Non-linear beam dynamics

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- Lagrangian Formalism
  - Lagrange mechanics
  - From the Lagrangian to the Hamiltonian
- Hamiltonian Formalism
  - Hamilton's equations
  - Properties of the Hamiltonian flow
  - Poisson brackets and their properties
- Canonical transformations
  - Preservation of phase volume and examples
- Single particle relativistic Hamiltonian
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- Secular perturbation theory
  - Third order resonance
  - Fixed points for general multi-pole
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  - Resonance overlap
- Summary

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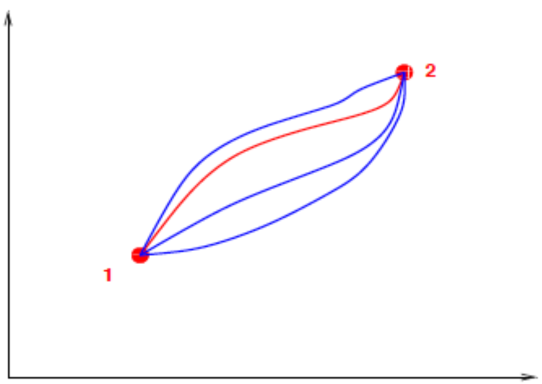
Lagrangian formalism

- Describe motion of particles in q_n coordinates (***n* degrees of freedom**) from time t_1 to time t_2
- Describe motion by the **Lagrangian function** $L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)$ with $(q_1,\ldots,q_n)$ the **generalized coordinates** and $(\dot{q}_1,\ldots,\dot{q}_n)$ the **generalized velocities**
- The Lagrangian function defined as $L = T - V$, i.e. difference between kinetic and potential energy
- The integral $W = \int L(q_i,\dot{q}_i,t)dt$ defines the **action**
- **Hamilton's principle**: system evolves so as the action becomes extremum (principle of **stationary action**)



Lagrange equations

- The variation of the action can be written as

$$\delta W = \int_{t_1}^{t_2} \left(L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q,\dot{q},t) \right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt$$

- Taking into account that $\delta \dot{q} = \dfrac{d\delta q}{dt}$, the 2nd part of the integral can be integrated by parts giving

$$\delta W = \left| \frac{\partial L}{\partial \dot{q}}\delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt = 0$$

- The first term is zero because $\delta q(t_1) = \delta q(t_2) = 0$ so the second integrant should also vanish, providing the following differential equations for each degree of freedom, the **Lagrange equations**

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0}$$

Lagrangian mechanics

- For a simple **force law** contained in a potential function, governing motion among interacting particles, the Lagrangian is (or as Landau-Lifshitz put it "experience has shown that…")

$$L = T - V = \sum_{i=1}^{n} \frac{1}{2} m_i q_i^2 - V(q_1, \ldots, q_n)$$

- For velocity independent potentials, Lagrange equations become

$$m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i} \ ,$$

i.e. **Newton's equations**.

From Lagrangian to Hamiltonian

- Some **disadvantages** of the Lagrangian formalism:
  - **No uniqueness**: different Lagrangians can lead to same equations
  - **Physical significance** not straightforward (even its basic form given more by "experience" and the fact that it actually works that way!)
- Lagrangian function provides in general n second order differential equations (coordinate space)
- We already observed the advantage to move to a system of $2n$ first order differential equations, which are more straightforward to solve (**phase space**)
- These equations can be derived by the **Hamiltonian** of the system

Contents of the 2nd lecture

Hamiltonian formalism

- The **Hamiltonian** of the system is defined as the **Legendre transformation** of the Lagrangian

$$H(\mathbf{q},\mathbf{p},t) = \sum_i \dot{q}_i p_i - L(\mathbf{q},\dot{\mathbf{q}},t)$$

where the **generalised momenta** are $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

- The **generalised velocities** can be expressed as a function of the **generalised momenta** if the previous equation is invertible, and thereby define the Hamiltonian of the system
- **Example**: consider $L(\mathbf{q},\dot{\mathbf{q}}) = \dfrac{1}{2}\sum_i m_i \dot{q}_i^2 - V(q_1,\ldots,q_n)$
- From this the momentum can be determined as $p_i = \dfrac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$

which can be trivially inverted to provide the Hamiltonian

$$H(\mathbf{q},\mathbf{p}) = \sum_i \frac{p_i^2}{2m_i} + V(q_1,\ldots,q_n)$$

Hamilton's equations

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian ("least" action) but also by simply taking the differential of the Hamiltonian

$$dH = \sum_i \cancel{p_i d\dot{q}_i} + \dot{q}_i dp_i - \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i} - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \frac{\partial L}{\partial t} dt$$

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or

$$dH(q,p,t) = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

- By equating terms, **Hamilton's equations** are derived

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ , \quad \dot{p}_i = -\frac{\partial H}{\partial q}\ , \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n+2$ equations describing the motion in the **"extended" phase space** $(q_i, \ldots, q_n, p_1, \ldots, p_n, t, -H)$

Properties of Hamiltonian flow

- The variables $(q_i, \ldots, q_n, p_1, \ldots, p_n, t, -H)$ are called **canonically conjugate** (or canonical) and define the evolution of the system in phase space
- These variables have the special property that they preserve volume in phase space, i.e. satisfy the well-known **Liouville's theorem**
- The variables used in the Lagrangian do not necessarily have this property
- Hamilton's equations can be written in **vector form** $\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$ with $\mathbf{z} = (q_i, \ldots, q_n, p_1, \ldots, p_n)$ and $\nabla = (\partial q_i, \ldots, \partial q_n, \partial p_1, \ldots, \partial p_n)$
- The $2n \times 2n$ matrix $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is called the **symplectic** matrix

Poisson brackets

- Crucial step in study of Hamiltonian systems is identification of **integrals of motion**
- Consider a time dependent function of phase space. Its time evolution is given by

$$\frac{d}{dt}f(\mathbf{p},\mathbf{q},t) = \sum_{i=1}^{n}\left(\frac{dq_i}{dt}\frac{\partial f}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i}\right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right) + \frac{\partial f}{\partial t} = [H,f] + \frac{\partial f}{\partial t}$$

where $[H,f]$ is the **Poisson bracket** of f with H

- If a quantity is explicitly **time-independent** and its Poisson bracket with the Hamiltonian vanishes (i.e. **commutes** with the H), it is a **constant** (or **integral**) of motion (as an **autonomous** Hamiltonian itself)

Poisson brackets' properties

- The Poisson brackets between two functions of a set of canonical variables can be defined by the differential operator

$$[f,g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \right)$$

- From this definition, and for any three given functions, the following properties can be shown

$$[af + bg, h] = a[f,h] + b[g,h] \; , a, b \in \mathbb{R}$$ **bilinearity**

$$[f,g] = -[g,f]$$ **anticommutativity**

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$ **Jacobi's identity**

$$[f,gh] = [f,g]h + g[f,h]$$

- Poisson brackets operation satisfies a **Lie algebra**

Contents of the 2nd lecture

Canonical Transformations

- Find a **function** for transforming the Hamiltonian from variable $(\mathbf{q},\mathbf{p})$ to $(\mathbf{Q},\mathbf{P})$ so system becomes **simpler** to study
- This transformation should be **canonical** (or **symplectic**), so that the Hamiltonian properties of the system are preserved
- These "mixed variable" **generating** functions are derived by

$$F_1(\mathbf{q},\mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i}, \ \ P_i = -\frac{\partial F_1}{\partial Q_i} \quad F_3(\mathbf{Q},\mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i}, \ \ P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F_2(\mathbf{q},\mathbf{P}) : p_i = \frac{\partial F_2}{\partial q_i}, \ \ Q_i = \frac{\partial F_2}{\partial P_i} \quad F_4(\mathbf{p},\mathbf{P}) : q_i = -\frac{\partial F_4}{\partial p_i}, \ \ Q_i = \frac{\partial F_4}{\partial P_i}$$

- A general non-autonomous Hamiltonian is transformed to

$$H(\mathbf{Q},\mathbf{P},t) = H(\mathbf{q},\mathbf{p},t) + \frac{\partial F_j}{\partial t} \ , \quad j = 1,2,3,4$$

- One generating function can be constructed by the other through **Legendre transformations**, e.g.

$$F_2(\mathbf{q},\mathbf{P}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{Q}\cdot\mathbf{P} \ , \quad F_3(\mathbf{Q},\mathbf{p}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{q}\cdot\mathbf{p} \ , \quad \ldots$$

with the inner product define as $\mathbf{q}\cdot\mathbf{p} = \sum_i q_i p_i$

Preservation of Phase Volume

- A fundamental property of canonical transformations is the **preservation of phase space volume**
- This volume preservation in phase space can be represented in the old and new variables as

$$\int \prod_{i=1}^{n} dp_i dq_i = \int \prod_{i=1}^{n} dP_i dQ_i$$

- The volume element in old and new variables are related through the Jacobian

$$\prod_{i=1}^{n} dp_i dq_i = \frac{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)} \prod_{i=1}^{n} dP_i dQ_i$$

- These two relationships imply that the Jacobian of a canonical transformation should have determinant equal to 1

$$\left|\frac{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)}\right| = \left|\frac{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)}{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}\right| = 1$$

Examples of transformations

- The transformation $Q = -p \ , \quad P = q$, which **interchanges conjugate variables** is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

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- On the other hand, the transformation from **Cartesian to polar** coordinates $q = P\cos Q \ , \quad p = P\sin Q$ is not, since

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} -P\sin Q & P\cos Q \\ \cos Q & \sin Q \end{vmatrix} = -P$$

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- There are actually **"polar" coordinates** that are **canonical**, given by $q = -\sqrt{2P}\cos Q\ ,\quad p = \sqrt{2P}\sin Q$ for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P}\sin Q & \sqrt{2P}\cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

Contents of the 2nd lecture

Single-particle relativistic Hamiltonian

- Neglecting self fields and radiation, motion can be described by a "single-particle" Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x},t)$$

  - $\mathbf{x} = (x, y, z)$ Cartesian positions
  - $\mathbf{p} = (p_x, p_y, p_z)$ conjugate momenta
  - $\mathbf{A} = (A_x, A_y, A_z)$ magnetic vector potential
  - $\mathbf{\Phi}$ electric scalar potential

- The ordinary kinetic momentum vector is written

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

with $\mathbf{v}$ the velocity vector and $\gamma = (1 - v^2/c^2)^{-1/2}$ the relativistic factor

Single-particle relativistic Hamiltonian

- It is generally a **3 degrees of freedom** one **plus time** (i.e. **4 degrees of freedom**)
- The Hamiltonian represents the total energy

$$H \equiv E = \gamma mc^2 + e\Phi$$

- The total kinetic momentum is

$$P = \left(\frac{H^2}{c^2} - m^2 c^2 \right)^{1/2}$$

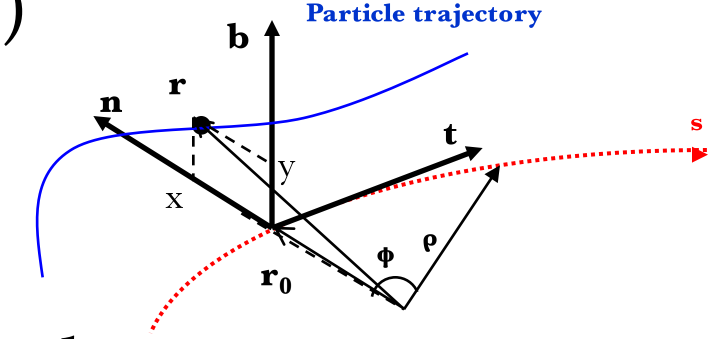
- Using Hamilton's equations

$$(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = [(\mathbf{x}, \mathbf{p}), H]$$

it can be shown that motion is governed by **Lorentz equations**

From Cartesian to "curved" coordinates

- It is useful (especially for rings) to transform the Cartesian coordinate system to the **Frenet-Serret system** moving to a closed curve, with path length s



- The position coordinates in the two systems are connected by $\mathbf{r} = \mathbf{r_0}(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z}$
- The **Frenet-Serret unit vectors** and their derivatives are defined as $(\mathbf{t},\mathbf{n},\mathbf{b}) = (\frac{d}{ds}\mathbf{r_0}(s), -\rho(s)\frac{d^2}{ds^2}\mathbf{r_0}(s), \mathbf{t}\times\mathbf{n})$

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho(s)} & 0 \\ 0 & 0 & \tau(s) \\ \frac{1}{\rho(s)} & 0 & -\tau(s)\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}$$

with $\rho(s)$ the radius of curvature and $\tau(s)$ the torsion which vanishes in case of planar motion

From Cartesian to "curved" variables

- We are seeking a canonical transformation between

$$
\begin{aligned}
(\mathbf{q},\mathbf{p}) &\mapsto (\mathbf{Q},\mathbf{P}) \text{ or} \\
(x,y,z,p_x,p_y,p_z) &\mapsto (X,Y,s,P_x,P_y,P_s)
\end{aligned}
$$

- The generating function is

$$
(\mathbf{q},\mathbf{P}) = -\left(\frac{\partial F_3(\mathbf{p},\mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_3(\mathbf{p},\mathbf{Q})}{\partial \mathbf{Q}}\right)
$$

- By using the relationship between the positions, the generating function is

$$
F_3(\mathbf{p},\mathbf{Q}) = -\mathbf{p}\cdot\mathbf{r} + \overline{F_3}(\mathbf{Q}) = -\mathbf{p}\cdot\mathbf{r}
$$

From Cartesian to "curved" variables

- for planar motion, the momenta are

$$\mathbf{P} = (P_X, P_Y, P_s) = \mathbf{p}\cdot(\mathbf{n}, \mathbf{b}, (1+\frac{X}{\rho})\mathbf{t})$$

- Taking into account that the **vector potential** is also transformed in the same way

$$(A_X, A_Y, A_s) = \mathbf{A}\cdot(\mathbf{n}, \mathbf{b}, (1+\frac{X}{\rho})\mathbf{t})$$

the **new Hamiltonian** is given by

$$\mathcal{H}(\mathbf{Q},\mathbf{P},t) = c\sqrt{(P_X - \frac{e}{c}A_X)^2 + (P_Y - \frac{e}{c}A_Y)^2 + \frac{(P_s - \frac{e}{c}A_s)^2}{(1+\frac{X}{\rho(s)})^2} + m^2c^2} + e\Phi$$

Changing of the independent variable

- It is more convenient to use the **path length** s , instead of the time as **independent variable**
- The Hamiltonian can be considered as having **4 degrees of freedom**, where the **4th "position"** is **time** and its conjugate momentum is $P_t = -\mathcal{H}$

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- The Hamiltonian can be considered as having **4 degrees of freedom**, where the **4th "position"** is **time** and its conjugate momentum is $P_t = -\mathcal{H}$
- In the same way, the new Hamiltonian with the path length as the independent variable is just $P_s = -\tilde{\mathcal{H}}(X, Y, t, P_X, P_Y, P_t, s)$ with

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{(\frac{P_t + e\Phi}{c})^2 - m^2c^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- It can be proved that this is indeed a **canonical transformation**
- Note the existence of the **reference orbit** for zero vector potential, for which $(X, Y, P_X, P_Y, P_s) = (0, 0, 0, 0, P_0)$

Neglecting electric fields

- Due to the fact that **longitudinal** (synchrotron) motion is **much slower** than the **transverse** (betatron) one, the electric field can be set to zero and the Hamiltonian is written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{\underbrace{(\frac{\mathcal{H}}{c})^2 - m^2c^2}_{P^2} - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- The Hamiltonian is then written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{(P^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- If **static** magnetic fields are considered, the time dependence is also dropped, and the system is having **2 degrees of freedom + "time"** (path length)

Momentum rescaling

- Due to the fact that **total momentum** is **much larger** than the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$(\mathbf{Q},\mathbf{P}) \quad \mapsto \quad (\bar{\mathbf{q}},\bar{\mathbf{p}}) \ \text{ or}$$

$$(X,Y,t,P_X,P_Y,P_t) \quad \mapsto \quad (\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = (X,Y,-c\,t,\frac{P_X}{P_0},\frac{P_Y}{P_0},-\frac{P_t}{P_0 c})$$

- The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$\bar{\mathcal{H}}(\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = \frac{\tilde{\mathcal{H}}}{P_0} = -e\bar{A}_s - \left(1+\frac{\bar{x}}{\rho(s)}\right)\sqrt{\bar{p}_t^2 - \frac{m^2c^2}{P_0} - (\bar{p}_x - e\bar{A}_x)^2 - (\bar{p}_y - e\bar{A}_y)^2}$$

with $\quad (\bar{A}_x,\bar{A}_y,\bar{A}_z) = \dfrac{1}{P_0\,c}(A_x,A_y,A_s)$

and $\quad \dfrac{m^2c^2}{P_0} = \dfrac{1}{\beta_0^2\gamma_0^2}$

Moving the reference frame

- Along the reference trajectory $\bar{p}_{t0} = \frac{1}{\beta_0}$ and $\frac{d\bar{t}}{ds}|_{P=P_0} = \frac{\partial \bar{H}}{\partial \bar{p}_t}|_{P=P_0} = -\bar{p}_{t0} = -\frac{1}{\beta_0}$
- It is thus useful to **move** the **reference frame** to the **reference trajectory** for which another canonical transformation is performed

$$(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \quad \mapsto \quad (\hat{\mathbf{q}}, \hat{\mathbf{p}}) \ \text{ or}$$

$$(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) \quad \mapsto \quad (\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = (\hat{x}, \hat{y}, \bar{t} + \frac{s - s_0}{\beta_0}, \hat{p}_x, \hat{p}_y, \bar{p}_t - \frac{1}{\beta_0})$$

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- The mixed variable generating function is

$(\hat{\mathbf{q}}, \bar{\mathbf{p}}) = (\frac{\partial F_2(\bar{\mathbf{q}},\hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_2(\bar{\mathbf{q}},\hat{\mathbf{p}})}{\partial \bar{\mathbf{q}}})$ providing

$$F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}}) = \bar{x}\hat{p}_x + \bar{y}\hat{p}_y + (\bar{t} + \frac{s - s_0}{\beta_0})(\hat{p}_t + \frac{1}{\beta_0})$$

- The Hamiltonian is then

$$\hat{\mathcal{H}}(\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \frac{1}{\beta_0}(\frac{1}{\beta_0} + \hat{p}_t) - e\hat{A}_s - \left(1 + \frac{\hat{x}}{\rho(s)}\right)\sqrt{(\hat{p}_t + \frac{1}{\beta_0})^2 - \frac{1}{\beta_0^2\gamma_0^2} - (\hat{p}_x - e\hat{A}_x)^2 - (\hat{p}_y - e\bar{A}_y)^2}$$

- First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
- In the **ultra-relativistic limit** $\beta_0 \to 1$, $\frac{1}{\beta_0^2\gamma^2} \to 0$ and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the "hats" are dropped for simplicity

Relativistic and transverse field approximations

- First note that $\hat{p}_t = \bar{p}_t - \dfrac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \dfrac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
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where the "hats" are dropped for simplicity

- If we consider **only transverse field** components, the **vector potential** has **only** a **longitudinal** component and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

High-energy, large ring approximation

- It is useful for study purposes (especially for finding an "integrable" version of the Hamiltonian) to make an extra approximation
- For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller** than **1**, i.e. the square root can be expanded.
- Considering also the large machine approximation $x << \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x(1+\delta)}{\rho(s)} - e\hat{A}_s$$

- This expansion may **not** be **a good idea**, especially for **low energy**, **small** size **rings**

Linear magnetic fields

- Assume a simple case of **linear transverse magnetic fields,**

$$\begin{aligned} B_x &= \quad b_1(s)y \\ B_y &= -\,b_0(s) + b_1(s)x \end{aligned} ,$$

  - main bending field $\quad -B_0 \equiv b_0(s) = \frac{P_0 c}{e\rho(s)} \;[\mathrm{T}]$
  - normalized quadrupole gradient $\quad K(s) = b_1(s)\frac{e}{cP_0} = \frac{b_1(s)}{B\rho} \;[1/\mathrm{m}^2]$
  - magnetic rigidity $\quad B\rho = \dfrac{P_0 c}{e} \;[\mathrm{T}\cdot\mathrm{m}]$

Linear magnetic fields

- Assume a simple case of **linear transverse magnetic fields,**

$$B_x = b_1(s)y$$
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  - main bending field $\quad -B_0 \equiv b_0(s) = \frac{P_0 c}{e\rho(s)}$ [T]
  - normalized quadrupole gradient $\quad K(s) = b_1(s)\frac{e}{cP_0} = \frac{b_1(s)}{B\rho}$ [1/m^2]
  - magnetic rigidity $\quad B\rho = \dfrac{P_0 c}{e}$ [T $\cdot$ m]

- The vector potential has only a longitudinal component which in curvilinear coordinates is

$$B_x = -\frac{1}{1+\frac{x}{\rho(s)}}\frac{\partial A_s}{\partial y} \; , \quad B_y = \frac{1}{1+\frac{x}{\rho(s)}}\frac{\partial A_s}{\partial x}$$

- The previous expressions can be integrated to give

$$A_s(x,y,s) = \frac{P_0 c}{e}\left[-\frac{x}{\rho(s)} - \left(\frac{1}{\rho(s)^2} + K(s)\right)\frac{x^2}{2} + K(s)\frac{y^2}{2}\right] = P_0 c\, \hat{A}_s(x,y,s)$$

The integrable Hamiltonian

- The Hamiltonian for linear fields can be finally written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{K(s)}{2}(x^2-y^2)$$

- Hamilton's equation are

$$\frac{dx}{ds} = \frac{p_x}{1+\delta}, \quad \frac{dp_x}{ds} = \frac{\delta}{\rho(s)} - \left(\frac{1}{\rho^2(s)} + K(s)\right)x$$

$$\frac{dy}{ds} = \frac{p_y}{1+\delta}, \quad \frac{dp_y}{ds} = K(s)y$$

and they can be written as two second order uncoupled differential equations, i.e. Hill's equations

$$x'' + \frac{1}{1+\delta}\overbrace{\left(\frac{1}{\rho(s)^2} + K(s)\right)}^{K_x} x = \frac{\delta}{\rho(s)}$$

$$y'' - \frac{1}{1+\delta}\underbrace{K(s)}_{K_y} y = 0$$

with the usual solution for $\delta = 0$ and $u = x, y$

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$$

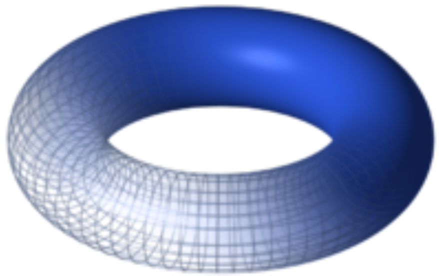
$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0) + \alpha(s)\cos(\psi(s)+\psi_0)\right)$$

Action-angle variables

- There is a canonical transformation to some **optimal set** of variables which can simplify the phase-space motion
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral $\mathbf{J} = \oint \mathbf{p} d\mathbf{q}$ over closed paths in phase space.
- An **integrable Hamiltonian** is written as a function of only the actions, i.e. $H_0 = H_0(\mathbf{J})$. Hamilton's equations give

$$\dot{\phi}_i = \frac{\partial H_0(\mathbf{J})}{\partial J_i} = \omega_i(\mathbf{J}) \Rightarrow \phi_i = \omega_i(\mathbf{J})t + \phi_{i0}$$

$$\dot{J}_i = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_i} = 0 \Rightarrow J_i = \text{const.}$$



  i.e. the **actions are integrals of motion** and the **angles** are **evolving linearly with time**, with **constant frequencies** which depend on the actions
- The actions define the surface of an **invariant torus**, topologically equivalent to the product of n circles

Accelerator Hamiltonian in action-angle variables

- Considering **on-momentum** motion, the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2} + \frac{K_x(s)x^2 - K_y(s)y^2}{2}$$

- The generating function from the original to action angle variables is

$$F_1(x,y,\phi_x,\phi_y;s) = -\frac{x^2}{2\beta_x(s)}\left[\tan\phi_x(s) + a_x(s)\right] - \frac{y^2}{2\beta_y(s)}\left[\tan\phi_y(s) + a_y(s)\right]$$

Accelerator Hamiltonian in action-angle variables

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- The **old variables** with respect to **actions and angles** are

$$u(s) = \sqrt{2\beta_u(s)J_u}\cos\phi_u(s)\ , \quad p_u(s) = -\sqrt{\frac{2J_u}{\beta_u(s)}}\left(\sin\phi_u(s)+\alpha_u(s)\cos\phi_u(s)\right)$$

and the Hamiltonian takes the form

$$\mathcal{H}_0(J_x,J_y,s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}$$

- The "time" (longitudinal position) dependence can be eliminated by the transformation to **normalized coordinate**

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{2J}\begin{pmatrix}\cos(\nu\phi)\\ \sin(\nu\phi)\end{pmatrix} \text{ with } \nu = \frac{1}{2\pi}\oint\frac{du}{\beta(s)}$$

General non-linear Accelerator Hamiltonian

- Considering the **general expression** of the the **longitudinal component** of the **vector potential** is
  - In curvilinear coordinates (curved elements)

$$A_s = (1 + \frac{x}{\rho(s)}) B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + i a_n}{n+1} (x+iy)^{n+1}$$

  - In Cartesian coordinates $A_s = B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + i a_n}{n+1} (x+iy)^{n+1}$

  with the **multipole coefficients** being written as

$$a_n = \frac{1}{B_0\, n!} \frac{\partial^n B_x}{\partial x^n}\Big|_{x=y=0} \quad \text{and} \quad b_n = \frac{1}{B_0\, n!} \frac{\partial^n B_y}{\partial x^n}\Big|_{x=y=0}$$

- The **general non-linear Hamiltonian** can be written as

$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}_0(x, y, p_x, p_y, s) + \sum_{k_x, k_y} h_{k_x, k_y}(s) x^{k_x} y^{k_y}$$

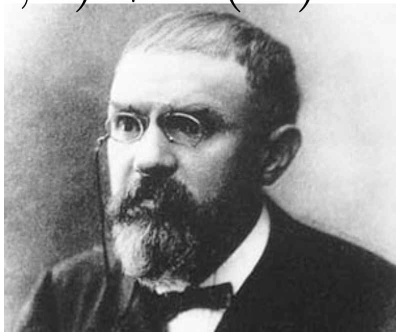
with the **periodic functions** $h_{k_x,k_y}(s) = h_{k_x,k_y}(s + C)$

Contents of the 2nd lecture

- Lagrangian Formalism
  - Lagrange mechanics
  - From the Lagrangian to the Hamiltonian
- Hamiltonian Formalism
  - Hamilton's equations
  - Properties of the Hamiltonian flow
  - Poisson brackets and their properties
- Canonical transformations
  - Preservation of phase volume and examples
- Single particle relativistic Hamiltonian
  - Canonical transformations and approximations
  - Linear magnetic fields and integrable Hamiltonian
  - Action-angle variables
  - General non-linear Hamiltonian
- Canonical perturbation theory
  - Form of the generating function
  - Small denominators and KAM theory
  - Perturbation treatment for a sextupole
  - Second order sextupole tune-shift
  - Resonance driving terms, tune-shift and tune-spread
- Secular perturbation theory
  - Third order resonance
  - Fixed points for general multi-pole
  - 4th order resonance
  - Onset of chaos
  - Resonance overlap
- Summary

Canonical perturbation theory

- Consider a general Hamiltonian with n degrees of freedom
$$H(\boldsymbol{J},\boldsymbol{\varphi},\theta) = H_0(\mathbf{J}) + \epsilon H_1(\boldsymbol{J},\boldsymbol{\varphi},\theta) + \mathcal{O}(\epsilon^2)$$
- The non-integrable part $H_1(\boldsymbol{J},\boldsymbol{\varphi},\theta)$ is 2π -periodic on the angles φ and the “time” θ
- Provided that ϵ is sufficiently small, **tori** should still **exist** but they are **distorted**
- We seek a **canonical transformation** that could **“straighten up"** the **tori**, i.e. it could transform the non-integrable part of the Hamiltonian (at first order in ϵ) to a **function only** of some **new actions** $\bar{H}(\bar{\boldsymbol{J}})$ plus higher orders in ϵ
- This can be performed by a **mixed variable** close to identity **generating function** $S(\bar{\boldsymbol{J}},\boldsymbol{\varphi},\theta) = \bar{\boldsymbol{J}}\cdot\boldsymbol{\varphi} + \epsilon S_1(\bar{\boldsymbol{J}},\boldsymbol{\varphi},\theta) + \mathcal{O}(\epsilon^2)$ for transforming old variables to new ones $(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$
- In principle, this procedure can be carried to arbitrary powers of the perturbation



Canonical perturbation theory

- By the canonical transformation equations, the **old action** and **new angle** can be also represented by a power series in ϵ

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \boldsymbol{\varphi}} + \mathcal{O}(\epsilon^2) \qquad \boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

or

$$\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2) \qquad \boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} - \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

- The previous equations expressing the old as a function of the new variables assume that there is possibility to **invert** the equation on the left, so that $S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)$ becomes a function of the new variables

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- The previous equations expressing the old as a function of the new variables assume that there is possibility to **invert** the equation on the left, so that $S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)$ becomes a function of the new variables
- The **new Hamiltonian** is then

$$\bar{H}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta) = H(\boldsymbol{J}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}), \boldsymbol{\varphi}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}), \theta) + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \mathcal{O}(\epsilon^2)$$

- The second term is appearing because of the "time dependence through θ

Canonical perturbation theory

- **Expand** term by term the Hamiltonian $H(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta)$ to leading order in ϵ

$$H_0(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})) = H_0(\bar{\boldsymbol{J}}) + \epsilon \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\epsilon H_1(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta) = \epsilon H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) + \mathcal{O}(\epsilon^2)$$

- The **new Hamiltonian** can also be **expanded** in orders of ϵ

$$\bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 + \dots$$

Canonical perturbation theory

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- The **new Hamiltonian** can also be **expanded** in orders of ϵ

$$\bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 + \dots$$

- **Equating** the terms of **equal order**, we obtain
  - Zero order $\bar{H}_0 = H_0(\bar{\boldsymbol{J}})$
  - First order $\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \bar{\boldsymbol{\varphi}}} + H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$

  where the frequency vector is $\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$

Canonical perturbation theory

- From the **1st order Hamiltonian**, the angles have to be eliminated. For this purpose, it can be **split** in two parts:
  - **Average** part: $\langle H_1 \rangle_{\bar{\varphi}} = \left(\frac{1}{2\pi}\right)^n \oint H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) d\bar{\boldsymbol{\varphi}}$
  - **Oscillating** part: $\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\varphi}}$
- The 1st order perturbation part of the Hamiltonian then becomes

$$\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} + \{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\}$$

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- Thus, the **generating function** should be **chosen** such that the **angle** dependence is **eliminated**, for which

$$\bar{H}_1(\bar{\boldsymbol{J}}) = \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} \quad \text{and} \quad \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} = -\{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\}$$

- The **new Hamiltonian** is a **function** only of the **new actions**

$$\bar{H}(\bar{\boldsymbol{J}}) = H_0(\bar{\boldsymbol{J}}) + \epsilon \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} + \mathcal{O}(\epsilon^2)$$

with the new frequency vector

$$\bar{\boldsymbol{\omega}}(\bar{\boldsymbol{J}}) = \frac{\partial \bar{H}(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} = \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + \epsilon \frac{\partial \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}}}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

Form of the generating function

- The **question** that remains to be answered is whether a generating function can be found that eliminates the angle dependence
- The **oscillating part** of the perturbation and the **generating function** can be expanded in Fourier series

$$\{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\} = \sum_{\boldsymbol{k},p} H_{1\mathbf{k}}(\bar{\boldsymbol{J}}) e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} \quad \text{and}$$

$$S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta) = \sum_{\boldsymbol{k},p} S_{1\mathbf{k}}(\bar{\boldsymbol{J}}) e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)}$$

- Following the **relationship** for the **angle elimination**, the Fourier coefficients of the generating function should satisfy $\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}} = k_1\bar{\varphi}_1 + \cdots + k_n\bar{\varphi}_n$ with

$$S_{1k}(\bar{\boldsymbol{J}}) = i\frac{H_{1k}(\bar{\boldsymbol{J}})}{\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p} \qquad \text{with} \qquad \boldsymbol{k}, p \neq \mathbf{0}$$

Form of the generating function

- Finally the generating function can be written as

$$S(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) = \bar{\boldsymbol{J}}\cdot\bar{\boldsymbol{\varphi}} + \epsilon i \sum_{\mathbf{k}\neq\mathbf{0}} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p} e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} + \mathcal{O}(\epsilon^2)$$

- The denominator is composed by the frequency vector

$$\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} \quad \text{and the integers} \quad \boldsymbol{k}, p \neq \mathbf{0}$$

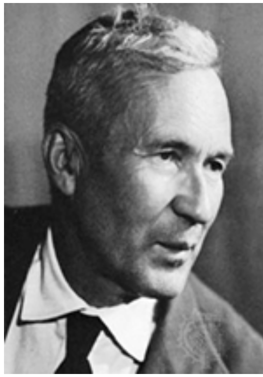
- If the denominator vanishes, i.e. for the **resonance condition** $\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p=0$, the Fourier series coefficients (**driving terms**) become **infinite**
- It actually implies that even at **first order** in the perturbation parameter and in the vicinity of a resonance, it is **impossible** to construct a **generating function** for seeking some **approximate integrals of motion**

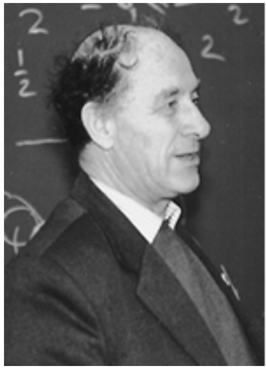
Problem of small denominators

- In principle, the **technique works** for **arbitrary order**, but the **disentangling** of **variables** becomes difficult even to 2nd order!!!
- The solution was given in the late 60s by introducing the **Lie transforms** (e.g. see Deprit 1969), which are **algorithmic** for **constructing generating functions** and were adapted to beam dynamics by Dragt and Finn (1976)
- On the other hand, the problem of **small denominators** due to **resonances** is not just a mathematical one. The inability to construct solutions close to a **resonance** has to do with the **un-predictable nature of motion** and the **onset** of **chaos**
- **KAM theory** developed the mathematical framework into which local solutions could be constructed provided some general conditions on the size of the perturbation and the distance of the system from resonances are satisfied

KAM theory

- Original idea of **Kolmogorov** (1954) (super-convergent series expansion) later proved by **Arnold** (1963) and **Moser** (1962)
- If a Hamiltonian system is subjected to **weak nonlinear perturbation**, **some invariant tori** are **deformed and survive**
- **Trajectories** starting on one of these tori **remain** on it thereafter, executing **quasi-periodic motion** with a **fixed frequency vector** depending only on the torus.
- The **family** of tori is **parameterized** over a **Cantor set** of frequency vectors, while in the **gaps** of the Cantor set **chaotic behavior** can occur
- The **KAM theorem** specifies **quantitatively** the **size of** the **perturbation** for this to be true.
- The **KAM tori** that survive are those that have **"sufficiently irrational"** frequencies
- The **conditions** of the KAM theorem become **increasingly difficult** to satisfy for **systems with more degrees of freedom**. As the number of dimensions of the system increases, the volume occupied by the tori decreases
- A complement of KAM theory for the stability of dynamical systems were given by **Nekhoroshev** (1971) who proved that if the **density** of **tori** is **large**, all **solutions** will **stay close** to the **tori** for **exponentially long times** showing **practical stability** of motion





Perturbation treatment for a sextupole

- Consider the simple case of a **periodic sextupole perturbation** and restrict the study only to one plane. The **Hamiltonian** is written as,

$$H(x, p_x, s) = \frac{p_x^2 + K(s)x^2}{2} + \frac{K_s(s)x^3}{3}$$

where $K(s)$ and $K_s(s)$ are periodic functions of time.

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where $K(s)$ and $K_s(s)$ are periodic functions of time.

- We proceed to the **transformation** in **action angle variables** to write the Hamiltonian in the form

$$H = H_0(J) + H_1(\phi, J) = \frac{J}{\beta(s)} + \frac{2\sqrt{2}K_s(s)}{3}\left(J\beta(s)\right)^{3/2}\cos^3\phi = \frac{J}{\beta(s)} + \frac{K_s(s)}{3\sqrt{2}}\left(J\beta(s)\right)^{3/2}\left(\cos 3\phi + 3\cos\phi\right)$$

Perturbation treatment for a sextupole

- Consider the simple case of a **periodic sextupole perturbation** and restrict the study only to one plane. The **Hamiltonian** is written as,

$$H(x, p_x, s) = \frac{p_x^2 + K(s)x^2}{2} + \frac{K_s(s)x^3}{3}$$

where $K(s)$ and $K_s(s)$ are periodic functions of time.

- We proceed to the **transformation** in **action angle variables** to write the Hamiltonian in the form

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- It can be shown that the **average** of the **sextupole perturbation**, over the angles **vanishes**

$$\langle\frac{\partial H_1(\phi, J)}{\partial J}\rangle_\phi = \frac{K_s(s)\beta(s)}{4\sqrt{2}\pi}\left(J\beta(s)\right)^{1/2}\int_0^{2\pi}(\cos 3\phi + 3\cos\phi)d\phi = 0$$

- Sextupoles do not provide any **tune-shift** at **first order**

Perturbation treatment for a sextupole

- The close to identity generating function is written as

$$S(\bar{J},\bar{\phi},\theta) = \bar{J}\cdot\bar{\phi} + S_1(\bar{J},\bar{\phi},\theta) + \ldots$$

- Following the perturbation steps, the generating function has to be chosen such that the following relationship is satisfied $\frac{\partial S_1(\bar{J},\bar{\phi},\theta)}{\partial\theta} + \nu(\bar{J})\cdot\frac{\partial S_1(\bar{J},\bar{\phi},\theta)}{\partial\bar{\phi}} = -\{H_1(\bar{J},\bar{\phi})\}$ with

$$\{H_1\} = H_1 - \langle H_1\rangle_{\bar{\phi}} = H_1 = \frac{K_s(s)}{3\sqrt{2}}\left(\bar{J}\beta(s)\right)^{3/2}(\cos 3\phi + 3\cos\phi)$$

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$$\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\phi}} = H_1 = \frac{K_s(s)}{3\sqrt{2}} \left(\bar{J}\beta(s)\right)^{3/2} (\cos 3\phi + 3\cos\phi)$$

- Following the canonical perturbation procedure the **generating function** is

$$S(\bar{J}, \bar{\phi}) = \bar{J} \cdot \bar{\phi} + i \sum_{k,p \neq 0} \frac{H_{1k}(\bar{J})}{k \cdot \nu(\bar{J}) + p} e^{i(k \cdot \bar{\phi} + p\theta)} + \dots$$

- The **only non-zero coefficients** are for $k = 1, 3$ and

$$S(\bar{J}, \bar{\phi}) = \bar{J} \cdot \bar{\phi} + i \frac{K_s(s)}{6\sqrt{2}} \left(\bar{J}\beta(s)\right)^{3/2} \sum_{p=-\infty}^{\infty} \left(\frac{e^{i(3\bar{\phi} + p\theta)}}{3\nu + p} + \frac{3e^{i(\bar{\phi} + p\theta)}}{\nu + p} \right)$$

Perturbation treatment for more sextupoles

- Expand both the perturbation and generating function in Fourier series of the form

$$S_1(\bar{J},\bar{\phi},\theta) = \sum_k S_{1k}(\bar{J},\theta)e^{ik\bar{\phi}} \text{ and } \{H_1(\bar{J},\bar{\phi},\theta)\} = \sum_k H_{1k}(\bar{J},\theta)e^{ik\bar{\phi}}$$

- The equation relating the amplitudes is

$$i\ k\ \nu\ S_{1k} + \frac{\partial S_{1k}}{\partial \theta} = -H_{1k}$$

which can be solved yielding

$$S_{1k} = \frac{i}{2\sin(\pi k\nu)} \int_\theta^{\theta+2\pi} H_{1k} e^{ik\nu(\theta'-\theta-\pi)} d\theta'$$

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- Following the canonical perturbation procedure the generating function is

$$S_1 = \sum_k \frac{i}{2\sin(\pi k\nu)}\int_\theta^{\theta+2\pi} H_{1k}e^{ik[\phi+\nu(\theta'-\theta-\pi)]}d\theta'$$

- For the sextupole, and letting $\psi(s) = \int_0^s \frac{ds'}{\beta(s')}$ we have

$$S_1 = -\frac{\bar{J}^{3/2}}{2\sqrt{2}}\int_s^{s+C} K_s(s')\beta(s')^{3/2}\left[\frac{\sin(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)} + \frac{\sin 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{3\sin(3\pi\nu)}\right]ds'$$

Perturbation treatment for a sextupole

- We derived (with a lot of effort) the common result that sextupoles **at first order** excite **integer** and **third integer** resonances
- Again this is not generally true! It is known that sextupoles can drive **any resonance** (either if they are large enough, or if the particle is far away from the closed orbit)
- This can be shown again by pursuing the perturbation approach to **second order** (as for the tune-shift)
- A useful application is to use the generating function for computing the correction to the original invariant, as the new one should be an integral of motion (at first order)

$$J \approx \bar{J} + \frac{\partial S_1(\bar{J}, \varphi, \theta)}{\partial \varphi}$$

Second order sextupole tune-shift

- It can be shown that at second order in perturbation theory the Hamiltonian depending only on the actions can be written

$$\bar{H}_2(\bar{J}) = \langle \frac{1}{2} \frac{\partial^2 H_0}{\partial \bar{J}^2} \left(\frac{\partial S_1}{\partial \phi} \right)^2 + \frac{\partial H_1}{\partial \bar{J}} \frac{\partial S_1}{\partial \phi} \rangle_\phi$$

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- This can be simplified to $\bar{H}_2(\bar{J}) = \langle \frac{\partial H_1}{\partial \bar{J}}\frac{\partial S_1}{\partial \phi}\rangle_\phi$
- The two terms are $\frac{\partial H_1}{\partial \bar{J}} = \frac{K_s(s)}{2\sqrt{2}}\bar{J}^{1/2}\beta(s)^{3/2}(\cos 3\phi + 3\cos\phi)$

$$\frac{\partial S_1}{\partial \phi} = -\frac{\bar{J}^{3/2}}{2\sqrt{2}}\int_s^{s+C} K_s(s')\beta(s')^{3/2}\left[\frac{\cos(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(3\pi\nu)}\right] ds'$$

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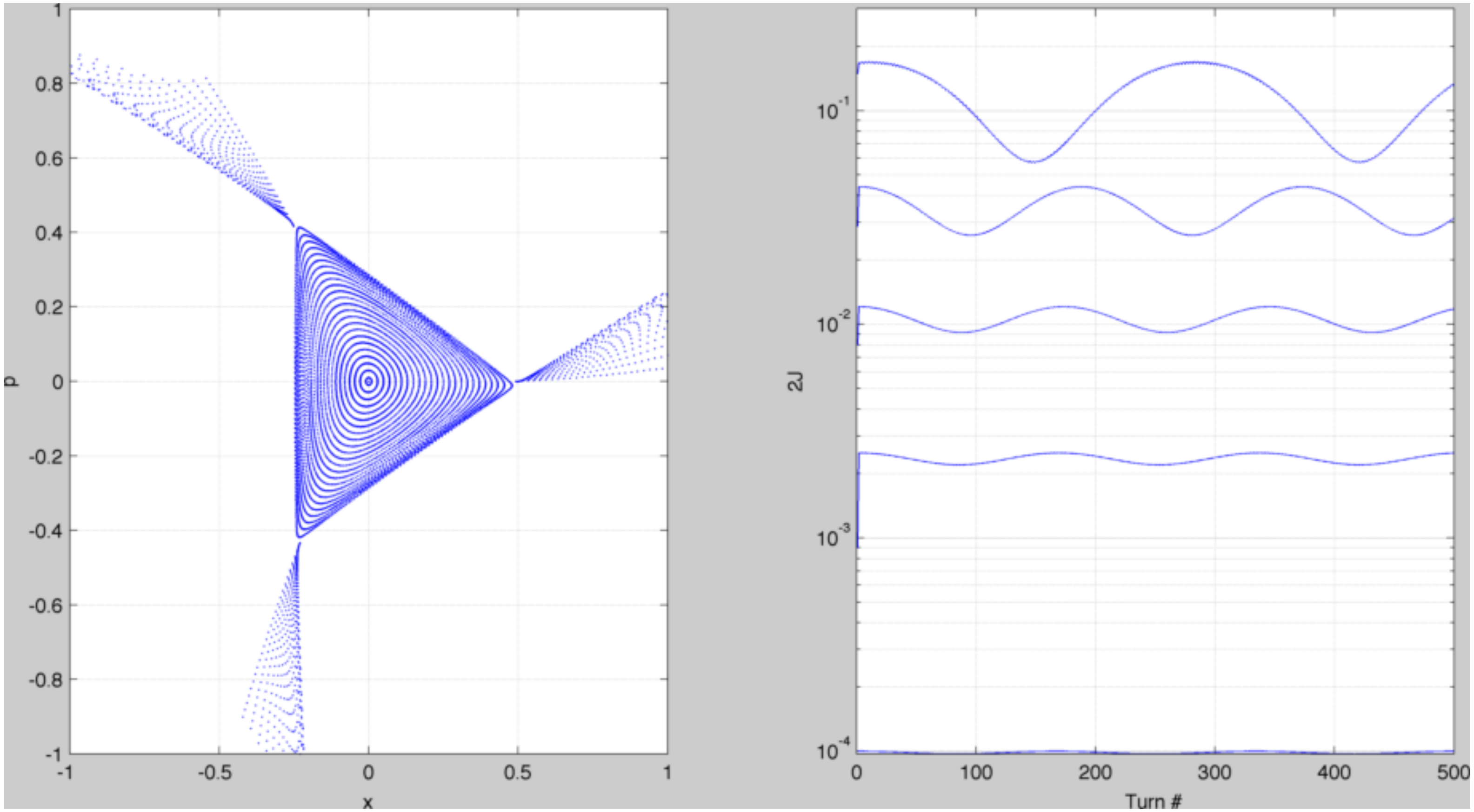
- The 2nd order Hamiltonian is given by the angle-averaged product of the last two terms.
- It is quadratic in the sextupole strength and the new action. The 2nd order tune-shift is the derivative in the action

$$\nu(\bar{J}) = \langle \frac{\partial H_2}{\partial \bar{J}}\rangle_{\phi,s} = -\frac{\bar{J}}{16\pi}\int_0^C ds K_s(s)\beta(s)^{3/2}\int_s^{s+C} K_s(s')\beta(s')^{3/2}$$
$$\times\left[\frac{\cos(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(3\pi\nu)}\right] ds'$$

Phase space for sextupole perturbation

- For small perturbations, the new action variable is almost an invariant but for larger ones phase space gets deformed
- Close to the integer or third integer resonance, canonical perturbation theory cannot be applied
- The solution is provided by **secular perturbation theory**



General accelerator Hamiltonian

- The general accelerator Hamiltonian is written as

$$\mathcal{H}(x,y,p_x,p_y,s) = \mathcal{H}_0(x,y,p_x,p_y,s) + \sum_{k_x,k_y} h_{k_x,k_y}(s) x^{k_x} y^{k_y}$$

- The transverse coordinated can be expressed in action-angle variables as

$$u(s) = \sqrt{\frac{J_u \beta_u(s)}{2}} \left(e^{i(\phi_u(s)+\theta_u(s))} + e^{-i(\phi_u(s)+\theta_u(s))} \right)$$

- The Hamiltonian in action-angle variables is

$$\mathcal{H}'(J_x,J_y,\phi_x,\phi_y) = H_0(J_x,J_y) + H_1(J_x,J_y,\phi_x,\phi_y)$$

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$$\mathcal{H}'(J_x,J_y,\phi_x,\phi_y) = H_0(J_x,J_y) + H_1(J_x,J_y,\phi_x,\phi_y)$$

  - The integrable part $H_0(J_x,J_y) = \frac{1}{R}(\nu_x J_x + \nu_y J_y)$
  - The perturbation

$$H_1(J_x,J_y,\phi_x,\phi_y;s) = \sum_{k_x,k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j^{k_x} \sum_l^{k_y} g_{j,k,l,m}(s) e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

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- The coefficients $g_{j,k,l,m}(s) = \frac{h_{k_x,k_y}(s)}{2^{\frac{j+k+l+m}{2}}} \binom{k_x}{j} \binom{k_y}{l} \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\theta_x(s) + (l-m)\theta_y(s)]}$ depend on the optics, with the indexes $k_x = j+k$, $k_y = l+m$

Resonance driving terms

- As the coefficients $h_{k_x,k_y}(s)$ are periodic, the perturbation can be expanded in Fourier series

$$H_1(J_x, J_y, \phi_x, \phi_y; \theta) = \sum_{k_x,k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j^{k_x} \sum_l^{k_y} \sum_{p=-\infty}^{\infty} g_{j,k,l,m;p} e^{i[(j-k)\phi_x + (l-m)\phi_y - p\theta]}$$

with the **resonance driving terms**

$$g_{j,k,l,m;p} = \binom{k_x}{j}\binom{k_y}{l} \frac{1}{2^{\frac{j+k+l+m}{2}}} \frac{1}{2\pi} \oint h_{k_x,k_y}(s) \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\phi_x(s) + (l-m)\phi_y(s) + p\theta]}$$

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- For $n_x = j - k$, $n_y = l - m$, resonance conditions appear for $n_x \nu_x + n_y \nu_y = p$
- Goal of accelerator design and correction systems is to minimize the resonance driving terms
  - Change magnet design so that $h_{k_x,k_y}(s)$ become smaller
  - Introduce magnetic elements capable of creating a cancelling effect
  - Sort magnets or non-linear elements in a way that phase terms are minimised

Tune-shift and tune-spread

- First order correction to the tunes is computed by the derivatives with respect to the action of the average part of perturbation. For a given term, $h_{k_x,k_y}(s)x^{k_x}y^{k_y}$ the leading order correction to the tunes are

$$\delta\nu_x = \frac{J_x^{k_x/2-1}J_y^{k_y/2}}{4\pi^2}\sum_j^{k_x}\sum_l^{k_y}\bar{g}_{j,k,l,m}\oint e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

$$\delta\nu_y = \frac{J_x^{k_x/2}J_y^{k_y/2-1}}{4\pi^2}\sum_j^{k_x}\sum_l^{k_y}\bar{g}_{j,k,l,m}\oint e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

where $\bar{g}_{j,k,l,m}$ is the average of $g_{j,k,l,m}(s)$ around the ring.

- In the accelerator jargon if $\delta\nu_{x,y}$ is independent of the action, it is referred to as **tune-shift**, whereas, if it depends on the action, it is called **tune-spread** (or amplitude detuning)

- At first order, $\delta\nu_{x,y} = 0$, for odd multi-poles $k_x = j + k$, $k_y = l + m$ (trigonometric functions give zero averages).

Contents of the 2nd lecture

- Lagrangian Formalism
  - Lagrange mechanics
  - From the Lagrangian to the Hamiltonian
- Hamiltonian Formalism
  - Hamilton's equations
  - Properties of the Hamiltonian flow
  - Poisson brackets and their properties
- Canonical transformations
  - Preservation of phase volume and examples
- Single particle relativistic Hamiltonian
  - Canonical transformations and approximations
  - Linear magnetic fields and integrable Hamiltonian
  - Action-angle variables
  - General non-linear Hamiltonian
- Canonical perturbation theory
  - Form of the generating function
  - Small denominators and KAM theory
  - Perturbation treatment for a sextupole
  - Second order sextupole tune-shift
  - Resonance driving terms, tune-shift and tune-spread
- **Secular perturbation theory**
  - Third order resonance
  - Fixed points for general multi-pole
  - 4th order resonance
  - Onset of chaos
  - Resonance overlap
- Summary

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Secular perturbation theory

CERN

- Consider a general two degrees of freedom Hamiltonian:

$$H(\mathbf{J},\boldsymbol{\varphi}) = H_0(\mathbf{J}) + \varepsilon H_1(\mathbf{J},\boldsymbol{\varphi})$$

with the perturbed part periodic in angles:

$$H_1(\mathbf{J},\boldsymbol{\varphi}) = \sum_{k_1,k_1} H_{k_1,k_2}(J_1,J_2)\exp[i(k_1\varphi_1 + k_2\varphi_2)]$$

- The resonance $n_1\omega_1 + n_2\omega_2 = 0$ prevents the convergence of the series

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- The resonance $n_1\omega_1 + n_2\omega_2 = 0$ prevents the convergence of the series
- A **canonical transformation** can be applied for eliminating one action: $(\mathbf{J},\boldsymbol{\varphi}) \longmapsto (\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}})$ using the generating function $F_r(\hat{\mathbf{J}},\boldsymbol{\varphi}) = (n_1\varphi_1 - n_2\varphi_2)\hat{J}_1 + \varphi_2\hat{J}_2$
- The relationships between new and old variables are

$$J_1 = n_1\hat{J}_1 \quad , \qquad J_2 = \hat{J}_2 - n_2\hat{J}_1$$

$$\hat{\varphi}_1 = n_1\varphi_1 - n_2\varphi_2 \quad , \qquad \hat{\varphi}_2 = \varphi_2$$

- This transformation put the system in a **rotating frame**, where the rate of change $\dot{\hat{\varphi}}_1 = n_1\dot{\varphi}_1 - n_2\dot{\varphi}_2$ measures the deviation from resonance

- The **transformed Hamiltonian** is $\hat{H}(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}}) = \hat{H}_0(\hat{\mathbf{J}}) + \varepsilon \hat{H}_1(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}})$ with the perturbation written as

$$\hat{H}_1(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}}) = \sum_{k_1,k_2} H_{k_1,k_2}(\hat{\mathbf{J}}) \exp\left\{\frac{i}{n_1}\left[k_1\hat{\varphi}_1 + (k_1 n_2 + k_2 n_1)\hat{\varphi}_1\right]\right\}$$

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- This transformation assumes that $\dot{\varphi}_2$ is the **slow frequency** and the Hamiltonian can be **averaged** over the corresponding angle to obtain

$\bar{H}(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}}) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_1(\hat{\mathbf{J}},\hat{\varphi}_1)$ with $\bar{H}_0(\hat{\mathbf{J}}) = \hat{H}_0(\hat{\mathbf{J}})$ and

$$\bar{H}_1(\hat{\mathbf{J}},\hat{\varphi}_1) = \langle \hat{H}_1(\hat{\mathbf{J}},\hat{\varphi}_1)\rangle_{\hat{\varphi}_2} = \sum_{p=-\infty}^{+\infty} H_{-pn_1,pn_2}(\hat{\mathbf{J}}) \exp(-ip\hat{\varphi}_1)$$

Secular perturbation theory

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- The averaging eliminated one angle and thus $\hat{J}_2 = J_2 + J_1\frac{n_2}{n_1}$ is an **invariant** of motion
- This means that the Hamiltonian has effectively only **one degree of freedom** and it is **integrable**

Secular perturbation theory

- Assuming that the **dominant Fourier harmonics** for $p = 0, \pm 1$ the Hamiltonian is written as
$$\bar{H}(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\phi}}_{\mathbf{1}}) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_{0,0}(\hat{\mathbf{J}}) + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$
- **Fixed points** $(\hat{J}_{10}, \hat{\phi}_{10})$ (i.e. periodic orbits) in phase space $(\hat{J}_1, \hat{\phi}_1)$ are defined by $\dfrac{\partial \bar{H}}{\partial \hat{J}_1} = 0 \,, \quad \dfrac{\partial \bar{H}}{\partial \hat{\phi}_1} = 0$

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- Introduce **moving reference** on **fixed point** and expand $\bar{H}(\hat{\mathbf{J}})$ around it $\Delta \hat{J}_1 = \hat{J}_1 - \hat{J}_{10}$
- Hamiltonian describing motion near a resonance:
  $$\bar{H}_r(\Delta \hat{J}_1, \hat{\phi}_1) = \left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\right|_{\hat{J}_1 = \hat{J}_{10}} \frac{(\Delta \hat{J}_1)^2}{2} + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

Secular perturbation theory

- Assuming that the **dominant Fourier harmonics** for $p = 0, \pm 1$ the Hamiltonian is written as

$$\bar{H}(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\phi}}_1) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_{0,0}(\hat{\mathbf{J}}) + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos\hat{\varphi}_1$$

- **Fixed points** $(\hat{J}_{10}, \hat{\phi}_{10})$ (i.e. periodic orbits) in phase space $(\hat{J}_1, \hat{\phi}_1)$ are defined by $\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0\,, \quad \frac{\partial \bar{H}}{\partial \hat{\phi}_1} = 0$
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- Motion near a typical resonance is like the one of the **pendulum**!!! The **libration frequency** and the resonance half width are

$$\hat{\omega}_1 = \left(2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\right|_{\hat{J}_1 = \hat{J}_{10}} \right)^{1/2} \qquad \Delta\hat{J}_{1\,max} = 2\left(\frac{2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}})}{\left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\right|_{\hat{J}_1 = \hat{J}_{10}}} \right)^{1/2}$$

Secular perturbation theory for the 3rd order resonance

- We first introduce the **distance** to the resonance

$$\nu = \frac{p}{3} + \delta \;, \quad \delta << 1$$

- It is convenient then to **eliminate** the "**time**" dependence by passing on a "1-turn" frame, using the generating function

$$F_2(\phi, J_1, s) = \phi J_1 + J_1 \left(\frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta(s')} \right) = (\phi + \chi(s)) J_1$$

with the new angle $\psi_1 = \phi - \chi(s)$ providing the Hamiltonian

$$H_1 = \frac{\nu}{R} J_1 + \frac{2\sqrt{2}}{3} K_s(s) (J_1 \beta)^{3/2} \cos^3(\psi_1 + \chi(s))$$

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- The perturbation can be expanded in a Fourier series, where as before, only the resonant term is kept or,

$$\hat{H}_1 = \nu J_1 + J_1^{3/2} A_{3p} \cos(3\psi_1 - p\theta)$$

in the rotating frame on top of the resonance

$$\hat{H}_2 = \delta J_2 + J_2^{3/2} A_{3p} \cos(3\psi_2)$$

Fixed points for 3rd order resonance

- By setting the Hamilton's equations equal to zero, three fixed points can be found at $\psi_{20} = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \quad J_{20} = \left(\frac{2\delta}{3A_{3p}}\right)^2$
- For $\frac{\delta}{A_{3p}} > 0$ all three points are unstable

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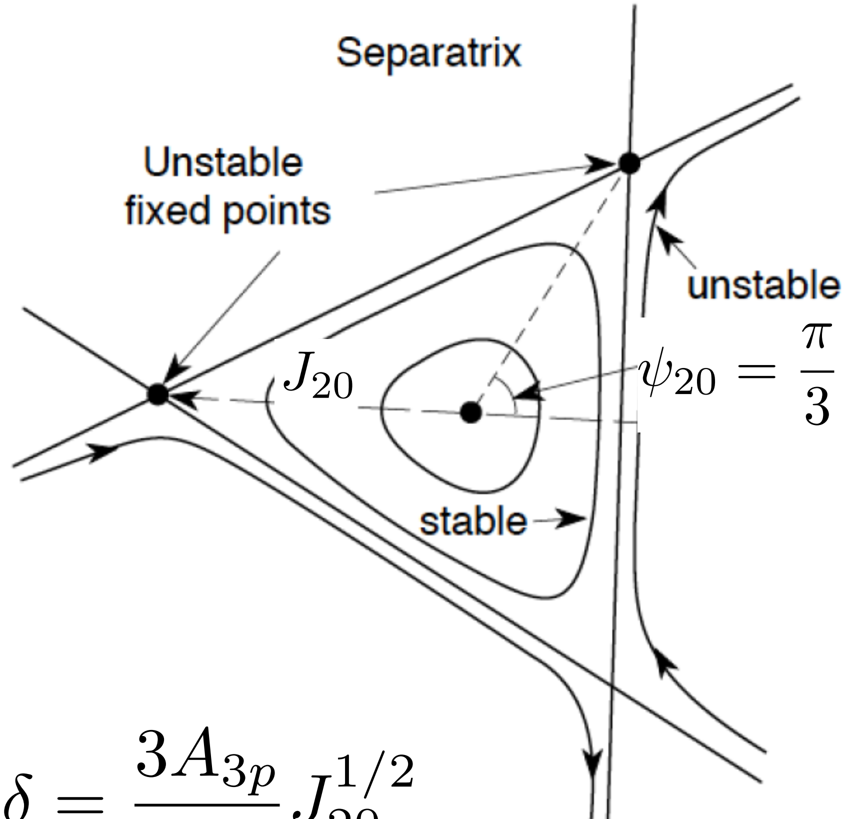
- For $\frac{\delta}{A_{3p}} > 0$ all three points are unstable
- Close to the elliptic one at $\psi_{20} = 0$ the motion in phase space is described by circles that they get more and more distorted to end up in the "triangular" **separatrix** uniting the unstable fixed points
- The tune separation from the resonance (**stop-band width**) is $\delta = \frac{3A_{3p}}{2} J_{20}^{1/2}$



Single resonance for accelerator Hamiltonian

- The single resonance accelerator Hamiltonian (Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

$$H(J_x, J_y, \phi_x, \phi_y, s) = \frac{1}{R}(\nu_x J_x + \nu_y J_y) + g_{n_x,n_y} \frac{2}{R} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

with $g_{n_x,n_y} e^{i\phi_0} = g_{j,k,l,m;p}$

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- From the **generating function** $F_r(\phi_x, \phi_y, \hat{J}_x, \hat{J}_y, s) = (n_x\phi_x + n_y\phi_y - p\theta)\hat{J}_x + \phi_y\hat{J}_y$ the relationships between old and new variables are

$$\hat{\phi}_x = (n_x\phi_x + n_y\phi_y - p\theta) \ , \quad J_x = n_x\hat{J}_x$$
$$\hat{\phi}_y = \phi_y \ , \qquad J_y = n_y\hat{J}_x + \hat{J}_y$$

- The following Hamiltonian is obtained

$$\hat{H}(\hat{J}_x, \hat{J}_y, \hat{\phi}_x) = \frac{(n_x\nu_x + n_y\nu_y - p)\hat{J}_x + \hat{J}_y}{R} + g_{n_x,n_y}\frac{2}{R}(n_x\hat{J}_x)^{\frac{k_x}{2}}(n_y\hat{J}_x + \hat{J}_y)^{\frac{k_y}{2}}\cos(\hat{\phi}_x + \phi_0)$$

Resonance widths

- There are two integrals of motion
  - The Hamiltonian, as it is **independent** on "time"
  - The **new action** $\hat{J}_y$ as the Hamiltonian is independent on $\hat{\phi}_y$
- The **two invariants** in the **old variables** are written as:

$$c_1 = \frac{J_x}{n_x} - \frac{J_y}{n_y}$$

$$c_2 = (\nu_x - \frac{p}{n_x + n_y})J_x + (\nu_y - \frac{p}{n_x + n_y})J_y + 2g_{n_x,n_y} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

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- Two cases can be distinguished
  - n_x , n_y have **opposite** sign, i.e. **difference** resonance, the motion is the one of an ellipse, so bounded
  - n_x , n_y have the **same** sign, i.e. **sum** resonance, the motion is the one of an hyperbola, so **not** bounded
- These are **first order** perturbation theory considerations
- The **distance** from the resonance is obtained as

$$\Delta = \frac{g_{n_x,n_y}}{R} J_x^{\frac{k_x-2}{2}} J_y^{\frac{k_y-2}{2}} (k_x n_x J_x + k_y n_y J_y)$$

Fixed points for general multi-pole

- For **any polynomial perturbation** of the form x^k the "resonant" Hamiltonian is written as

$$\hat{H}_2 = \delta J_2 + \alpha(J_2) + J_2^{k/2} A_{kp} \cos(k\psi_2)$$

- Note now that **in contrast** to the sextupole there is a non-linear detuning term $\alpha(J_2)$

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- Note now that **in contrast** to the sextupole there is a non-linear detuning term $\alpha(J_2)$

- The conditions for the fixed points are

$$\sin(k\psi_2) = 0\ , \quad \delta + \frac{\partial\alpha(J_2)}{\partial J_2} + \frac{k}{2} J_2^{k/2-1} A_{kp} \cos(k\psi_2) = 0$$

- There are k **fixed points** for which $\cos(k\psi_{20}) = -1$ and the fixed points are **stable** (elliptic). They are surrounded by ellipses

- There are also k **fixed points** for which $\cos(k\psi_{20}) = 1$ and the fixed points are **unstable** (hyperbolic). The trajectories are hyperbolas

Fixed points for an octupole

- The resonant Hamiltonian close to the **4th order resonance** is written as

$$\hat{H}_2 = \delta J_2 + c J_2^2 + J_2^2 A_{kp} \cos(4\psi_2)$$

- The **fixed points** are found by taking the derivative over the two variables and setting them to zero, i.e.

$$\sin(4\psi_2) = 0 \ , \quad \delta + 2cJ_2 + 2J_2 A_{kp} \cos(4\psi_2) = 0$$

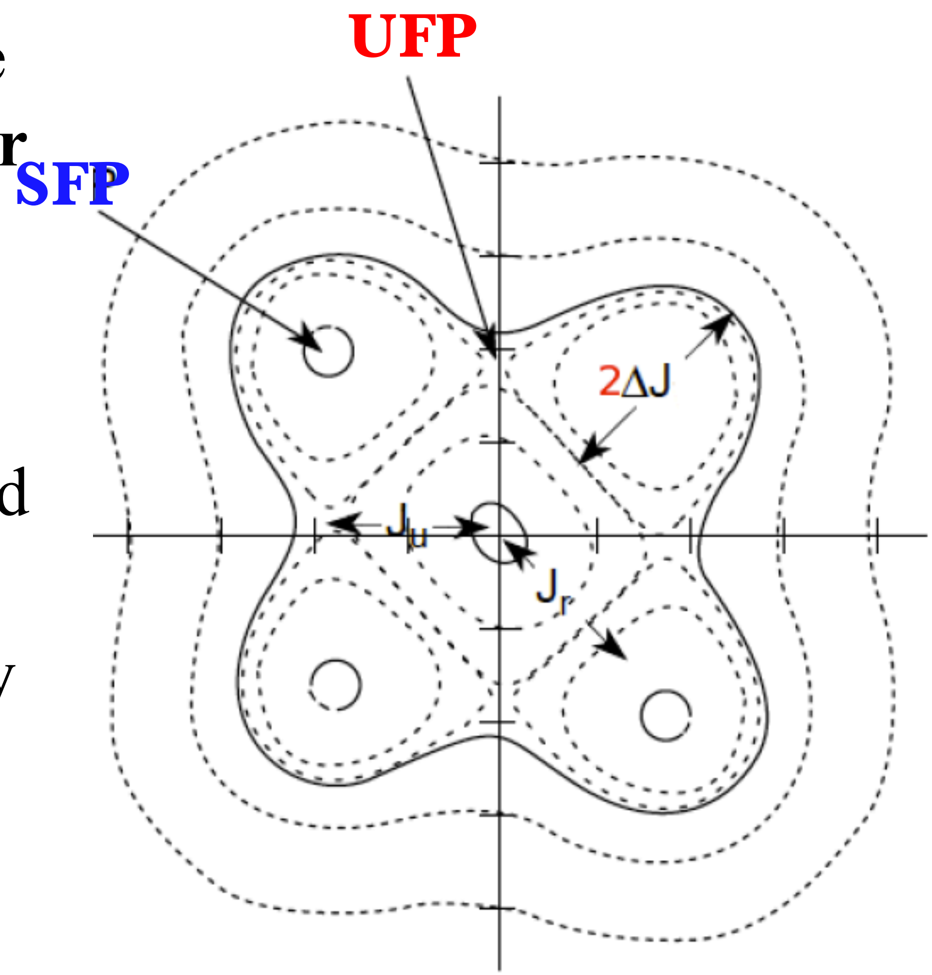
- The fixed points are at

$$\psi_{20} = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$$

- For **half** of them, there is a minimum in the potential as $\cos(4\psi_{20}) = -1$ and they are **elliptic** and **half** of them they are **hyperbolic** as $\cos(4\psi_{20}) = 1$

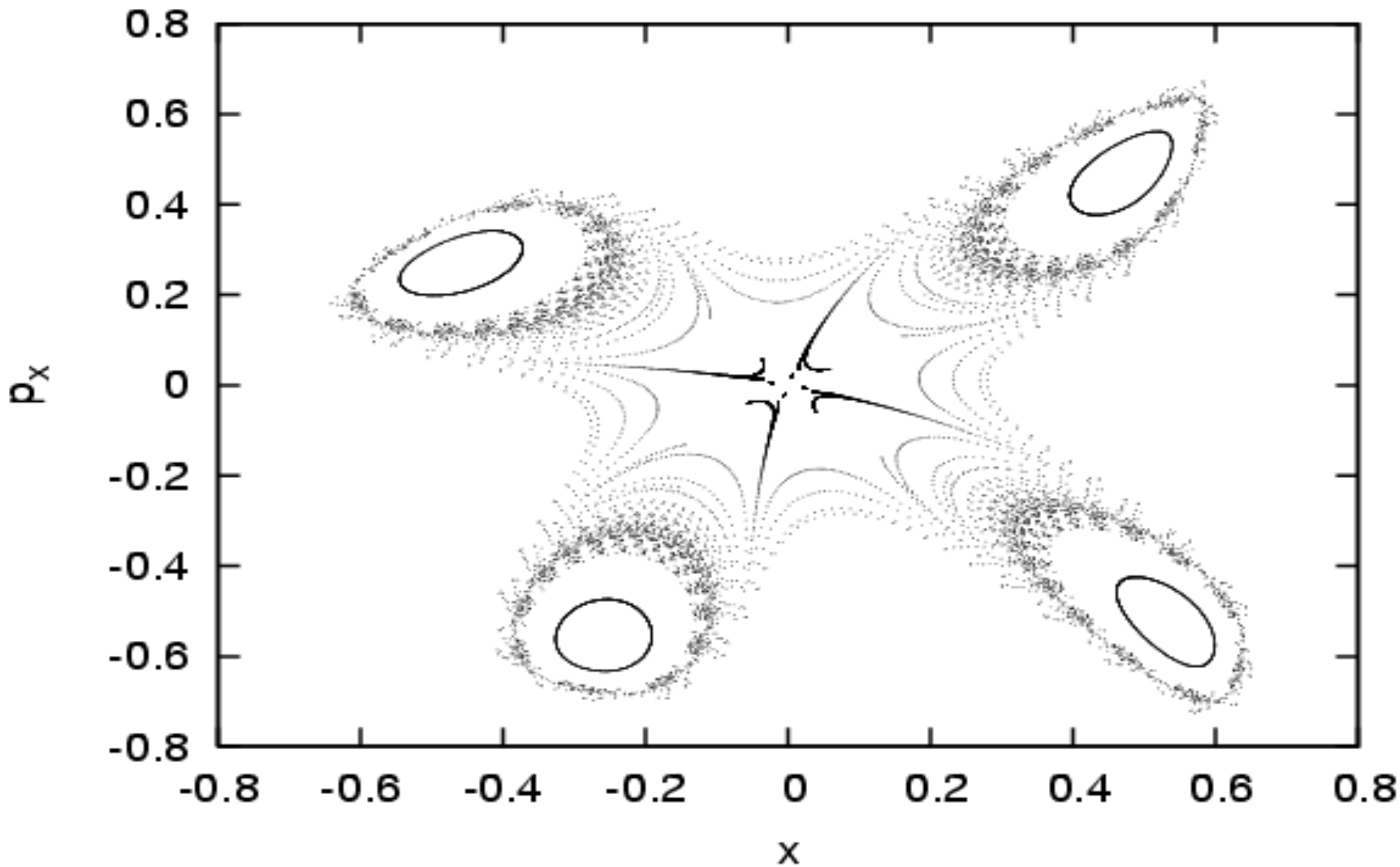
Topology of an octupole resonance

- **Regular motion** near the center, with curves getting more deformed towards a **rectangular shape**
- The **separatrix** passes through 4 unstable fixed points, but motion seems well contained
- **Four stable fixed points** exist and they are surrounded by stable motion (**islands of stability**)
- Question: Can the **central fixed point** become **hyperbolic** (answer in the appendix)



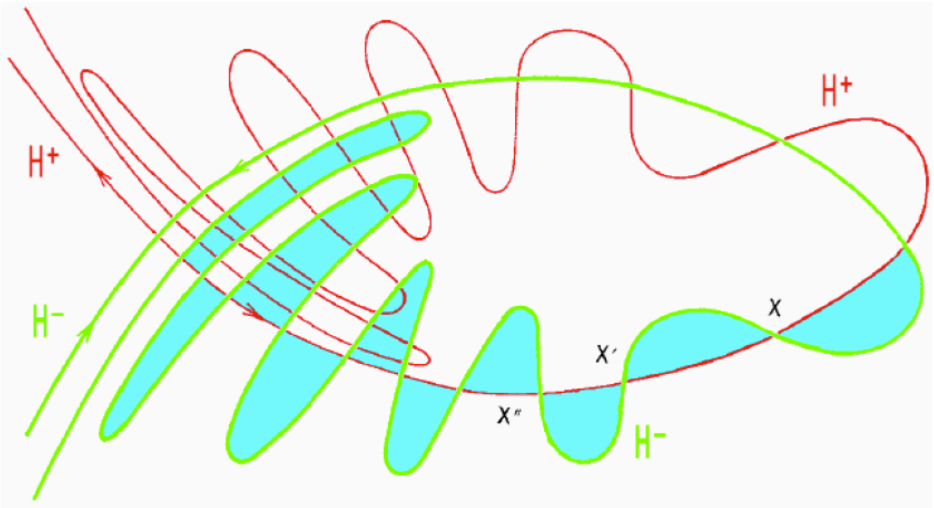
Octupole with hyperbolic central fixed point

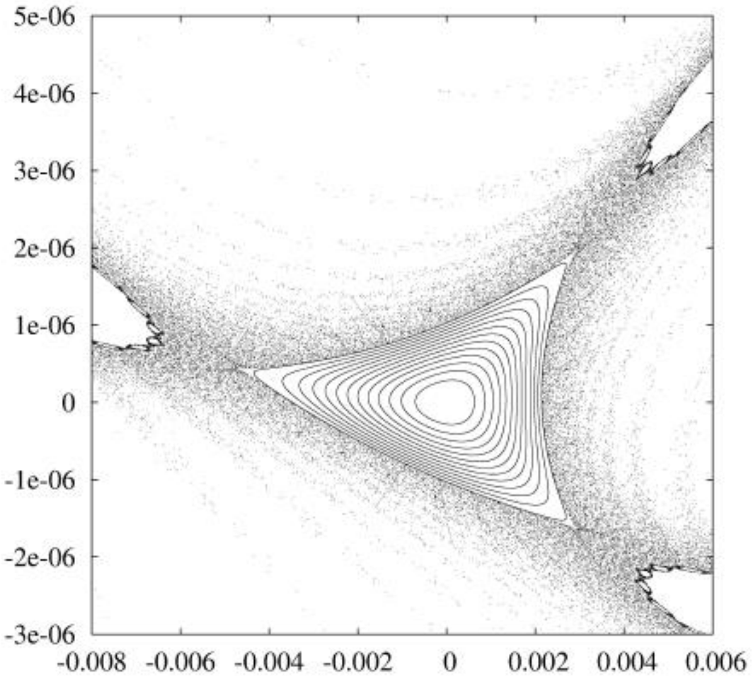
- Now, if $c = 0$ the solution for the action is $J_{20} = 0$
- So there is **no minima** in the potential, i.e. the central fixed point is **hyperbolic**



Path to chaos

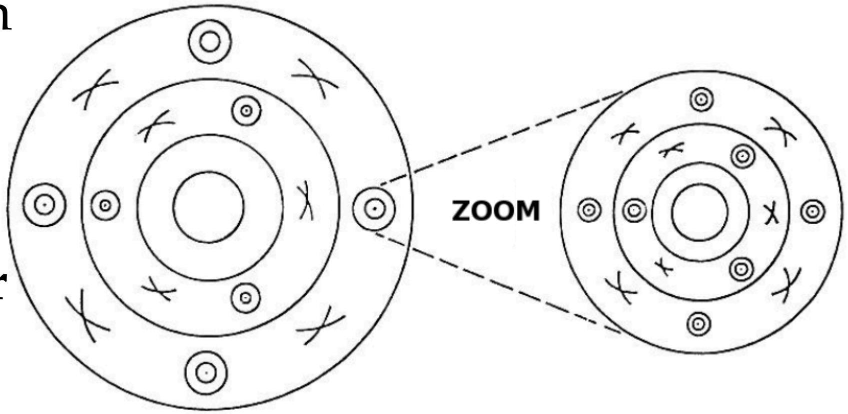
- When **perturbation** becomes **higher**, motion around the **separatrix** becomes **chaotic** (producing **tongues** or **splitting** of the separatrix)
- **Unstable** fixed points are indeed the **source of chaos** when a perturbation is added

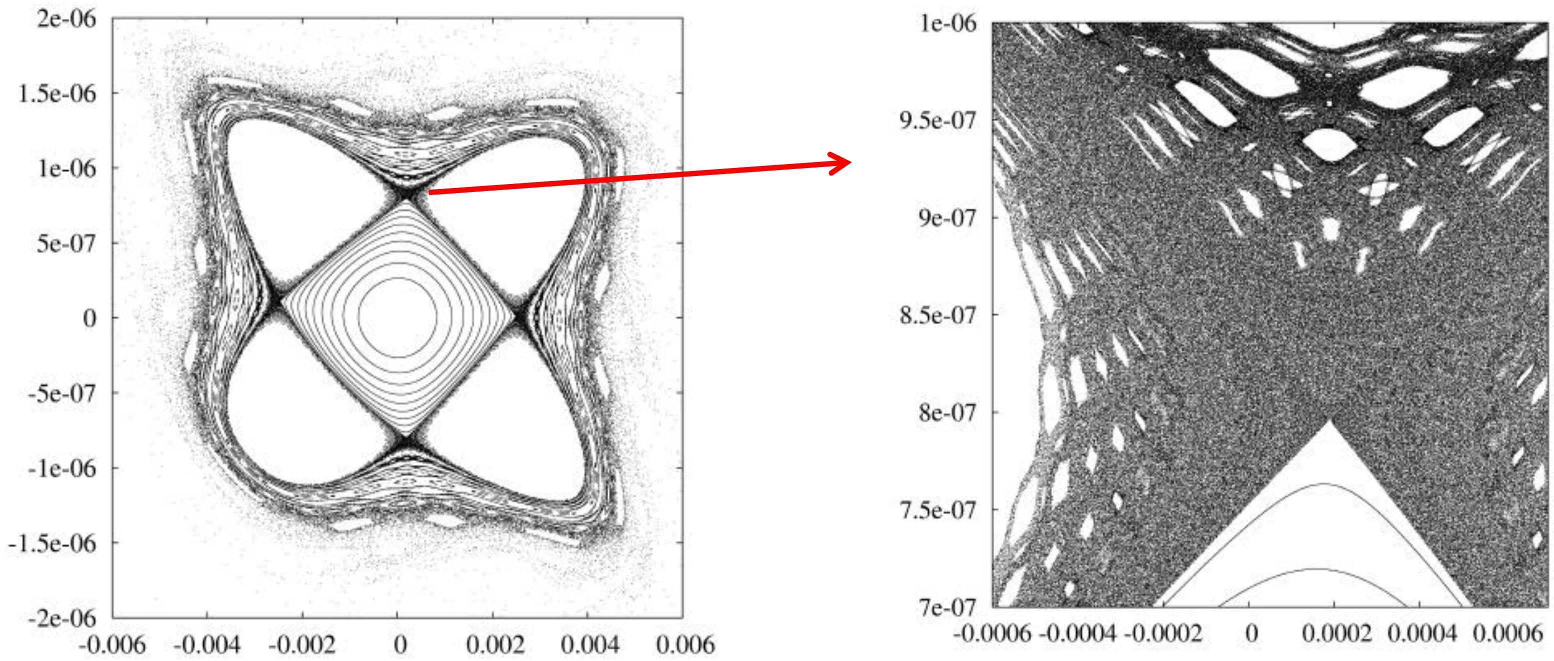




Chaotic motion

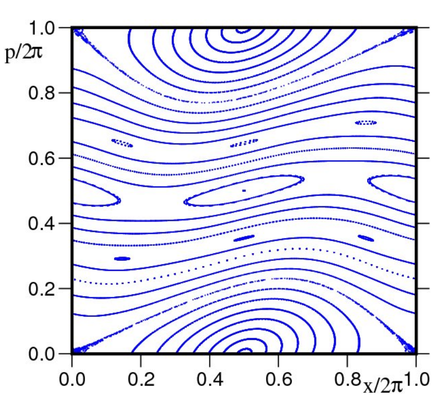
- **Poincare-Birkhoff** theorem states that under **perturbation** of a **resonance** only an **even number** of **fixed points** survives (**half stable** and the other **half unstable**)
- **Themselves** get **destroyed** when perturbation gets **higher**, etc. (**self-similar** fixed points)
- Resonance **islands grow** and **resonances** can **overlap** allowing diffusion of particles

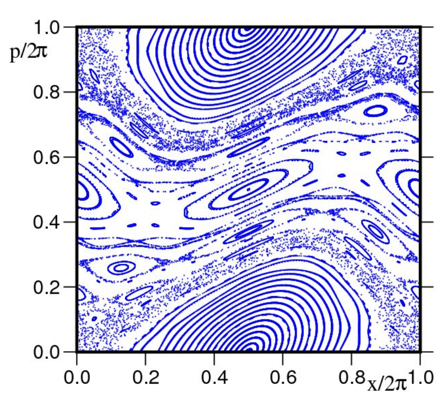


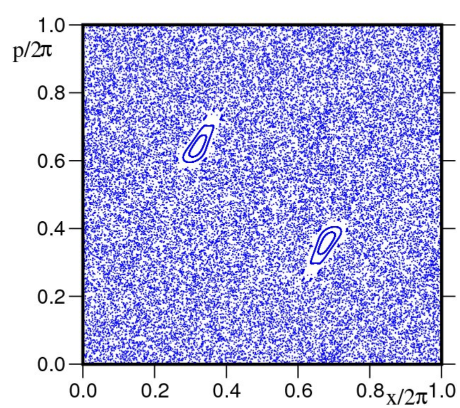


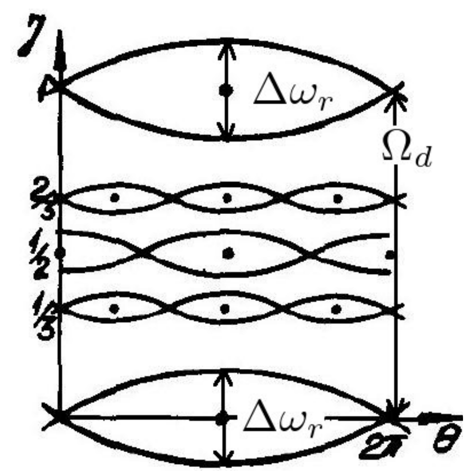
Resonance overlap criterion

- When perturbation grows, the resonance island width grows
- **Chirikov** (1960, 1979) proposed a **criterion** for the **overlap** of two **neighboring resonances** and the onset of orbit diffusion
- The **distance** between two resonances is $$\delta \hat{J}_{1\ n,n'} = \frac{2\left(\frac{1}{n_1+n_2} - \frac{1}{n'_1+n'_2}\right)}{\left|\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\Big|_{\hat{J}_1=\hat{J}_{10}}\right|}$$
- The **simple overlap criterion** is
  $$\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \delta \hat{J}_{n,n'}$$
- Considering the **width** of **chaotic layer** and **secondary islands**, the "two thirds" rule apply $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \frac{2}{3}\delta \hat{J}_{n,n'}$
- The main limitation is the **geometrical nature** of the criterion (**difficulty** to be **extended** for **> 2 degrees of freedom**)









Contents of the 2nd lecture

Summary

- **Hamiltonian formalism** provides the **natural framework** for studying non-linear dynamics
- The **relativistic Hamiltonian** is **non-linear by construction** and can only be transformed to an **integrable** one after a series of **approximations**
- **Action-angle** is the set of **variables** adequate for studying **integrable** systems, as motion evolves on **multi-dimensional tori**
- **Perturbation** of integrable Hamiltonian **distorts tori** and canonical perturbation theory looks for an appropriate canonical transformation to **"straighten" tori**
- **Small denominators** (resonances) appear **preventing** the **convergence** of generating functions
- **Secular perturbation** theory enables the analysis of the phase space close to a **resonance**, which is similar to the motion of a pendulum
- **Appearance** of **fixed points** (periodic orbits) determine **topology** of the phase space
- **Perturbation** of **unstable** (hyperbolic points) opens the **path** to **chaotic motion**
- **Resonance** can **overlap** enabling the rapid **diffusion** of **orbits**