Non-linear beam dynamics

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Contents of the 3rd lecture

- Lie formalism and symplectic maps
  - Hamiltonian generators and Lie operators
  - Map for the quadrupole
  - Map for the general multi-pole
  - Map concatenation
- Symplectic integrators
  - Taylor map for the quadrupole
  - Restoration of symplecticity
  - 3-kick symplectic integrator
  - Higher order symplectic integrators
  - Accurate symplectic integrators with positive kicks ($SABA_2C$)
- Normal forms
  - Effective Hamiltonian
  - Normal form for a perturbation
  - Example for a thin octupole
- Summary
- Appendix : $SABA_2C$ integrator for accelerator elements

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Symplectic maps

- Consider two sets of canonical variables $\mathbf{z}$, $\bar{\mathbf{z}}$ which may be even considered as the evolution of the system between two points in phase space
- A transformation from the one to the other set can be constructed through a **map** $\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$
- The **Jacobian matrix** of the map $M = M(\mathbf{z}, t)$ is composed by the elements $M_{ij} \equiv \dfrac{\partial \bar{z}_i}{\partial z_j}$
- The map is **symplectic** if $M^T J M = J$ where $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$
- It can be shown that the variables defined through a symplectic map $[\bar{z}_i, \bar{z}_j] = [z_i, z_j] = J_{ij}$ which is a known relation satisfied by canonical variables
- In other words, symplectic maps **preserve** Poisson brackets
- Symplectic maps provide a very useful framework to represent and analyze motion through an accelerator

Lie formalism

- The Poisson bracket properties satisfy what is mathematically called a **Lie** algebra
- They can be represented by (Lie) operators of the form $:f:g = [f,g]$ and $:f:^2 g = [f,[f,g]]$ etc.

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- For a Hamiltonian system $H(\mathbf{z},t)$ there is a **formal solution** of the equations of motion $\frac{d\mathbf{z}}{dt} = [H,\mathbf{z}] =: H : \mathbf{z}$ written as $\mathbf{z}(t) = \sum_{k=0}^{\infty} \frac{t^k :H:^k}{k!}\mathbf{z}_0 = e^{t:H:}\mathbf{z}_0$ with a symplectic map $\mathcal{M} = e^{:H:}$

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- The 1-turn accelerator map can be represented by the composition of the maps of each element $\mathcal{M} = e^{:f_2:}\, e^{:f_3:}\, e^{:f_4:}\, \ldots$ where f_i (called the generator) is the Hamiltonian for each element, a polynomial of degree m in the variables $z_1, \ldots, z_n$

Magnetic element Hamiltonians

- Dipole:

$$H = \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Lie operators for simple elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$ $p = p_0$	$\exp(:-\frac{1}{2}Lp^2:)$
Thin-lens Quadrupole	$x = x_0$ $p = p_0 - \frac{1}{f}x_0$	$\exp(:-\frac{1}{2f}x^2:)$
Thin-lens Multipole	$x = x_0$ $p = p_0 + \lambda n x^{n-1}$	$\exp(:\lambda x^n:)$
Thin-lens kick	$x = x_0$ $p = p_0 + f(x)$	$\exp(:\int_0^x f(x')dx':)$
Thick focusing quad	$x = x_0 \cos kL + \frac{p_0}{k}\sin kL$ $p = -kx_0 \sin kL + p_0 \cos kL$	$\exp[:-\frac{1}{2}L(k^2x^2 + p^2):]$
Thick defocusing quad	$x = x_0 \cosh kL + \frac{p_0}{k}\sinh kL$ $p = kx_0 \sinh kL + p_0 \cosh kL$	$\exp[:\frac{1}{2}L(k^2x^2 - p^2):]$
Coordinate shift	$x = x_0 - b$ $p = p_0 + a$	$\exp(:ax + bp:)$
Coordinate rotation	$x = x_0 \cos\mu + p_0 \sin\mu$ $p = -x_0 \sin\mu + p_0 \cos\mu$	$\exp[:-\frac{1}{2}\mu(x^2 + p^2):]$
Scale change	$x = e^{-\lambda}x_0$ $p = e^{\lambda}p_0$	$\exp(:\lambda xp:)$

Formulas for Lie operators

$$:a: = 0, \qquad e^{:a:} = 1$$

$$:f:a = 0, \qquad e^{:f:}a = a$$

$$:f:f = 0, \qquad e^{:f:}f = f$$

$$\{:f:, :g:\} = :[f,g]:$$

$$e^{:f:}g(X) = g(e^{:f:}X)$$

$$e^{:\tilde{C}X:}g(X) = g(X - SC)$$

$$e^{:f:}G(:g:)e^{-:f:} = G(:e^{:f:}g:)$$

Map for quadrupole

- Consider the 1D quadrupole Hamiltonian

$$H = \frac{1}{2}(k_1 x^2 + p^2)$$

- For a quadrupole of length L, the map is written as

$$e^{\frac{L}{2}:(k_1 x^2 + p^2):}$$

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- Its application to the transverse variables is

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):}x = \sum_{n=0}^{\infty}\left(\frac{(-k_1 L^2)^n}{(2n)!}x + L\frac{(-k_1 L^2)^n}{(2n+1)!}p\right)$$

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):}p = \sum_{n=0}^{\infty}\left(\frac{(-k_1 L^2)^n}{(2n)!}p - \sqrt{k_1}\frac{(-k_1 L^2)^n}{(2n+1)!}p\right)$$

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- This finally provides the usual quadrupole matrix

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} x = \cos(\sqrt{k_1} L) x + \frac{1}{\sqrt{k_1}} \sin(\sqrt{k_1} L) p$$

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} p = -\sqrt{k_1} \sin(\sqrt{k_1} L) x + \cos(\sqrt{k_1} L) p$$

Map for general monomial

- Consider a monomial in the positions and momenta $x^n p^m$
- The map is written as $e^{a:x^n p^m:}$
- Its application to the transverse variables is
  - For $n \neq m$

$$e^{:\alpha x^n p^m:} x = x \left[1 + \alpha(n-m) x^{n-1} p^{m-1}\right]^{\frac{m}{m-n}}$$

$$e^{:\alpha x^n p^m:} p = p \left[1 + \alpha(n-m) x^{n-1} p^{m-1}\right]^{\frac{n}{n-m}}$$

  - For $n = m$

$$e^{:\alpha x^n p^n:} x = x e^{-\alpha n x^{n-1} p^{n-1}}$$

$$e^{:\alpha x^n p^n:} p = p e^{\alpha n x^{n-1} p^{n-1}}$$

Map Concatenation

- For combining together the different maps, the **Campbell-Baker-Hausdorff** theorem can be used. It states that for t_1, t_2 sufficiently small, and A, B real matrices, there is a real matrix C for which

$$e^{sA}e^{tB} = e^C$$

- For map composition through Lie operators, this is translated to $e^{:h:} = e^{:f:}e^{:g:}$ with

$$h = f + g + \frac{1}{2} : f : g + \frac{1}{12} : f :^2 g + \frac{1}{12} : g :^2 f + \frac{1}{24} : f :: g :^2 f - \frac{1}{720} : g :^4 f - \frac{1}{720} : f :^4 g + \dots$$

or

$$h = f + g + \frac{1}{2}[f, g] + \frac{1}{12}[f, [f, g]] + \frac{1}{12}[g, [g, f]] + \frac{1}{24}[f, [g, [g, f]]] - \frac{1}{720}[g, [g, [g, f]]] - \frac{1}{720}[f, [f, [f, g]]] + \dots$$

i.e. a series of Poisson bracket operations.

- Note that the **full map** is by "construction" symplectic.
- By **truncating the map** to a certain order, **symplecticity** is **lost**.

Example: The $-I$ transformer

- Consider two identical sextupoles in a beam line represented by a map $\mathcal{R}$
- The **sextupole map** can be represented at **second order** as

$$\mathcal{S}_2 = e^{-\frac{1}{2}L_s:H_d:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

with the **sextupole** effective **Hamiltonian** $H_s = \dfrac{1}{6}k_2(x^3 - 3xy^2)$ and H_d the **drift Hamiltonian**

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- The **total map** can be approximated at 2nd order by

$$\mathcal{M} = \mathcal{SRS} \approx \mathcal{S}_2\mathcal{R}\mathcal{S}_2 = e^{-\frac{1}{2}L_s:H_d:} e^{-L_s:H_s:} \bar{\mathcal{R}} e^{-L_s:H_s:} e^{-\frac{1}{2}L_s:H_d:}$$

with the map $\bar{\mathcal{R}} = e^{-\frac{1}{2}L_s:H_d:} \mathcal{R} e^{-\frac{1}{2}L_s:H_d:}$

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with the map $\bar{\mathcal{R}} = e^{-\frac{1}{2}L_s:H_d:}\mathcal{R}e^{-\frac{1}{2}L_s:H_d:}$

- Inserting the identity $\bar{\mathcal{R}}\bar{\mathcal{R}}^{-1} = \mathcal{I}$ and considering the **similarity transformation** $\bar{\mathcal{R}}^{-1}e^{-L_s:H_s:}\bar{\mathcal{R}} = e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}$, the map can be rewritten as

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

Example: The $-I$ transformer

- If the map $\bar{\mathcal{R}}$ is chosen such that $\bar{\mathcal{R}}^{-1}H_s = -H_s$ then the sextupole map Lie operators

$$e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}e^{-L_s:H_s:} = e^{L_s:H_s:}e^{-L_s:H_s:} = \mathcal{I}$$

- In that way, the **sextupole non-linearity** is getting **eliminated** in the final map

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- Inspecting the form of H_s (odd in x and even in y), this can be achieved if

$$\bar{\mathcal{R}}x = -x, \qquad \bar{\mathcal{R}}p_x = -p_x, \qquad \bar{\mathcal{R}}y = \pm y, \qquad \bar{\mathcal{R}}p_y = \pm p_y$$

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or in matrix form

$$\bar{\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_x + a_x \sin\mu_x & b_x \sin\mu_x & 0 & 0 \\ -c_x \sin\mu_x & \cos\mu_x - a_x \sin\mu_x & 0 & 0 \\ 0 & 0 & \cos\mu_y + a_y \sin\mu_y & b_y \sin\mu_y \\ 0 & 0 & -c_y \sin\mu_y & \cos\mu_y - a_y \sin\mu_y \end{pmatrix}$$

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- The horizontal part of the matrix is $-\mathcal{I}_2$ and the vertical part is $\pm\mathcal{I}_2$, which is obtained for phase advances

$$\mu_x = (2n_x + 1)\pi, \qquad \mu_y = n_y \pi$$

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Why symplecticity is important

- **Symplecticity** guarantees that the **transformations** in phase space are **area preserving**
- To understand what deviation from symplecticity produces consider the simple case of the **quadrupole** with the general matrix written as

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

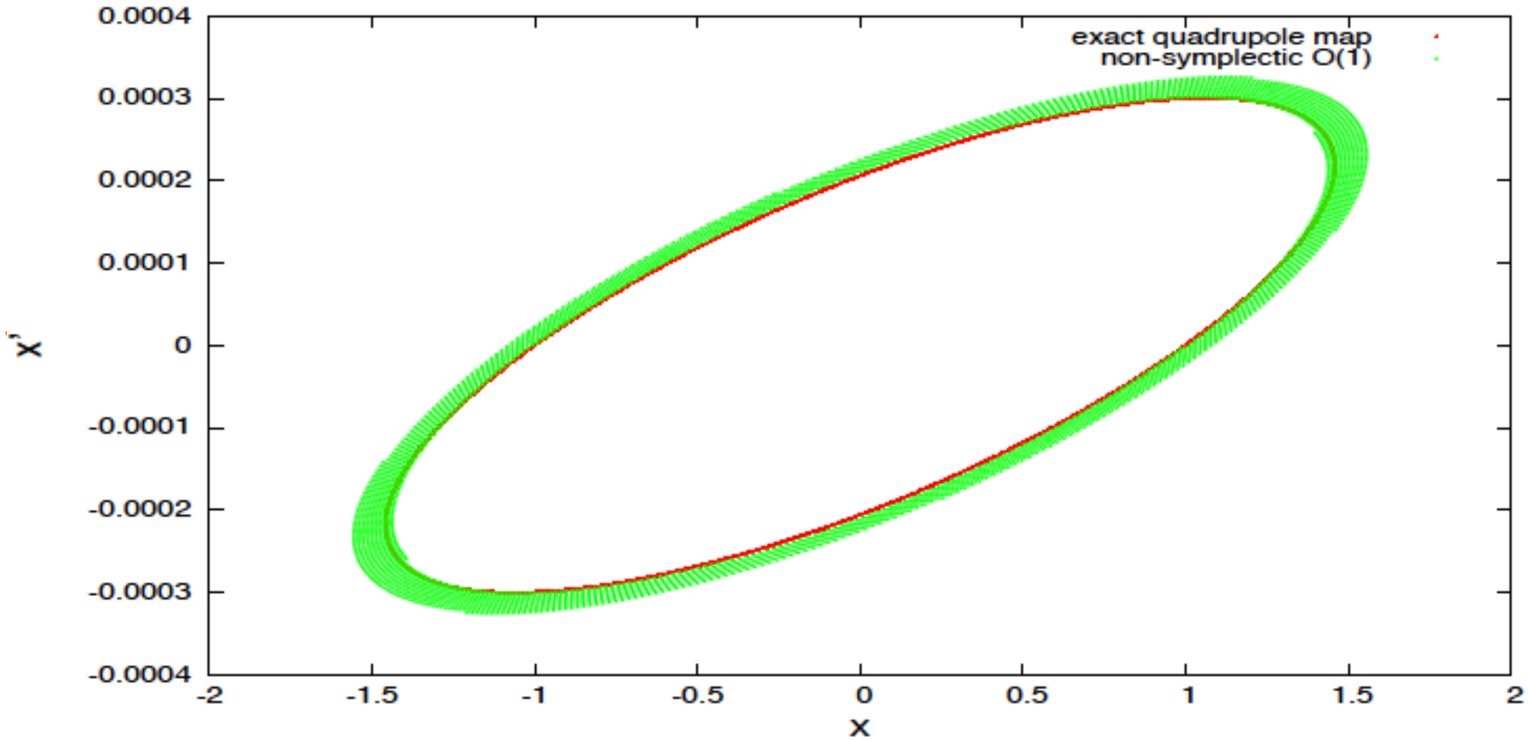
- Take the Taylor expansion for small lengths, up to first order

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1 & L \\ -kL & 1 \end{pmatrix} + O(L^2)$$

- This is indeed not symplectic as the determinant of the matrix is equal to $1 + kL^2$, i.e. there is a deviation from symplecticity at 2$^{\text{nd}}$ order in the quadrupole length

Phase portrait for non-symplectic matrix

- The iterated non-symplectic matrix does not provide the well-know elliptic trajectory in phase space
- Although the trajectory is very close to the original one, it **spirals outwards towards infinity**

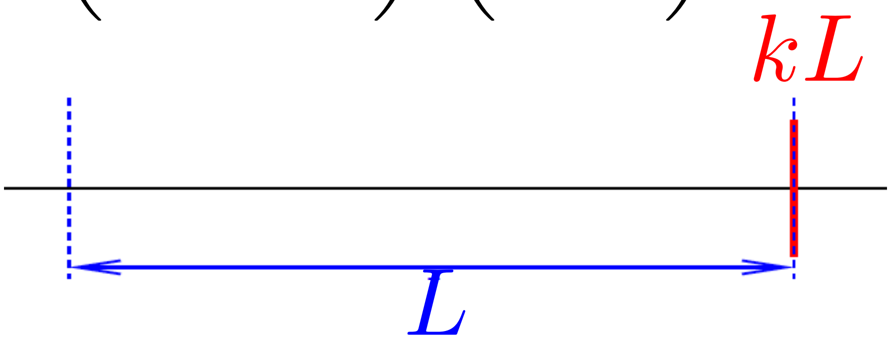


Restoring symplecticity

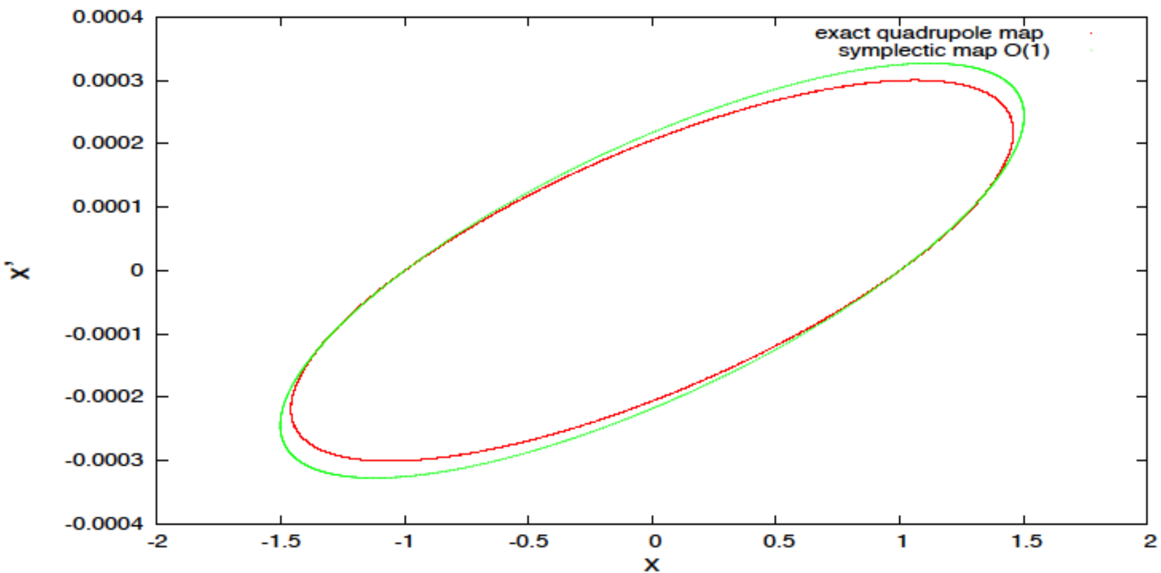
- **Symplecticity** be can **restored** by adding "artificially" a correcting term to the matrix to become

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1 & L \\ -kL & 1{\color{red}-kL^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

- In fact, the matrix now can be decomposed as a **drift** with a **thin quadrupole** at the end



- This representation, although not exact produces an **ellipse** in phase space

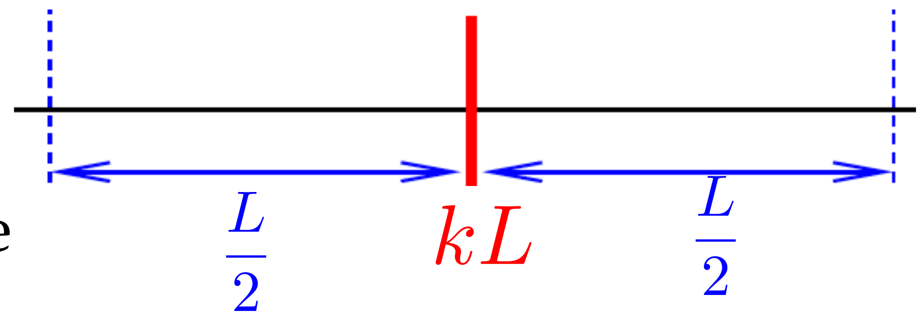


Restoring symplecticity II

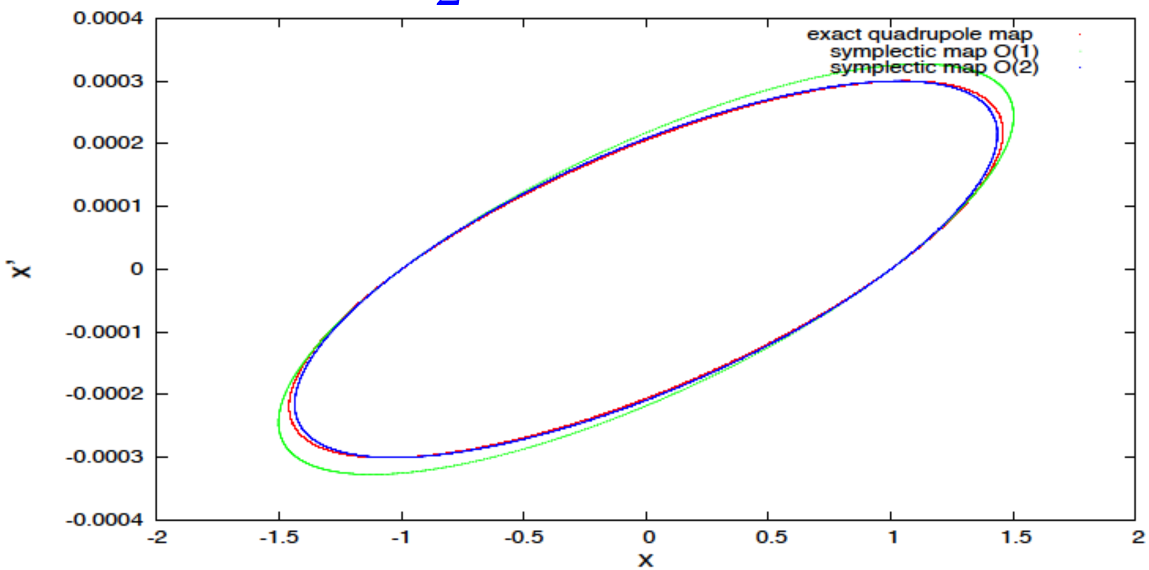
- The same approach can be continued to **2nd order** of the Taylor map, by adding a **3rd order correction**

$$\mathcal{M}_Q = \begin{pmatrix} 1-\frac{1}{2}kL^2 & L-\frac{1}{4}kL^3 \\ -kL & 1-\frac{1}{2}kL^2 \end{pmatrix} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}$$

- The matrix now can be decomposed as **two half drifts with a thin kick** at the center



- This representation is even **more exact** as the error now is at 3rd order in the length



3-kick symplectic integrator

- The idea is to distribute **three kicks with different strengths** so as to get a final map which is more accurate then the previous ones
- For the quadrupole, one can write

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1 & d_1 L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_1 k L & 1 \end{pmatrix} \begin{pmatrix} 1 & d_2 L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_2 k L & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_3 k L & 1 \end{pmatrix} \begin{pmatrix} 1 & d_4 L/2 \\ 0 & 1 \end{pmatrix}$$

which imposes $\sum d_i = \sum c_i = 1$.

- A symmetry condition of this form can be added

$$d_1 = d_4 \ , \quad d_2 = d_3 \ , \quad c_1 = c_3$$

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- This provides the **matrix** $\mathcal{M}_Q = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ with

$$m_{11} = m_{22} = -\frac{1}{2}kL^2 + c_1 d_2 (d_1 + \frac{c_2}{2}) k^2 L^4 - d_1 d_2^2 c_1^2 c_2 k^3 L^6$$

$$m_{12} = L - (\frac{c_2}{4} + d_1 d_2 + 2 d_1 d_2 c_1) kL^3 + 2 d_1 d_2 c_1 (d_1 d_2 + \frac{c_2}{2}) k^2 L^5 + d_1^2 d_2^2 c_1^2 c_2 k^3 L^7$$

$$m_{21} = -kL + c_1 d_2 (1 + c_2) k^2 L^3 - d_2^2 c_1^2 c_2 k^3 L^5$$

3-kick symplectic integrator

- By imposing that the determinant is 1, the following additional relations are obtained

$$c_1 d_2 (d_1 + \frac{c_2}{2}) = \frac{1}{24}$$

$$\frac{c_2}{4} + d_1 d_2 + 2 d_1 d_2 c_1 = \frac{1}{6}$$

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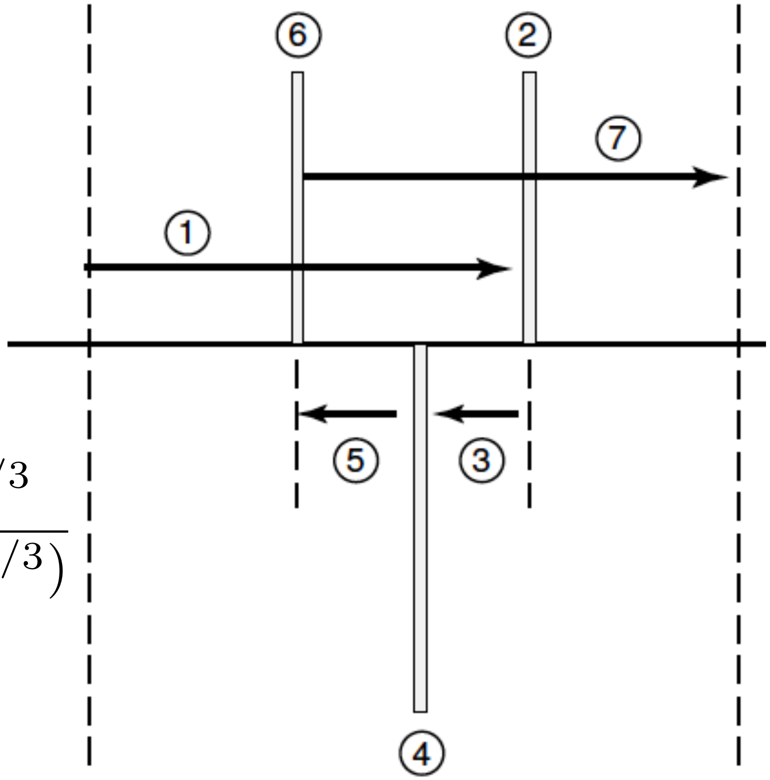

- This is actually the famous 7 step 4^{th} order symplectic integrator of **Forest**, **Ruth** and **Yoshida** (1990). It can be generalized for **any non-linear element**
- It imposes **negative drifts**…

Higher order integrators

- Yoshida has proved that a **general integrator map** of order $2k$ can be used to built a **map of order** $2k+2$

$$S_{2k+2}(t) = S_{2k}(x_1 t) \circ S_{2k}(x_0 t) \circ S_{2k}(x_1 t)$$

with $$x_0 = \frac{-2^{\frac{1}{2k+1}}}{2-2^{\frac{1}{2k+1}}}, \quad x_1 = \frac{1}{2-2^{\frac{1}{2k+1}}}$$

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- For example the **4th order scheme** can be considered as a **composition** of **three 2nd order ones** (single kicks)

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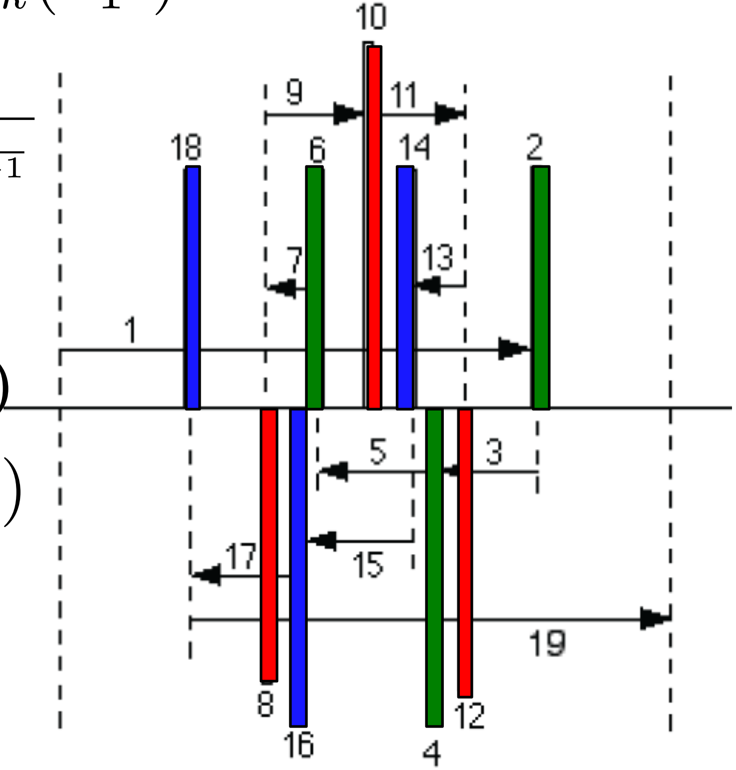

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- A **6th order** integrator can be produced by **three interleaved 4th order** ones (9 kicks)

$$S_6(t) = S_4(x_1 t) \circ S_4(x_0 t) \circ S_4(x_1 t)$$

with $\quad x_0 = \frac{-2^{\frac{1}{5}}}{2 - 2^{\frac{1}{5}}}, \quad x_1 = \frac{1}{2 - 2^{\frac{1}{5}}}$

Modern symplectic integration schemes

- Symplectic integrators with **positive** steps for Hamiltonian systems $H = A + \epsilon B$ with both A and B **integrable** were proposed by **McLachan** (1995).
- **Laskar** and **Robutel** (2001) derived all orders of such integrators
- Consider the formal solution of the Hamiltonian system written in the Lie representation

$$\vec{x}(t) = \sum_{n \geq 0} \frac{t^n}{n!} L_H^n \vec{x}(0) = e^{tL_H} \vec{x}(0).$$

- A symplectic integrator of order n from t to $t+\tau$ consists of approximating the Lie map $e^{\tau L_H} = e^{\tau(L_A + L_{\epsilon B})}$ by products of $e^{c_i \tau L_A}$ and $e^{d_i \tau L_{\epsilon B}}$, $i = 1, \ldots, n$ which integrate exactly A and B over the time-spans $c_i \tau$ and $d_i \tau$
- The constants c_i and d_i are chosen to reduce the error

SABA$_2$ integrator

- The SABA$_2$ integrator is written as

$$\mathrm{SABA}_2 = e^{c_1 \tau L_A} e^{d_1 \tau L_{\epsilon B}} e^{c_2 \tau L_A} e^{d_1 \tau L_{\epsilon B}} e^{c_1 \tau L_A},$$

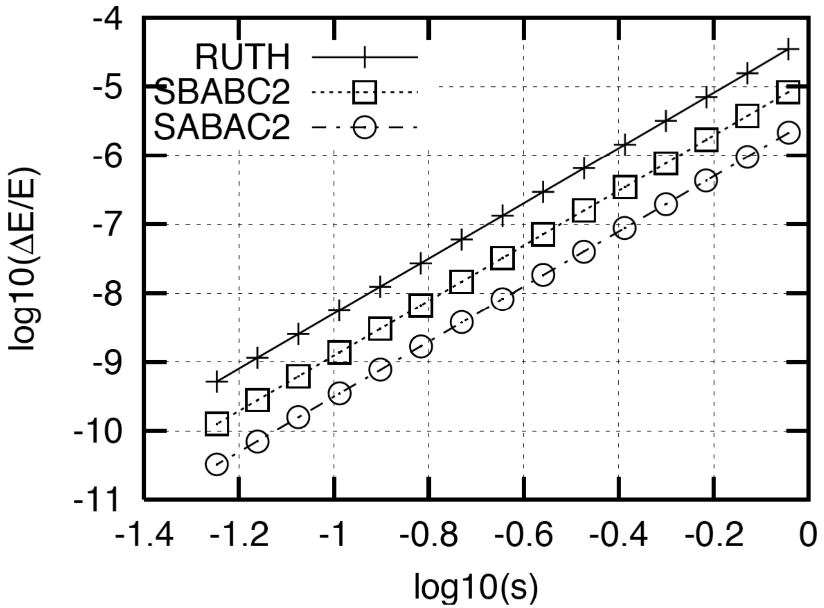
with $c_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$, $c_2 = \frac{1}{\sqrt{3}}$, $d_1 = \frac{1}{2}$.

- When $\{\{A, B\}, B\}$ is integrable, e.g. when A is quadratic in momenta and B depends only in positions, the accuracy of the integrator is improved by two small negative steps

$$\mathrm{SABA}_2\mathrm{C} = e^{-\tau^3 \epsilon^2 \frac{c}{2} L_{\{\{A,B\},B\}}} \left(\mathrm{SABA}_2\right) e^{-\tau^3 \epsilon^2 \frac{c}{2} L_{\{\{A,B\},B\}}}$$

with $c = (2 - \sqrt{3})/24$

- The accuracy of SABA$_2$C is one order of magnitude higher than the Forest-Ruth 4$^{\text{th}}$ order scheme



- The usual "drift-kick" scheme corresponds to the 2$^{\text{nd}}$ order inte

$$\mathrm{SABA}_1 = e^{\frac{\tau}{2} L_A} e^{\tau L_{\epsilon B}} e^{\frac{\tau}{2} L_A},$$

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Normal forms

- Normal forms consists of finding a canonical transformation of the 1-turn map, so that it becomes simpler to analyze
- In the linear case, the Floquet transformation is a kind a normal form as it turns **ellipses** into **circles**

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- The transformation can be written formally as

$$\begin{tikzcd} \mathbf{z} \arrow[r, "\mathcal{M}"] \arrow[d, "\Phi^{-1}"'] & \mathbf{z}' \arrow[d, "\Phi^{-1}"] \\ \boldsymbol{u} \arrow[r, "\mathcal{N}"'] & \boldsymbol{u}' \end{tikzcd}$$

with the original map $\mathcal{M} = \Phi^{-1} \circ \mathcal{N} \circ \Phi$
and its normal form

$$\mathcal{N} = \Phi \circ \mathcal{M} \circ \Phi^{-1} = e^{:h_{eff}:}$$

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$$\mathcal{N} = \Phi \circ \mathcal{M} \circ \Phi^{-1} = e^{:h_{eff}:}$$

- The transformation $\Phi = e^{:F:}$ is better suited in action angle variables, i.e. $\boldsymbol{\zeta} = e^{-:F_r:}\ \mathbf{h}$ taking the system from the original action-angle $h^{\pm}_{x,y} = \sqrt{2J_{x,y}}\ e^{\mp i\phi_{x,y}}$ to a new set $\zeta^{\pm}_{x,y}(N) = \sqrt{2I_{x,y}}\ e^{\mp i\psi_{x,y}(N)}$ with the angles being just simple rotations, $\psi_{x,y}(N) = 2\pi N \nu_{x,y} + \psi_{x,y_0}$ and the new effective Hamiltonian depends only on the new actions

Effective Hamiltonian

- The generating function can be written as a polynomial in the new actions, i.e.

$$F_r = \sum_{jklm} f_{jklm} {\zeta_x^+}^j {\zeta_x^-}^k {\zeta_y^+}^l {\zeta_y^-}^m = f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m}{2}} e^{-i\psi_{jklm}}$$

- There are **software tools** that built this transformation
- Once the "new" effective Hamiltonian is known, all interesting quantities can be derived
- This Hamiltonian is a function only of the new actions, and to 3rd order it is obtained as

$$\begin{aligned} h_{eff} =& \nu_x I_x + \nu_y I_y \\ & + \frac{1}{2}\alpha_c \delta^2 + c_{x1} I_x \delta + c_{y1} I_y \delta + c_3 \delta^3 \\ & + c_{xx} I_x^2 + c_{xy} I_x I_y + c_{yy} I_y^2 + c_{x2} I_x \delta^2 + c_{y2} I_y \delta^2 c_4 \delta^4 \end{aligned}$$

Effective Hamiltonian

- The correction of the tunes is given by

$$Q_x = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial I_x} = \frac{1}{2\pi}(\nu_x + 2c_{xx}I_x + c_{xy}I_y + c_{x1}\delta + c_{x2}\delta^2)$$

$$Q_y = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial I_y} = \frac{1}{2\pi}(\nu_y + 2c_{yy}I_y + c_{xy}I_x + c_{y1}\delta + c_{y2}\delta^2)$$

tunes **tune-shift with amplitude** **1st and 2nd order chromaticity**

- The correction to the path length is

$$\Delta s = \frac{\partial h_{eff}}{\partial \delta} = \alpha_c\delta + c_3\delta^2 + 4c_4\delta^3 + c_{x1}I_x + c_{y1}I_y + 2c_{x2}I_x\delta + 2c_{y2}I_y\delta$$

1st, 2nd and 3rd momentum compaction

Normal form for perturbation

- Using the BCH formula, one can prove that the composition of two maps with g small can be written as

$$e^{:f:}e^{:g:} = \exp\left[:f + \left(\frac{:f:}{1-e^{-:f:}}\right)g + \mathcal{O}(g^2):\right]$$

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- Consider a linear map (rotation) followed by a small perturbation $\mathcal{M} = e^{:f_2:}e^{:f_3:}$
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- This can be written as

$$\begin{aligned}\mathcal{N} &= e^{:f_2:}e^{-:f_2:}e^{:F:}e^{:f_2:}e^{:f_3:}e^{:-F:}\\ &= e^{:f_2:}e^{:e^{-:f_2:}F+f_3-F:}+\dots\\ &= e^{:f_2:}e^{:(e^{-:f_2:}-1)F+f_3:}+\dots\end{aligned}$$

$$F = \frac{f_3}{1-e^{-:f_2:}}$$

- This will **transform** the new **map** to a **rotation** to leading order

Example: Octupole

- Consider a linear map followed by an octupole

$$\mathcal{M} = e^{-\frac{\nu}{2}:x^2+p^2:}e^{:\frac{x^4}{4}:} = e^{:f_2:}e^{:\frac{x^4}{4}:}$$

- The **generating function** has to be chosen such as to make the following expression simpler

$$(e^{-:f_2:} - 1)F + \frac{x^4}{4}$$

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- We pass to the **action angle variable** (resonance basis)

$$h^{\pm} = \sqrt{2J}\, e^{\mp i\phi} = x \mp ip$$

- The perturbation is

$$x^4 = (h_+ + h_-)^4 = h^{\pm} = h_+^4 + 4h_+^3 h_- + 6h_+^2 h_-^2 + 4h_+ h_-^3 + h_-^4$$

Example: Octupole

- The **term** $6h_+^2h_-^2 = 24J^2$ is **independent** on the **angles**. Thus we may **choose** the **generating functions** such that the **other terms are eliminated**. It takes the form

$$F = \frac{1}{16}\left(\frac{h_+^4}{1-e^{4i\nu}} + \frac{4h_+^3h_-}{1-e^{2i\nu}} + \frac{4h_+h_-^3}{1-e^{2i\nu}} + \frac{h_-^4}{1-e^{4i\nu}}\right)$$

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- The map is now written as

$$\mathcal{M} = e^{-:F:} e^{:\nu J + \frac{3}{8}J^2:} e^{:F:}$$

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- The **new effective Hamiltonian** is depending only on the **actions** and contains the tune-shift terms
- The **generator** in the original variables is written as

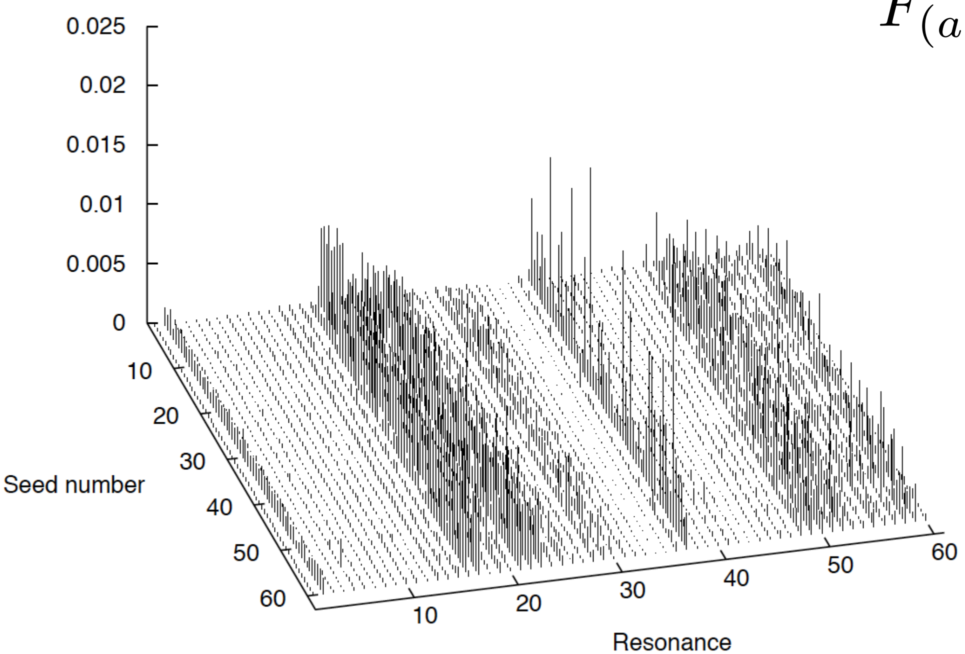
$$F = -\frac{1}{64}\left[-5x^4 + 3p^4 + 6x^2p^2 + 4x^3p(2\cot(\nu) + \cot(2\nu)) + 4xp^3(2\cot(\nu) - \cot(2\nu))\right]$$

- **Constant values** of the generator describe the **trajectories** in phase space

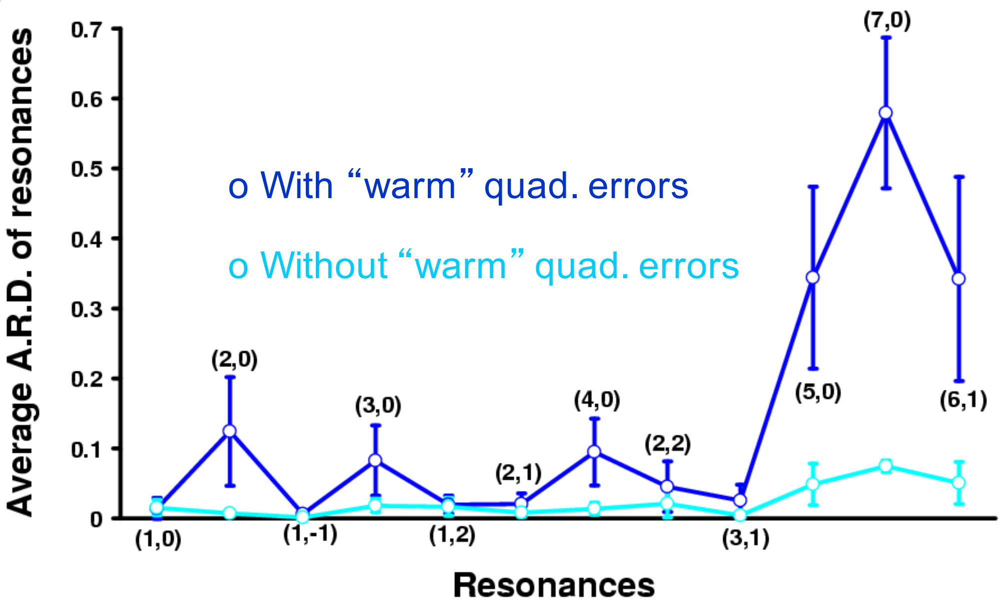
Graphical resonance representation

- It is possible by constructing the one turn map to built the generating (sometimes called "**distortion**") function $F_r \approx \sum_{jklm} f_{jklm} \, J_x^{\frac{j+k}{2}} \, J_y^{\frac{l+m}{2}} \, e^{-i\psi_{jklm}}$
- For any resonance $a\nu_x + bq_y = c$, and setting $\psi_{jklm} = 0$, the associated part of the functions is

$$F_{(a,b)} \approx \sum_{\substack{jklm \\ j+k+l+m \leq n \\ j+k=a \,,\, l+m=b}} f_{jklm} \, J_x^{\frac{a}{2}} \, J_y^{\frac{b}{2}}$$



Normal forms for LHC models



Phase	Type	DA (σ)	LHC Version 4	LHC Version 5 Nominal	LHC Version 5 Target
15°	Warm Quads switched ON	Average	10.0	9.1	10.4
		Minimum	8.5	7.4	8.6
	Warm Quads switched OFF	Average	10.7	11.6	12.4
		Minimum	9.6	10.3	11.3
45°	Warm Quads switched ON	Average	11.1	11.3	12.8
		Minimum	9.5	9.2	11.4
	Warm Quads switched OFF	Average	11.4	12.4	13.8
		Minimum	10.1	10.7	12.3

- In the LHC at injection (450 GeV), beam stability is necessary over a very large number of turns (10^7)
- Stability is reduced from random multi-pole imperfections mainly in the super-conducting magnets
- Area of stability (Dynamic aperture - DA) computed with particle tracking for a large number of random magnet error distributions
- Numerical tool based on normal form analysis (GRR) permitted identification of DA reduction reason (errors in the "warm" quadrupoles)

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Summary

- **Symplectic maps** are the natural way to represent accelerator dynamics
- They are obtained through **Lie transformations**
- **Truncation** of the map makes it deviate from symplecticity
- Symplecticity essential for preserving Hamiltonian structure of system (area preservation)
- **Use symplectic integrators** for tracking
- Even high order integrators with positive steps exist
- Normal form construction on the 1-turn map makes non-linear dynamics analysis straightforward

SABA$_2$C for accelerators

- The accelerator Hamiltonian in the small angle, "hard-edge" approximation is written as $H(x, y, l, p_x, p_y, \delta; s) = H_0 + V$,

  with the unperturbed part $H_0 = (1 + h\ x)\dfrac{p_x^2 + p_y^2}{2(1+\delta)}$,

  and the perturbation $V(x,y) = \displaystyle\sum_{n\geq 1}\sum_{j=0}^{n} a_{n,j} x^j y^{n-j}$

- The unperturbed part of the Hamiltonian can be integrated

$$
e^{sL_A}: \begin{cases}
x^f &= \dfrac{1}{h}\left\{(1+hx^i)\left(\cos\phi + \dfrac{p_x^i}{p_y^i}\sin\phi\right)^2 - 1\right\} \\
y^f &= y^i + \dfrac{1+hx^i}{h}\left\{\dfrac{{p_x^i}^2 + {p_y^i}^2}{{p_y^i}^2}\phi + \dfrac{{p_y^i}^2 - {p_x^i}^2}{2{p_y^i}^2}\sin(2\phi) + 2\dfrac{p_x^i}{p_y^i}\sin^2\phi\right\} \\
p_x^f &= p_y^i \dfrac{p_x^i - p_y^i\tan\phi}{p_y^i + p_x^i\tan\phi} \qquad \text{with} \quad \phi = \dfrac{p_y^i h s}{2(1+\delta)} \\
p_y^f &= p_y^i
\end{cases}
$$

SABA_2C for accelerators II

- The perturbation part of the Hamiltonian can be integrated

$$e^{sL_B}:\begin{cases} x^f = x^i \quad , \quad p_x^f = p_x^i - \left.\dfrac{\partial V}{\partial x}\right|_i s \\ y^f = y^i \quad , \quad p_y^f = p_y^i - \left.\dfrac{\partial V}{\partial y}\right|_i s \end{cases} \text{with} \quad \begin{aligned} \left.\frac{\partial V}{\partial x}\right|_i &= \sum_{n\geq 1}\sum_{j=1}^{n} j a_{n,j}(x^i)^{j-1}(y^i)^{n-j} \\ \left.\frac{\partial V}{\partial y}\right|_i &= \sum_{n\geq 1}\sum_{j=0}^{n}(n-j)a_{n,j}(x^i)^{j}(y^i)^{n-j-1} \end{aligned} .$$

- The corrector is expressed as

$$C = \{\{A,B\},B\} = \frac{1+hx}{1+\delta}\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right],$$

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$$e^{sL_C}:\begin{cases} x^f = x^i \\ y^f = y^i \\ p_x^f = p_x^i - \dfrac{1}{1+\delta}\left\{h\left[\left.\dfrac{\partial V}{\partial x}\right|_i^2 + \left.\dfrac{\partial V}{\partial y}\right|_i^2\right] + 2(1+hx^i)\left[\left.\dfrac{\partial V}{\partial x}\right|_i \left.\dfrac{\partial^2 V}{\partial x^2}\right|_i + \left.\dfrac{\partial V}{\partial y}\right|_i \left.\dfrac{\partial^2 V}{\partial x \partial y}\right|_i\right]\right\} s \\ p_y^f = p_y^i - \dfrac{2(1+hx^i)}{1+\delta}\left\{\left.\dfrac{\partial V}{\partial x}\right|_i \left.\dfrac{\partial^2 V}{\partial x \partial y}\right|_i + \left.\dfrac{\partial V}{\partial y}\right|_i \left.\dfrac{\partial^2 V}{\partial y^2}\right|_i\right\} s \end{cases} .$$