M}^\dagger \mathbf{S}_V \mathbf{M}$

$$\mathbf{S}_V = \begin{pmatrix} N^2 & 0 \\ 0 & \frac{N^2-1}{4} \end{pmatrix}$$

Iterating the insertion of soft virtual gluons builds up the resummed amplitude:

$$\mathbf{M} = \exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\,\mathbf{\Gamma}\right)\mathbf{M}_0$$

where the evolution matrix is

$$\mathbf{\Gamma} = \begin{pmatrix} \frac{N^2-1}{4N}\rho(Y,\Delta y) & \frac{N^2-1}{4N^2} i\pi \\ i\pi & -\frac{1}{N} i\pi + \frac{N}{2} Y + \frac{N^2-1}{4N}\rho(Y,\Delta y) \end{pmatrix}$$

$\Delta y =$ distance between jet centres $\quad Y =$ size of gap

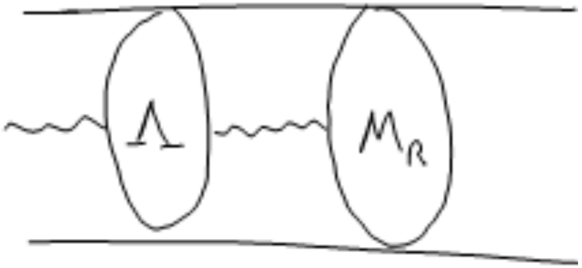
Further real emissions introduce new ingredients:

1) How to add a real gluon to the four-quark amplitude

$$\mathbf{M}_R = \mathbf{D} \cdot \mathbf{M}$$

2) How to evolve the resulting 5-parton amplitude

$$\mathbf{M}_R(Q_0) = \exp\left(-\frac{2\alpha_s}{\pi} \int_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \boldsymbol{\Lambda} \right) \mathbf{M}_R(k_T)$$



$$\mathbf{D}^\mu = \begin{pmatrix} \frac{1}{2}(-h_1^\mu - h_2^\mu + h_3^\mu + h_4^\mu) & \frac{1}{4N}(-h_1^\mu - h_2^\mu + h_3^\mu + h_4^\mu) \\ 0 & \frac{1}{2}(-h_1^\mu - h_2^\mu + h_3^\mu + h_4^\mu) \\ \frac{1}{2}(-h_1^\mu + h_2^\mu + h_3^\mu - h_4^\mu) & \frac{1}{4N}(h_1^\mu - h_2^\mu - h_3^\mu + h_4^\mu) \\ 0 & \frac{1}{2}(-h_1^\mu + h_2^\mu - h_3^\mu + h_4^\mu) \end{pmatrix} \qquad h_i^\mu = \frac{1}{2}k_T\frac{p_i^\mu}{p_i\cdot k}$$

$$\mathbf{\Lambda} = \begin{pmatrix} \frac{N}{4}(Y-i\pi)+\frac{1}{2N}i\pi & \left(\frac{1}{4}-\frac{1}{N^2}\right)i\pi & -\frac{N}{4}s_yY & 0 \\ i\pi & \frac{N}{4}\left(2Y-i\pi\right)-\frac{3}{2N}i\pi & 0 & 0 \\ -\frac{N}{4}s_yY & 0 & \frac{N}{4}(Y-i\pi)-\frac{1}{2N}i\pi & -\frac{1}{4}i\pi \\ 0 & 0 & -i\pi & \frac{N}{4}\left(2Y-i\pi\right)-\frac{1}{2N}i\pi \end{pmatrix}$$

$$+ \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & N \end{pmatrix} \frac{1}{4}\rho(Y, 2\left|y\right|)$$

$$+ \begin{pmatrix} C_F & 0 & 0 & 0 \\ 0 & C_F & 0 & 0 \\ 0 & 0 & C_F & 0 \\ 0 & 0 & 0 & C_F \end{pmatrix} \frac{1}{2}\rho(Y, \Delta y)$$

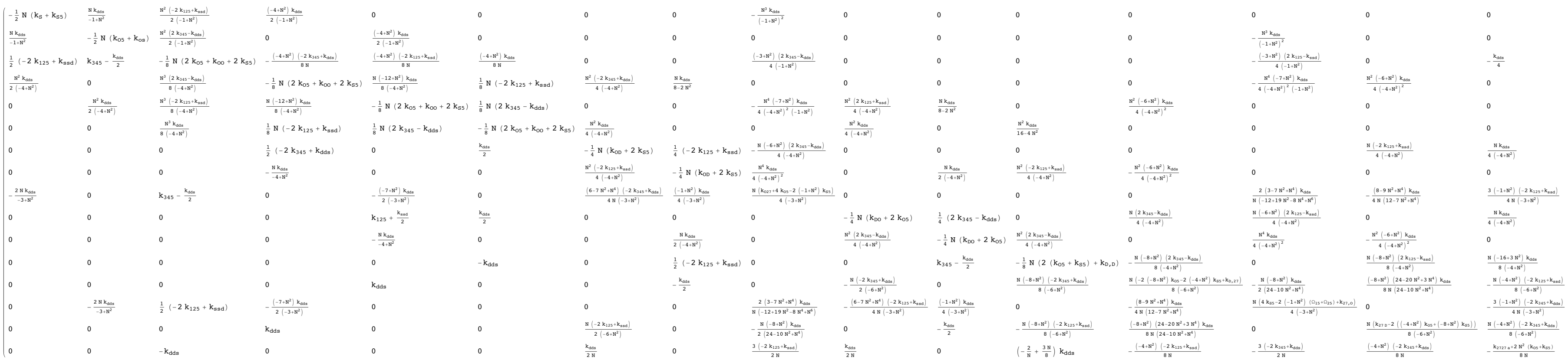
$$+ \begin{pmatrix} \frac{N}{4}\left(-\frac{1}{2}\lambda\right) & 0 & \frac{N}{4}\left(-\frac{1}{2}s_y\lambda\right) & \frac{1}{4}\left(\frac{1}{2}s_y\lambda\right) \\ 0 & \frac{N}{4}\left(-\frac{1}{2}\lambda\right) & 0 & \frac{N}{4}\left(\frac{1}{2}s_y\lambda\right) \\ \frac{N}{4}\left(-\frac{1}{2}s_y\lambda\right) & 0 & \frac{N}{4}\left(-\frac{1}{2}\lambda\right) & \frac{1}{4}\left(-\frac{1}{2}\lambda\right) \\ \frac{1}{2}s_y\lambda & \left(\frac{N}{4}-\frac{1}{N}\right)\left(\frac{1}{2}s_y\lambda\right) & -\frac{1}{2}\lambda & \frac{N}{4}\left(-\frac{1}{2}\lambda\right) \end{pmatrix}$$

Kyrieleis & Seymour: hep-ph/0510089

Has been extended to all five parton amplitudes:

e.g. gg → ggg

Sjödahl
arXiv:0807.0555



And then we need to allow for a second (and third etc.) real emission.

Very soon we need to keep track of very large matrices.

The general algorithm

$$\sigma = \sum_{n=0}^{\infty} \int \mathrm{d}\sigma_n \, F_n$$

$$\begin{aligned}
\mathrm{d}\sigma_0 &= \left\langle M^{(0)} \middle| \mathbf{V}^\dagger_{0,Q} \mathbf{V}_{0,Q} \middle| M^{(0)} \right\rangle \mathrm{d}\Pi_0 \\
\mathrm{d}\sigma_1 &= \left\langle M^{(0)} \middle| \mathbf{V}^\dagger_{q_{1T},Q} \mathbf{D}^\dagger_{1\mu} \mathbf{V}^\dagger_{0,q_{1T}} \mathbf{V}_{0,q_{1T}} \mathbf{D}^\mu_1 \mathbf{V}_{q_{1T},Q} \middle| M^{(0)} \right\rangle \mathrm{d}\Pi_0 \mathrm{d}\Pi_1 \\
\mathrm{d}\sigma_2 &= \left\langle M^{(0)} \middle| \mathbf{V}^\dagger_{q_{1T},Q} \mathbf{D}^\dagger_{1\mu} \mathbf{V}^\dagger_{q_{2T},q_{1T}} \mathbf{D}^\dagger_{2\nu} \mathbf{V}^\dagger_{0,q_{2T}} \mathbf{V}_{0,q_{2T}} \mathbf{D}^\nu_2 \mathbf{V}_{q_{2T},q_{1T}} \mathbf{D}^\mu_1 \mathbf{V}_{q_{1T},Q} \middle| M^{(0)} \right\rangle \mathrm{d}\Pi_0 \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 \\
&\text{etc.}
\end{aligned} \tag{1.1}$$

$$\mathbf{V}_{a,b} = \exp\left[-\frac{2\alpha_s}{\pi} \int_a^b \frac{\mathrm{d}k_T}{k_T} \sum_{i<j} (-\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{2} \left\{ \int \frac{\mathrm{d}y\,\mathrm{d}\phi}{2\pi} \omega_{ij} - i\pi\, \Theta(ij = II \text{ or } FF) \right\} \right], \tag{1.2}$$

$$\omega_{ij} = \frac{1}{2} k_T^2 \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

$$\mathbf{D}^\mu_i = \sum_j \mathbf{T}_j \, \frac{1}{2} q_{Ti} \frac{p^\mu_j}{p_j \cdot q_i},$$

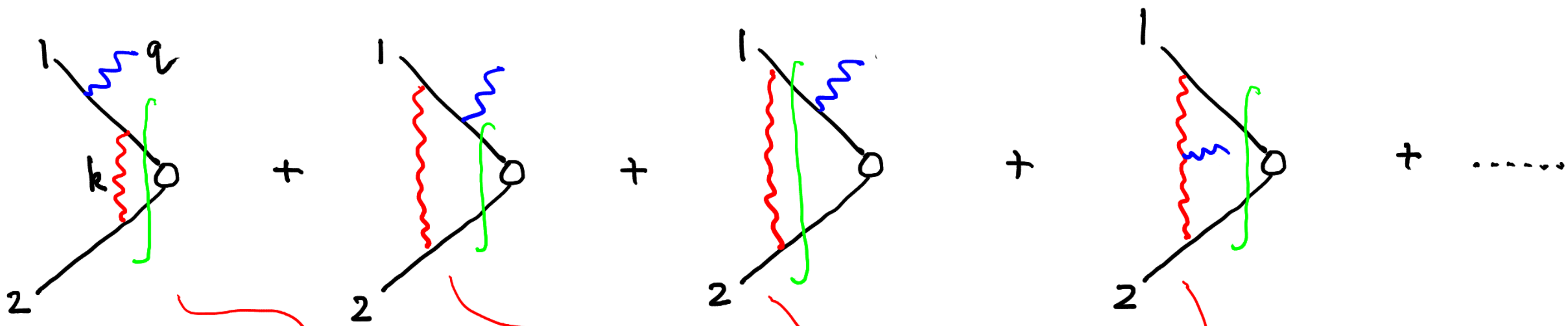
$$\mathrm{d}\Pi_i = -\frac{2\alpha_s}{\pi} \frac{\mathrm{d}q_{Ti}}{q_{Ti}} \frac{\mathrm{d}y_i \mathrm{d}\phi_i}{2\pi}.$$

Aim is to test the correctness of

$$\mathbf{V}_{0,q_{1T}} \mathbf{D}^\mu_1 \mathbf{V}_{q_{1T},Q} \quad \text{and} \quad \mathbf{V}_{0,q_{2T}} \mathbf{D}^\nu_2 \mathbf{V}_{q_{2T},q_{1T}} \mathbf{D}^\mu_1 \mathbf{V}_{q_{1T},Q}.$$

René Ángeles-Martínez: PhD thesis
Ángeles-Martínez , JRF, Seymour: arXiv:1510.07998

Start with one emission case

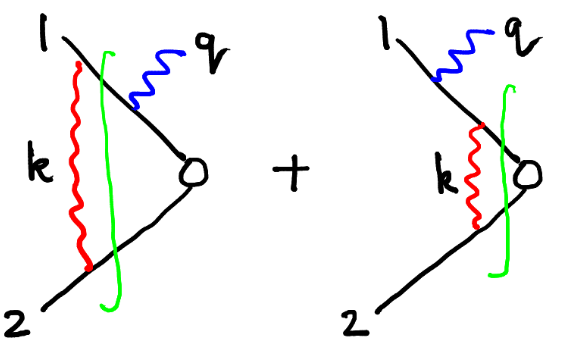


$$i\pi\int\frac{d^2k_\perp}{k_\perp^2}\,\frac{p_1\cdot\epsilon^*}{p_1\cdot q}\Big\{\Big[\,t_1\,(t_1\cdot t_2)\;-\;T_1\cdot T_2\,t_1\;+\;T_1\cdot T_2\,t_1\,\Big]$$

$$+\,i\pi\int\frac{d^2k_\perp}{k_\perp^2}\,\frac{q_\perp^2}{k_\perp^2+q_\perp^2}\,\frac{p_1\cdot\epsilon^*}{p_1\cdot q}\Big\{T_1\cdot T_2\,t_1\;-\;t_1\,t_1\cdot t_2\Big\}$$

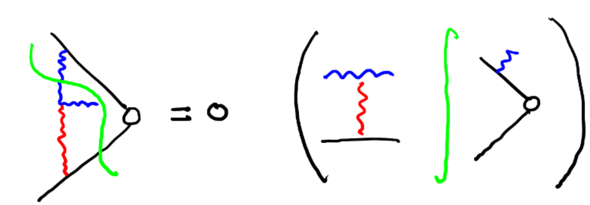
"switch"

=

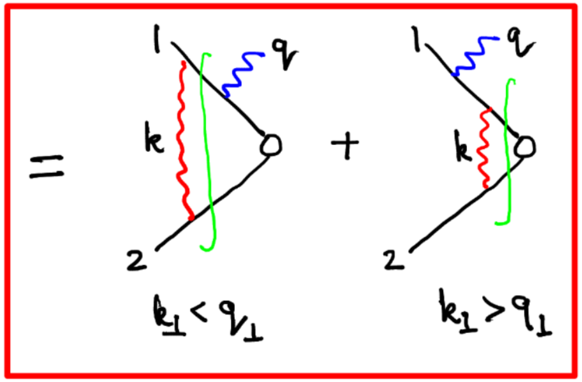


$k_\perp < q_\perp$ $\quad\quad$ $k_\perp > q_\perp$

nb.



That was a <u>non-trivial</u> test of $k_\perp$ ordering



$$\int \frac{dq_\perp}{q_\perp} \int^{q_\perp} \frac{dk_\perp}{k_\perp} + \int \frac{dk_\perp}{k_\perp} \int^{k_\perp} \frac{dq_\perp}{q_\perp}$$

$$= \Big(C(0, q_\perp)\, J(q_\perp) + J(q_\perp)\, C(q_\perp, Q) \Big) \times \Big(\succ\!\circ \Big)$$

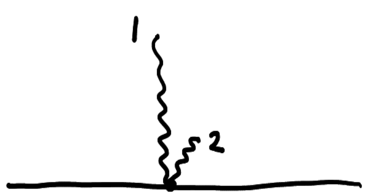
$q_{1T} \gg q_{2T}$

Double emission case

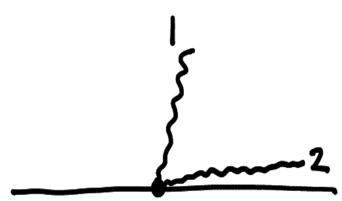
- Limit 1: Both emissions are at wide angle but one gluon is much softer than the other, i.e. $(q_1^\pm \sim q_{1T}) \gg (q_2^\pm \sim q_{2T})$. Specifically, we take $q_2 \to \lambda q_2$ and keep the leading term for small λ.

- Limit 2: One emission (q_2) collinear with p_i by virtue of its small transverse momentum and the other (q_1) at a wide angle, i.e. $q_2^+ \gg q_{2T}$ and $q_1^+ \sim q_{1T} \gg q_{2T}$. Specifically, we take $q_2 \to (q_2^+, \lambda^2 q_{2T}^2/(2q_2^+), \lambda \mathbf{q}_{2T})$ and keep the leading term for small λ.

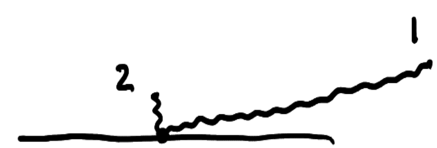
- Limit 3: One emission (q_1) collinear with p_i by virtue of its high energy and the other (q_2) at a wide angle, i.e. $q_1^+ \gg q_{1T}$ and $q_{1T} \gg q_{2T} \sim q_2^+$. Specifically, we take[3] $q_1 \to (q_1^+/\lambda, \lambda q_{1T}^2/(2q_1^+), \mathbf{q}_{1T})$ and $q_2 \to \lambda q_2$, and keep the leading term for small λ.



Limit 1



Limit 2



Limit 3

Relevant for SLL

"Eikonal" cuts

e.g. 1st row of graphs

$$G_{11} = \frac{q_1^-}{(q_2^- + q_1^-)} \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2} \qquad G_{13} = \frac{q_2^-}{(q_1^- + q_2^-)} \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

$$G_{11} + G_{13} = \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$ as expected

$$G_{12} = -\left[\int_0^{2q_1^- q_2^+} \frac{\mathrm{d}k_T^2}{k_T^2} + \frac{q_2^- - q_1^-}{q_2^- + q_1^-} \int_0^{2(q_1^+ + q_2^-)^2 q_2^+/q_1^-} \frac{\mathrm{d}k_T^2}{k_T^2}\right]$$

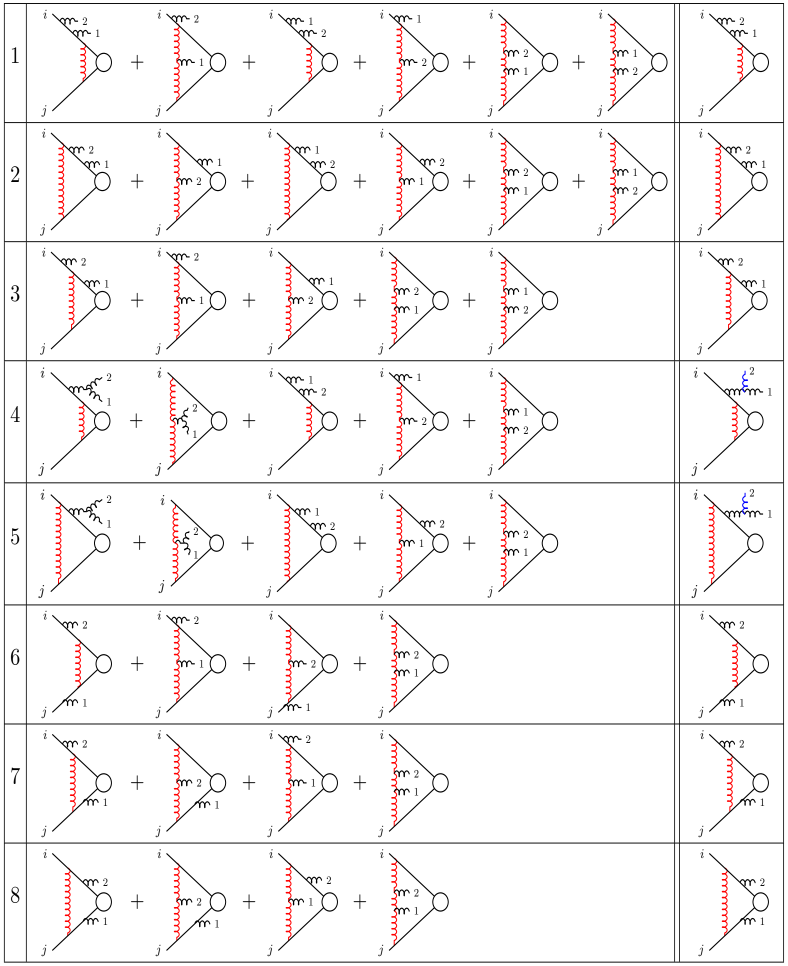
$$G_{12} + G_{14} = -\frac{1}{(q_1^- + q_2^-)}\left[q_2^- \int_0^{q_{2T}^2} \frac{\mathrm{d}k_T^2}{k_T^2} + q_1^- \int_0^{q_{1T}^2} \frac{\mathrm{d}k_T^2}{k_T^2}\right]$$

Subleading in limits 1 & 2

$$G_{15} + G_{16} \approx -\frac{q_2^-}{(q_1^- + q_2^-)} \int_{q_{2T}^2}^{q_{1T}^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

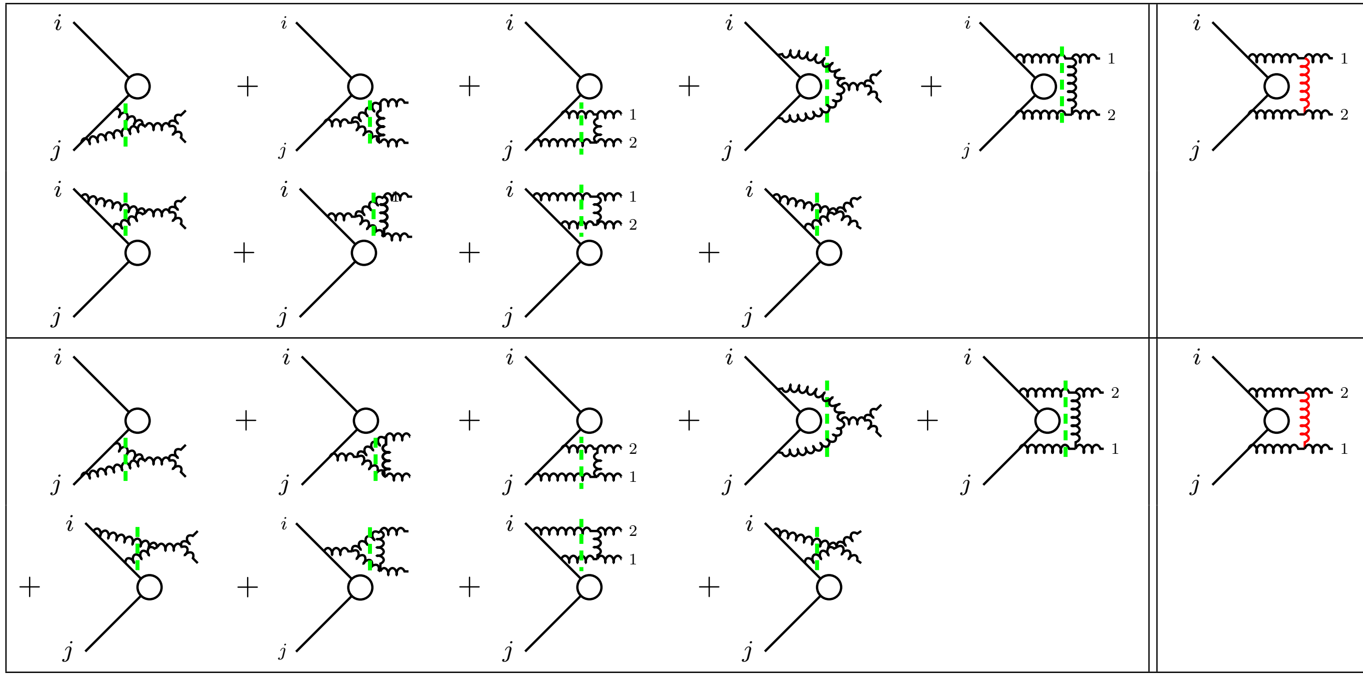
Only leading in limit 3

In all 3 limits sum is $$\int_{q_{1T}^2}^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$



"Soft gluon" cuts

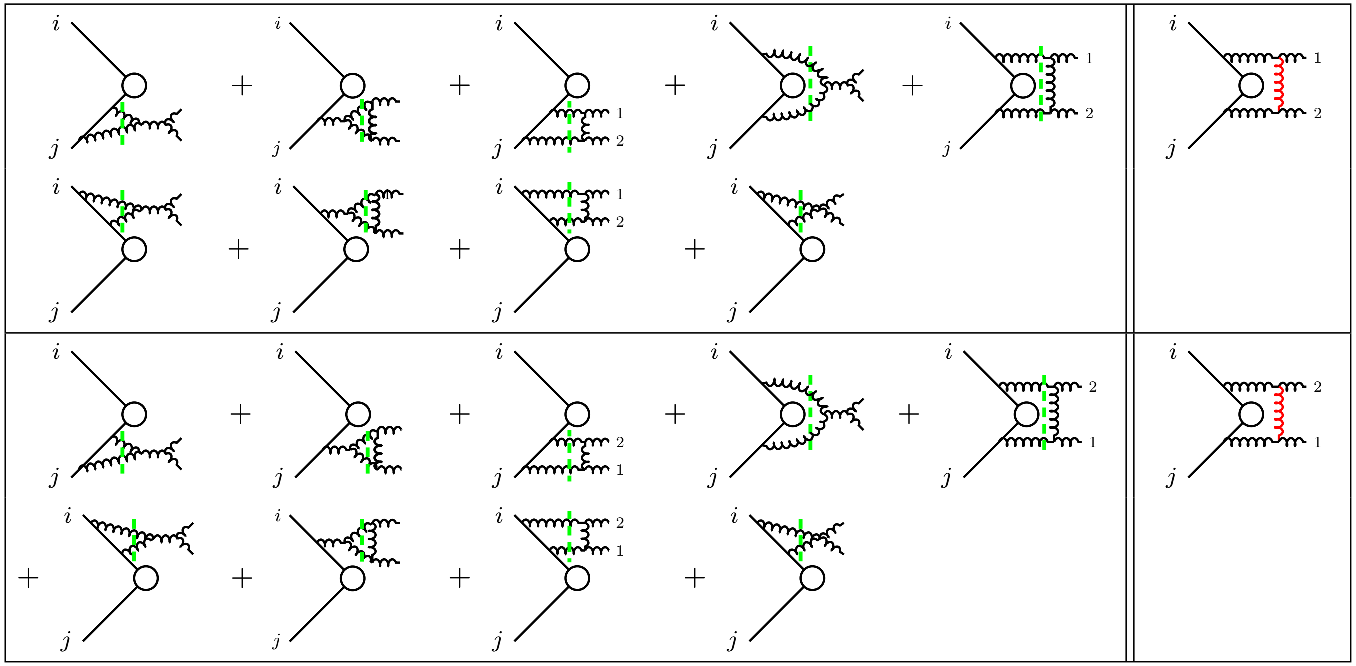
$$q_T^{(ab)} = \frac{2q \cdot p_a \, q \cdot p_b}{p_a \cdot p_b}$$



$$-\frac{i\pi}{8\pi^2}\left[\frac{p_j \cdot \varepsilon_1}{p_j \cdot q_1} - \frac{p_i \cdot \varepsilon_1}{p_i \cdot q_1}\right]\left\{\left[\frac{p_j \cdot \varepsilon_2}{p_j \cdot q_2} - \frac{q_1 \cdot \varepsilon_2}{q_1 \cdot q_2}\right]\left[-\frac{1}{\epsilon} + \ln\left(\frac{2q_2 \cdot q_1 \, p_j \cdot q_2}{q_1 \cdot p_j \, \mu^2}\right)\right]\mathbf{T}_j^d i f^{dc_2 b} i f^{bc_1 a}\mathbf{T}_i^a\right.$$
$$\left. - \left[\frac{p_i \cdot \varepsilon_2}{p_i \cdot q_2} - \frac{q_1 \cdot \varepsilon_2}{q_1 \cdot q_2}\right]\left[-\frac{1}{\epsilon} + \ln\left(\frac{2q_2 \cdot q_1 \, p_i \cdot q_2}{q_1 \cdot p_i \, \mu^2}\right)\right]\mathbf{T}_j^d i f^{dc_1 b} i f^{bc_2 a}\mathbf{T}_i^a\right\}\left|M^0\right\rangle .$$

$$\ln\left(\frac{2q_2 \cdot q_1 \, p_i \cdot q_2}{q_1 \cdot p_i \mu^2}\right) = \ln\left(\frac{q_{2T(i1)}^2}{\mu^2}\right) = \ln\left(\frac{q_{2T}^2}{\mu^2}\right) + \ln\left(\frac{2q_1 \cdot q_2}{q_{1T} q_{2T}}\right) + y_1 - y_2,$$
$$\ln\left(\frac{2q_2 \cdot q_1 \, p_j \cdot q_2}{q_1 \cdot p_j \mu^2}\right) = \ln\left(\frac{q_{2T(j1)}^2}{\mu^2}\right) = \ln\left(\frac{q_{2T}^2}{\mu^2}\right) + \ln\left(\frac{2q_1 \cdot q_2}{q_{1T} q_{2T}}\right) + y_2 - y_1,$$

$$\left\{-\frac{i\pi}{8\pi^2} i f^{c_1 e a_1} i f^{c_2 e a_2}\left[-\frac{1}{\epsilon} + \ln\left(\frac{q_{2T}^2}{\mu^2}\right) + \ln\left(\frac{2q_1 \cdot q_2}{q_{1T} q_{2T}}\right)\right]\right\} \mathbf{K}_2^{a_1 a_2}(q_1, q_2)\left|M^{(0)}\right\rangle$$



e.g. the $-\frac{i\pi}{8\pi^2}\frac{p_j\cdot\varepsilon_1}{p_j\cdot q_1}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}$ terms in limit 1

$$G_{1c} = -\frac{3}{2}\int_{p_j\cdot q_1}^{p_j\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} - \frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{1d} = \frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{1e} = -\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} + \frac{1}{2}\int_{p_j\cdot q_1}^{p_j\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$\text{sum} = -\int_0^{(q_2^{(1j)})^2}\frac{\mathrm{d}l_T^2}{l_T^2}$$

$$G_{2c} = \frac{3}{4}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

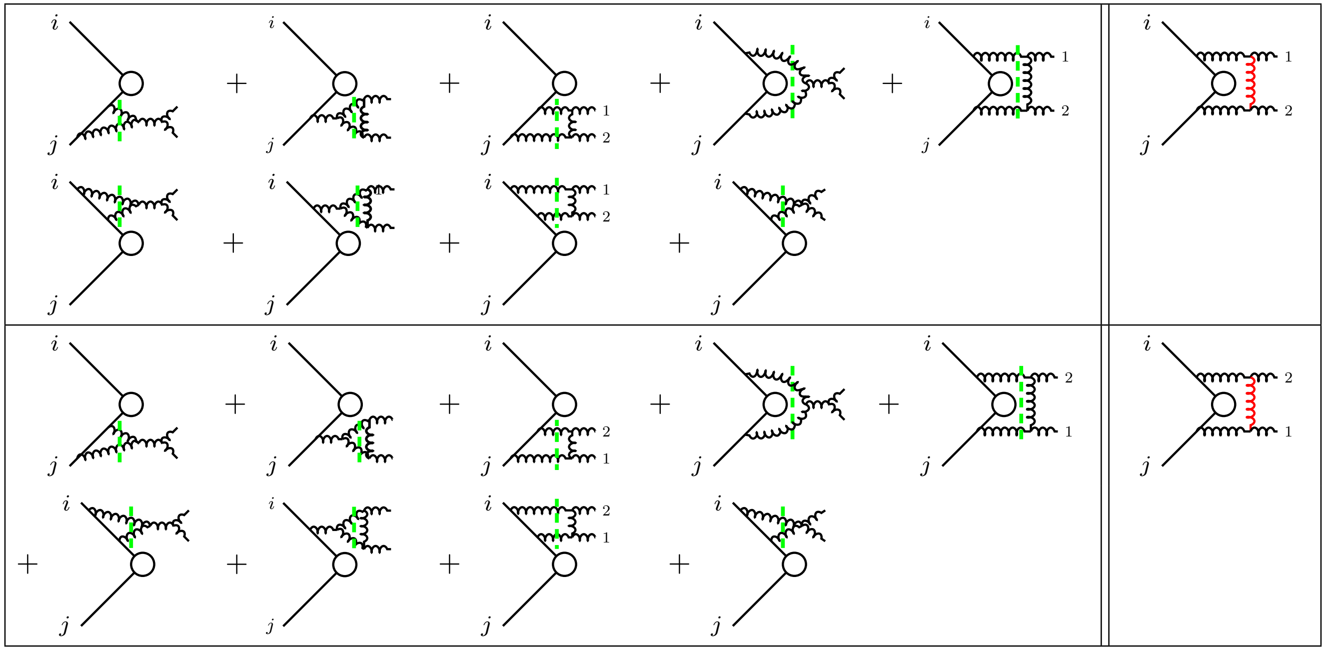
$$G_{2d} = -\frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{2e} = \frac{7}{4}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} + \int_{p_i\cdot q_1}^{p_i\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$\text{sum} = \int_0^{(q_2^{(1i)})^2}\frac{\mathrm{d}l_T^2}{l_T^2}$$

Limit 2

q_2 collinear with p_i



Only graphs G_{2e} and G_{2h} are leading.

$$G_{2e} = -\frac{i\pi}{8\pi^2}\left[\frac{1}{2}\frac{p_i\cdot\varepsilon_1}{p_i\cdot q_1}\frac{p_i\cdot\varepsilon_2}{p_i\cdot q_2}\right]\int_0^{q_{2T}^2}\frac{\mathrm{d}l_T^2}{l_T^2}$$

$$G_{2h} = -\frac{i\pi}{8\pi^2}\left[\frac{1}{2}\frac{p_i\cdot\varepsilon_1}{p_i\cdot q_1}\frac{p_i\cdot\varepsilon_2}{p_i\cdot q_2} - \frac{p_i\cdot\varepsilon_2}{p_i\cdot q_2}\frac{p_j\cdot\varepsilon_1}{p_j\cdot q_1}\right]\int_0^{q_{2T}^2}\frac{\mathrm{d}l_T^2}{l_T^2}$$

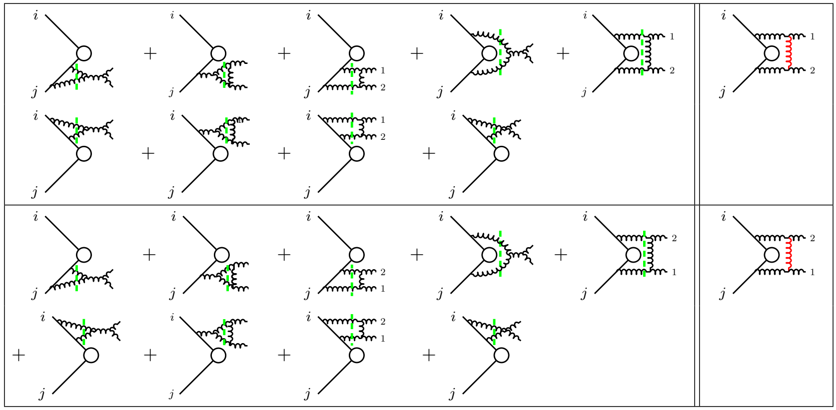
$$q_{2T} \equiv q_{2T}^{(1i)}$$

Limit 3 $\qquad q_1$ collinear with p_i

The leading contributions to the second colour structure cancel.

The first colour structure receives leading contributions to the following two Lorentz structures:

$$-\frac{i\pi}{8\pi^2}\left\{\frac{\varepsilon_1^-}{q_1^-}\frac{\varepsilon_2^+}{q_2^+}, \frac{\varepsilon_1^-}{q_1^-}\frac{\varepsilon_2^-}{q_2^-}\right\}.$$



Only graph G_{1e} contributes to the first and it gives

$$-\int_0^{q_{2T}^2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

Graphs $\{G_{1a}, G_{1b}, G_{1c}, G_{1d}, G_{1e}, G_{1i}\}$ contribute to the second. The contributions of graphs G_{1a}, G_{1b} cancel whilst

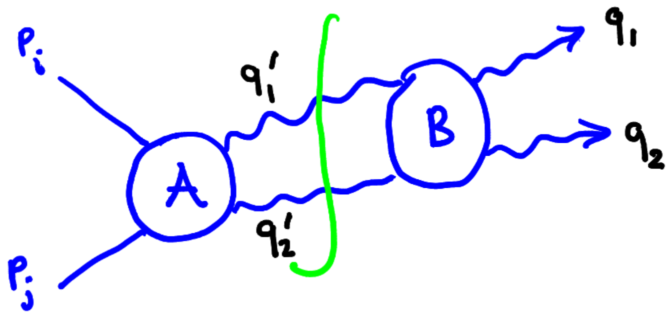
$$\begin{aligned}
G_{1c} &= \left[\frac{3(q_1^-)^2+3q_1^-q_2^-+2(q_2^-)^2}{4(q_1^-+q_2^-)^2}\right]\int_0^{q_1^+q_2^-}\frac{\mathrm{d}l_T^2}{l_T^2}-\frac{1}{2}\int_{p_i\cdot q_1}^{p_i\cdot(q_1+q_2)}\frac{\mathrm{d}l_T^2}{l_T^2},\\
G_{1d} &= -\frac{3q_1^-+2q_2^-}{2(q_1^-+q_2^-)}\int_0^{q_1^+q_2^-}\frac{\mathrm{d}l_T^2}{l_T^2}+\int_{p_i\cdot q_1}^{p_i\cdot(q_1+q_2)}\frac{\mathrm{d}l_T^2}{l_T^2},\\
G_{1e} &= \frac{7q_1^-+6q_2^-}{4(q_1^-+q_2^-)}\int_0^{q_1^+q_2^-}\frac{\mathrm{d}l_T^2}{l_T^2}-\frac{1}{2}\int_{p_i\cdot q_1}^{p_i\cdot(q_1+q_2)}\frac{\mathrm{d}l_T^2}{l_T^2}-\int_{p_j\cdot q_2}^{p_j\cdot q_1}\frac{\mathrm{d}l_T^2}{l_T^2}.
\end{aligned}$$

The sum of these three contributions is

$$\int_0^{q_{2T}^2}\frac{\mathrm{d}l_T^2}{l_T^2}-\frac{q_1^-q_2^-}{2(q_1^-+q_2^-)^2}\int_0^{q_1^+q_2^-}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$q_{2T}\equiv q_{2T}^{(1j)}$$

Finally, the four-gluon vertex graph G_{1i} exactly cancels the second term of this expression.



A more physical derivation of the same result

Sum over physical polarizations

$$i\mathcal{M} = \int \frac{\mathrm{d}^{d-2}k_T}{8q_1\cdot q_2(2\pi)^{d-2}\sqrt{1-2k_T^2/(q_1\cdot q_2)}} \times \frac{1}{2}(i\mathcal{A}^{\mu\nu})\left(\sum_{p_1'} \varepsilon^*_{1'\mu;p_1'}\varepsilon_{1'\sigma;p_1'} \sum_{p_2'} \varepsilon^*_{2'\nu;p_2'}\varepsilon_{2'\lambda;p_2'}\right)(i\mathcal{B}^{\sigma\lambda}) \,,$$

$$\varepsilon_{1'\sigma;p_1'}\varepsilon_{2'\lambda;p_2'}\mathcal{B}^{\sigma\lambda} \overset{k_T^2\ll 2q_1\cdot q_2}{\longrightarrow} g_s^2\mu^{2\epsilon}\frac{4q_1\cdot q_2}{k_T^2}\,\varepsilon_{1';p_1'}\cdot\varepsilon^*_{1;p_1}\,\varepsilon_{2';p_2'}\cdot\varepsilon^*_{2;p_2}\,,$$

$$\mathcal{M} = \frac{ig_s^2\mu^{2\epsilon}}{8\pi^2}\int\frac{\mathrm{d}^{d-2}k_T}{k_T^2(2\pi)^{d-4}}\sum_{p_1'p_2'}\left(\mathcal{A}^{\mu\nu}\varepsilon^*_{1'\mu;p_1'}\varepsilon^*_{2'\nu;p_2'}\right)\varepsilon_{1';p_1'}\cdot\varepsilon^*_{1;p_1}\,\varepsilon_{2';p_2'}\cdot\varepsilon^*_{2;p_2}\,.$$

Gluon 1: momentum and polarization unchanged by the Coulomb scattering.

$$\sum_{p_1'}\mathcal{A}^{\mu}\varepsilon^*_{1'\mu;p_1'}\,\varepsilon_{1';p_1'}\cdot\varepsilon^*_{1;p_1} = -\mathcal{A}^{\mu}\varepsilon^*_{1\mu;p_1}\,.$$

$$\mathcal{M} = -\frac{ig_s^2\mu^{2\epsilon}}{8\pi^2}\int\frac{\mathrm{d}^{d-2}k_T}{k_T^2(2\pi)^{d-4}}\sum_{p_2'}\left(\mathcal{A}^{\mu\nu}\varepsilon^*_{1\mu;p_1}\,\varepsilon^*_{2'\nu;p_2'}\right)\varepsilon_{2';p_2'}\cdot\varepsilon^*_{2;p_2}.$$

$$\mathcal{M} = \left(\mathcal{A}^{\mu\nu} \varepsilon^*_{1\mu;p_1}\, \varepsilon^*_{2\nu;p_2}\right) \frac{i g_s^2 \mu^{2\epsilon}}{8\pi^2} \int \frac{k_T^{d-3} \mathrm{d}k_T\, \mathrm{d}\phi\, \sin^{-2\epsilon}\phi\, \mathrm{d}^{d-4}\Omega}{k_T^2 (2\pi)^{d-4}} \frac{1 + \frac{k_T}{q_{2T(i1)}} \cos\phi}{1 + 2\frac{k_T}{q_{2T(i1)}} \cos\phi + \frac{k_T^2}{q^2_{2T(i1)}}} \,.$$

$$k_{T\mu} = k_T \sin\phi\, \varepsilon_{1\mu;\perp} + k_T \cos\phi\, \varepsilon_{1\mu;\parallel}$$

In four dimensions, the ϕ integral yields an **exact** Θ-function:

$$\mathcal{M} = \left(\mathcal{A}^{\mu\nu} \varepsilon^*_{1\mu;p_1}\, \varepsilon^*_{2\nu;p_2}\right) \frac{i g_s^2}{4\pi} \int_0^{q_{2T(i1)}} \frac{\mathrm{d}k_T}{k_T} \,.$$

$$q^2_{2T(i1)} = \frac{2 q_2 \cdot q_1\, p_i \cdot q_2}{q_1 \cdot p_i}$$

The k_T of the Coulomb gluon is exactly limited by the transverse momentum of the softer of the two gluons it is exchanged between, as measured in the rest frame of harder of the two gluons and the parent of the softer gluon.

Has been generalized to hard processes with any number of hard partons

$$
\begin{aligned}
|M_N^{(1)}\rangle \quad = \quad & \sum_{m=0}^{N} \sum_{i=2}^{p} \sum_{j=1}^{i-1} (g_s\mu^\epsilon)^{N-m}\, \mathbf{J}^{(0)}(q_N)\cdots \mathbf{J}^{(0)}(q_{m+1})\, \mathbf{I}_{ij}(\tilde{q}_{m+1},\tilde{q}_m)\ |M_m^{(0)}\rangle \\
+ & \sum_{m=1}^{N} \sum_{j=1}^{n+m-1} \sum_{k=1}^{n+m-1} (g_s\mu^\epsilon)^{N-m}\, \mathbf{J}^{(0)}(q_N)\cdots \mathbf{J}^{(0)}(q_{m+1})\, \mathbf{I}_{n+m,j}(\tilde{q}_{m+1}, q_m^{(jk)})\, \mathbf{d}_{jk}(q_m)\, |M_{m-1}^{(0)}\rangle \ ,
\end{aligned}
$$

$$
|M_m^{(0)}\rangle = (g_s\mu^\epsilon)^m\, \mathbf{J}^{(0)}(q_m)\, \mathbf{J}^{(0)}(q_{m-1})\cdots \mathbf{J}^{(0)}(q_1)\, |M_0^{(0)}\rangle
$$

$$
(q^{(ij)})^2 = \frac{2\, q\cdot p_i\ q\cdot p_j}{p_i\cdot p_j}
$$

$$
\mathbf{I}_{ij}(a,b) = \frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon^2}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[\left(\frac{b^2}{4\pi\mu^2}\right)^{-\epsilon}\left(1+i\pi\epsilon\,\tilde{\delta}_{ij}-\epsilon\ln\frac{2p_i\cdot p_j}{b^2}\right)-\left(\frac{a^2}{4\pi\mu^2}\right)^{-\epsilon}\left(1+i\pi\epsilon\,\tilde{\delta}_{ij}-\epsilon\ln\frac{2p_i\cdot p_j}{a^2}\right)\right]
$$

$$
= \frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon^2}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[-\frac{1}{2}\ln^2\frac{2p_i\cdot p_j}{b^2}+\frac{1}{2}\ln^2\frac{2p_i\cdot p_j}{a^2}-i\pi\tilde{\delta}_{ij}\ln\frac{b^2}{a^2}\right] \qquad a^2, b^2 > 0
$$

$$
\mathbf{d}_{ij}(q) = \mathbf{T}_j\left(\frac{p_j\cdot\varepsilon}{p_j\cdot q}-\frac{p_i\cdot\varepsilon}{p_i\cdot q}\right)
$$

$$
\sum_j \mathbf{d}_{ij}(q) = \sum_j \mathbf{T}_j\frac{p_j\cdot\varepsilon}{p_j\cdot q} = \mathbf{J}^{(0)}(q)
$$

Ángeles Martínez , JRF, Seymour: arXiv:1602.00623

The general algorithm

$$\sigma = \sum_{n=0}^{\infty} \int d\sigma_n F_n$$

$$\begin{aligned}
d\sigma_0 &= \left\langle M^{(0)} \middle| \mathbf{V}^\dagger_{0,Q} \mathbf{V}_{0,Q} \middle| M^{(0)} \right\rangle d\Pi_0 \\
d\sigma_1 &= \left\langle M^{(0)} \middle| \mathbf{V}^\dagger_{q_{1T},Q} \mathbf{D}^\dagger_{1\mu} \mathbf{V}^\dagger_{0,q_{1T}} \mathbf{V}_{0,q_{1T}} \mathbf{D}^\mu_1 \mathbf{V}_{q_{1T},Q} \middle| M^{(0)} \right\rangle d\Pi_0 d\Pi_1 \\
d\sigma_2 &= \left\langle M^{(0)} \middle| \mathbf{V}^\dagger_{q_{1T},Q} \mathbf{D}^\dagger_{1\mu} \mathbf{V}^\dagger_{q_{2T},q_{1T}} \mathbf{D}^\dagger_{2\nu} \mathbf{V}^\dagger_{0,q_{2T}} \mathbf{V}_{0,q_{2T}} \mathbf{D}^\nu_2 \mathbf{V}_{q_{2T},q_{1T}} \mathbf{D}^\mu_1 \mathbf{V}_{q_{1T},Q} \middle| M^{(0)} \right\rangle d\Pi_0 d\Pi_1 d\Pi_2 \\
&\text{etc.}
\end{aligned} \tag{1.1}$$

$$\mathbf{V}_{a,b} = \exp\left[-\frac{2\alpha_s}{\pi} \int_a^b \frac{dk_T}{k_T} \sum_{i<j} (-\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{2} \left\{ \int \frac{dy\, d\phi}{2\pi} \omega_{ij} - i\pi\, \Theta(ij = II \text{ or } FF) \right\} \right], \tag{1.2}$$

$$\omega_{ij} = \frac{1}{2} k_T^2 \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

$$\mathbf{D}^\mu_i = \sum_j \mathbf{T}_j \frac{1}{2} q_{Ti} \frac{p^\mu_j}{p_j \cdot q_i},$$

$$d\Pi_i = -\frac{2\alpha_s}{\pi} \frac{dq_{Ti}}{q_{Ti}} \frac{dy_i d\phi_i}{2\pi}.$$

Confirmed at least at low orders in perturbation theory – with a surprising statement on how it can be extended to include (almost?) all of the sub-leading contribution computed in the eikonal approximation.

Summary

- All order effects in QCD are crucial to describe many features of the final state in collider physics.

- Colour-mixing, Coulomb gluons and other sub-leading N_c terms can be included in an amplitude-level shower via evolution in dipole transverse momentum (= work underway with Simon Plätzer, Mike Seymour, Matthew de Angelis and René Ángeles Martínez).

- General algorithm should contain a lot of physics because of the remarkable accuracy of "dipole transverse momentum" evolution.