Random Matrix Models for Finite Density QCD

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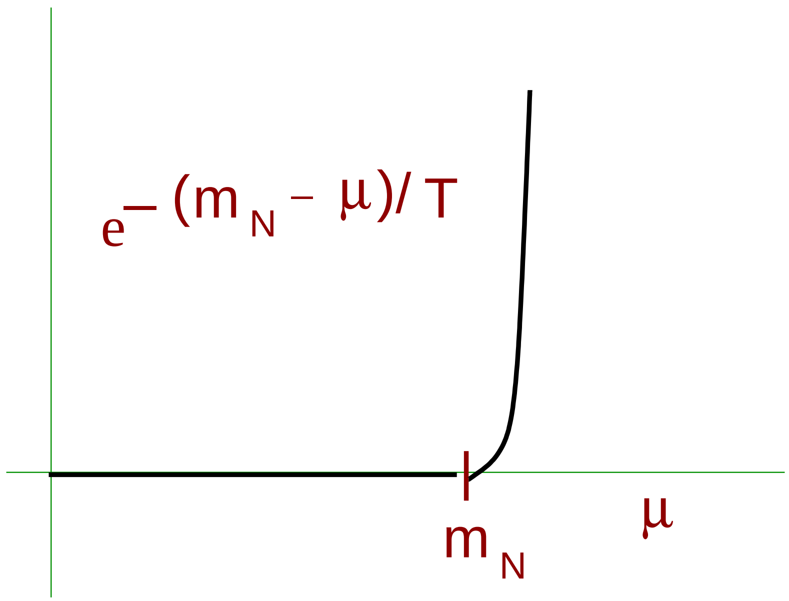
QCD Partition Function

$$Z_{\rm QCD}(m,\mu) = \sum_{{\rm states},k} e^{-\beta(E_k-\mu N_k)}.$$

N_k is the charge of the state k and E_k is the relativistic energy $E_k = \sum_i \sqrt{p_{k_i^2} + m_{k_i^2}}$.

At low temperatures the partition function does not depend on μ for $\mu < m_N$.
So observables do not depend on μ in this parameter domain.



Dirac Operator

The QCD partition function the trace of the time evolution operator at imaginary time $t = i\beta,$ which can be written as a path integral,

$$Z_{\mathrm{QCD}}(m,\mu) \quad = \quad \langle \prod_k^{N_f} \det(D + m + \mu\gamma_0) \rangle_{\mathrm{YM}}.$$

Dirac operator

quark mass

chemical potential

Phase Quenched QCD

The QCD partition function where the fermion determinant has been replaced by its absolute value (the phase quenched QCD partition function) can be written as

$$Z_{|\mathrm{QCD}|} = \langle |\det(D + m + \mu\gamma_0)|^2 \rangle = \langle \det(D + m + \mu\gamma_0)\det(D + m - \mu\gamma_0) \rangle.$$

Therefore, μ can be interpreted as an isospin chemical potential. Goldstone bosons made out of quarks and conjugate anti-quarks are charged with respect to the chemical potential.

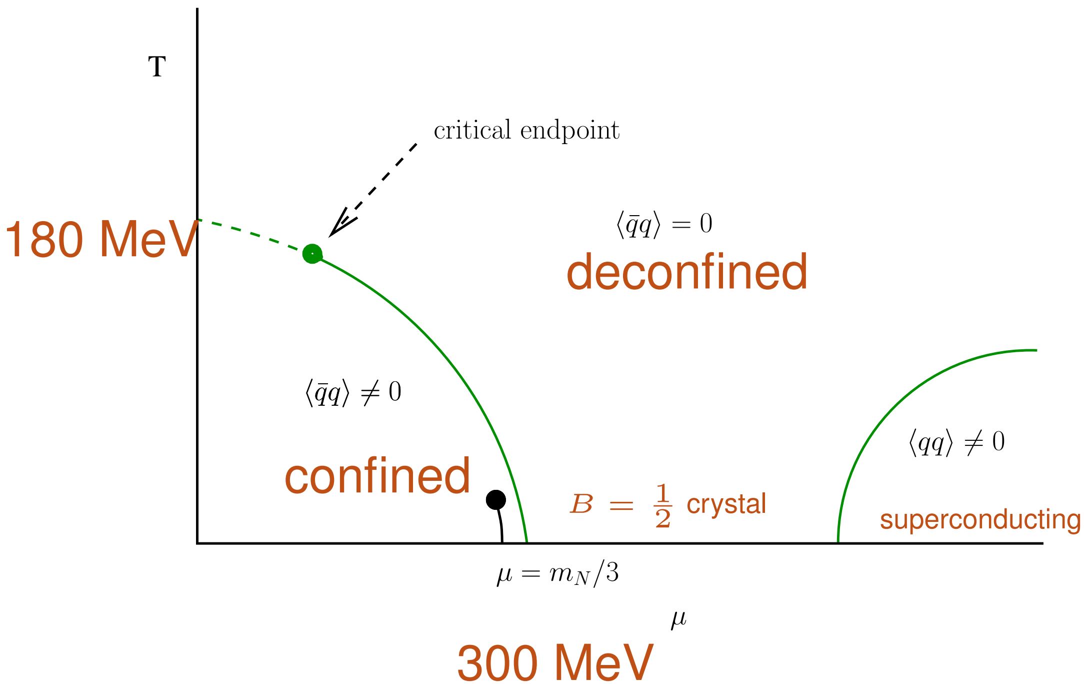
Alford-Kapustin-Wilczek-1999

The mass of the Goldstone bosons is given by $M_k - 2\mu q_k$ with q_k the charge of the Goldstone bosons.

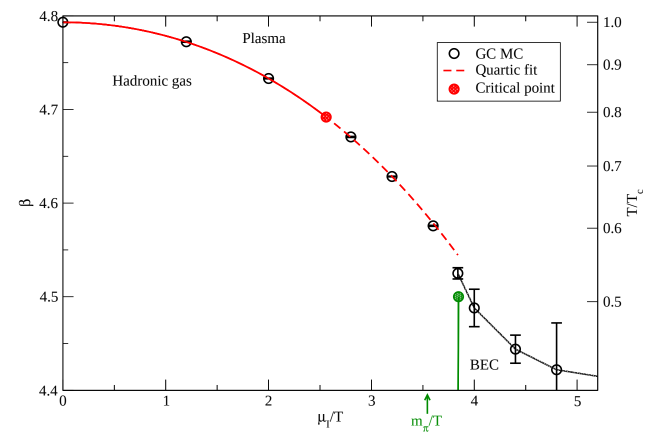
A phase transition to a Bose condensed phase takes place at $\mu = m_\pi/2$.

KSTVZ-2000, Toublan-JV-2000, Son-Stephanov-2000

Phase Diagram QCD and |QCD|



Schematic QCD phase diagram.



Phase diagram of phase quenched QCD (de Forcrand-Stephanov-Wenger-2007).

The high temperature expansion of the free energy can be obtained by a Taylor expansion (Allton-et-al-2003, Gavai-Gupta-2003), reweighting (Fodor-Katz-2002) or from an extrapolation from imaginary μ (Lombardo-2000, de Forcrand-Philipsen-2002, D'Elia-Lombardo-2002).

Sign Problem for $\mu \neq 0$

Because the Dirac operator at nonzero μ is nonhermitean, the fermion determinant is complex

$$\det(D + \mu\gamma_0 + m) = e^{i\theta} |\det(D + \mu\gamma_0 + m)|.$$

The *fundamental* problem is that the average phase factor may vanish in the thermodynamic limit, so that Monte-Carlo simulations are not possible (sign problem).

The severity of the sign problem can be measured by the ratio

$$\langle e^{2i\theta} \rangle_{1+1^*} \equiv \frac{\langle \det^2(D + m + \mu\gamma_0) \rangle}{\langle |\det(D + m + \mu\gamma_0)|^2 \rangle} \sim e^{-V(F_{N_f=2} - F_{pq})}.$$

full QCD
partition function

phase quenched
partition function

Splittorff-JV-2006

Sign Problem

- Because of the severity of the sign problem the QCD partition function cannot be simulated by Monte-Carlo methods at low temperatures and nonzero baryon densities.
- To get some information on what is going on in QCD at nonzero chemical potential one needs to construct models that can be solved.
- If such models can be solved analytically they can be used to test new algorithms.

I. Random Matrix Models

Three Random Matrix Models

Testing Algorithms

Why do Random Matrix Theories work?

Random Matrix Model at Nonzero Chemical Potential

QCD partition function

$$Z(m,\mu) = \langle \det(D + m + \mu\gamma_0) \rangle = \left\langle \det \begin{pmatrix} m & id^\dagger + \mu \\ id + \mu & m \end{pmatrix} \right\rangle$$

The matrix elements of d are random variables because the the gauge fields are stochastic variables distributed according to the Yang-Mills action.

$$Z^{\mathrm{RMT}}(m,\mu) = \int dW\, dW^* P(W) \det D.$$

- The radical proposal is to replace the matrix elements of d by independently distributed Gaussian random variables.
- QCD is strongly interacting and random matrix theories are the ultimate strongly interacting theories.

Random Matrix Model

$$D = \begin{pmatrix} m & iW + \mu \\ iW^\dagger + \mu & m \end{pmatrix},$$

where μ can be arbitrary complex and the $n \times n$ matrix W is distributed according to

$$P(W)dWdW^* = e^{-n\Sigma^2 \mathrm{Tr} W^\dagger W}.$$

The model has one parameter, Σ , which is the chiral condensate.

- Imaginary chemical potential Jackon-JV–1994
- Real chemical potential Stephanov-1996

Solution of the Random Matrix Model

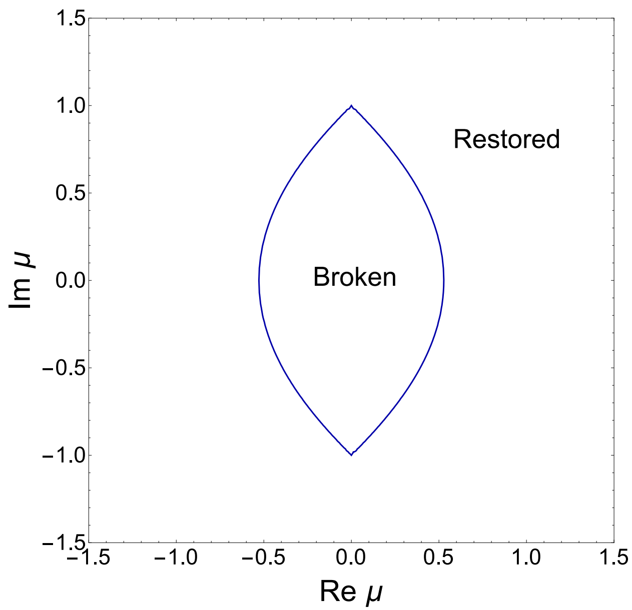
The random matrix partition function can be evaluated exactly at any finite n Halasz-Jackson-JV-1998

$$Z_\nu(m,\mu) = \int_0^\infty ds s^{\nu+1} I_\nu(2mns\Sigma)(s^2-\mu^2)^n e^{-n\Sigma^2(s^2-\mu^2+m^2)}.$$

This expression is valid for arbitrary complex μ and is an analytic function of μ.

This supports the idea of getting around the sign problem by analytical continuation to imaginary chemical potential or by Taylor expansion.

Lombardo-2000, D'Elia-Lombardo-2002, de Forcrand-Philipsion-2002, Allton-et-al-2003, Gavai-Gupta-2003



Successes of the Random Matrix Model

The chiral phase transition with increasing temperature

The nature of the quenched approximation

The phase diagram of QCD at $\mu \neq 0$

The spectrum of the Dirac operator

Finite Temperature Chiral Phase Transition

Replace the time derivative by only the lowest Matsubara frequency

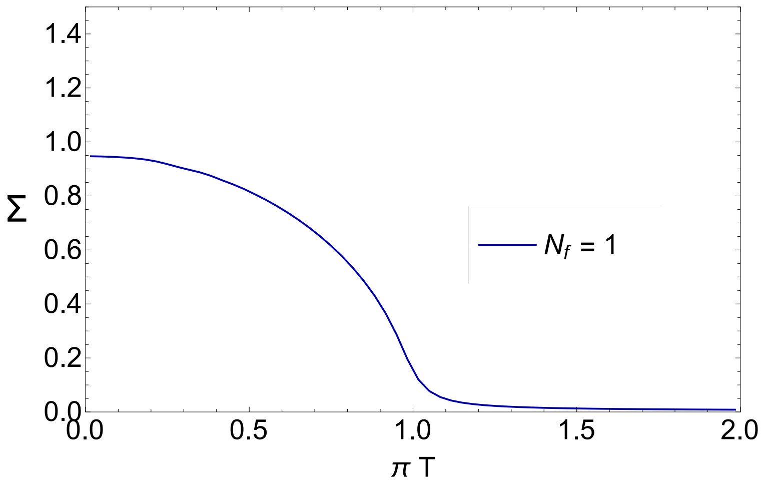
$$\partial_0 \to i\pi T\sigma_3$$

$$\det\begin{pmatrix} m & iW + i\pi T\sigma_3 \\ iW^\dagger + i\pi T\sigma_3 & m \end{pmatrix} = \det\begin{pmatrix} m & iW\sigma_3 + i\pi T \\ i\sigma_3 W^\dagger + i\pi T & m \end{pmatrix}$$

and σ_3 can be absorbed in the probability distribution.

This model has a chiral phase transition at $T = 1/\pi\Sigma$. Jackson-JV-1994

πT can also be interpreted as imaginary chemical potential.



The Nature of Quenching

The quenched partition function is given by

$$\lim_{k\to 0}\langle \det{}^k(D+m+\mu\gamma_0)\det{}^k(D+m+\mu\gamma_0)^\dagger\rangle$$
$$=\lim_{k\to 0}\langle \det{}^k(D+m+\mu\gamma_0)\det{}^k(D+m-\mu\gamma_0)\rangle.$$

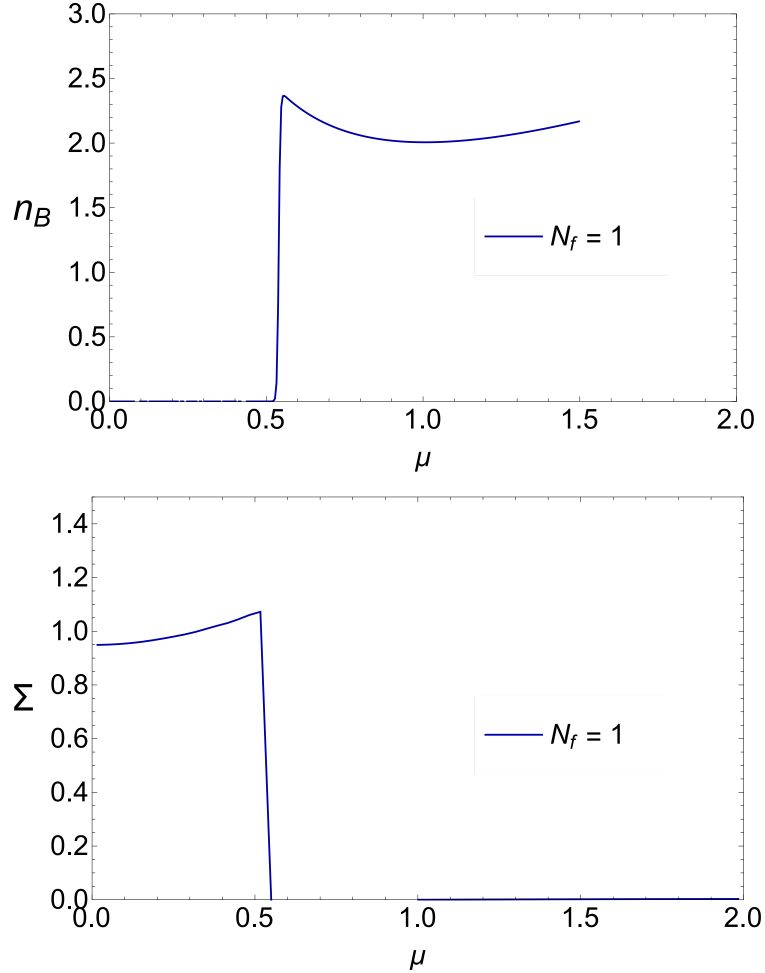
Therefore, the quenched theory has the physics of the phase quenched theory and has a pion condensate phase transition when $\mu > m_\pi/2$.

Stephanov-1996

Model for the Transition to Nonzero Baryon Density

For increasing chemical potential, this model has a first order phase transition to a phase with vanishing chiral condensate and nonzero baryon density.

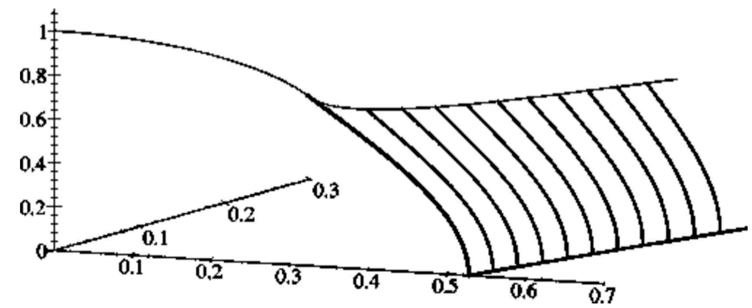
For large n the results are given by the mean field approximation and do not depend on the number of flavors.



Model for the Phase Diagram of QCD

By including the temperature at nonzero chemical potential through the lowest Masubare frequency, we obtain a random matrix model that allows us to calculate the phase diagram in the $\mu - T - m$-plane.

This works because the random matrix model is equivalent to a sixth order Landau-Ginsberg potential.



Halasz-Jackson-Stephanov-Shrock-JV-1998

Other Models

Osborn model

BBKSV model

The Osborn Model

Two Matrix Model

$$Z(m,\mu) = \int d\Phi_1 d\Phi_2 \det{}^n \begin{pmatrix} m & i\Phi_1 + \mu\Phi_2 \\ i\Phi_1^\dagger + \mu\Phi_2^\dagger & m \end{pmatrix} e^{-n\Sigma^2 \mathrm{Tr}(\Phi_1^\dagger\Phi_1 + \Phi_2^\dagger\Phi_2)}.$$

- This model has a much larger invariance group, U(n)×U(n), which is the reason that is completely solvable.
- The eigenvalue density can be obtained analytically.
- Duality: $Z(\Sigma,\mu) = Z(\Sigma\mu, 1/\mu)$.
- $Z(m,\mu) = (1-\mu^2)^n Z(\frac{m}{\sqrt{1-\mu^2}}, 0)$ for $\mu < 1$.
- There is no phase transition at $\mu = 1$.

Solution of the Osborn Model

Solution for $\mu < 1$ (in units where $\Sigma = 1$)

$$Z(m,\mu) \quad = \quad (\mu^2-1)^{2n+\nu+1} e^{-\frac{m^2 n}{1-\mu^2}} \int_0^\infty s ds s^{\nu+1} (s^2)^n e^{-ns^2|1-\mu^2|} I_\nu(2mns)$$

Osborn-2004

Solution of the JSV model

$$Z_\nu(m,\mu) = e^{-n(m^2-\mu^2)} \int_0^\infty ds s^{\nu+1} I_\nu(2mns)(s^2-\mu^2)^n e^{-ns^2}.$$

The BBKSV Model

$$Z(m,\mu) = \int d\Phi_1 d\Phi_2 \det \begin{pmatrix} m & e^{\mu}\Phi_1 - e^{-\mu}\Phi_2^\dagger \\ -e^{-\mu}\Phi_1^\dagger + e^{\mu}\Phi_2 & m \end{pmatrix} e^{-2n\Sigma^2 \mathrm{Tr}(\Phi_1\Phi_1^\dagger + \Phi_2\Phi_2^\dagger)}$$

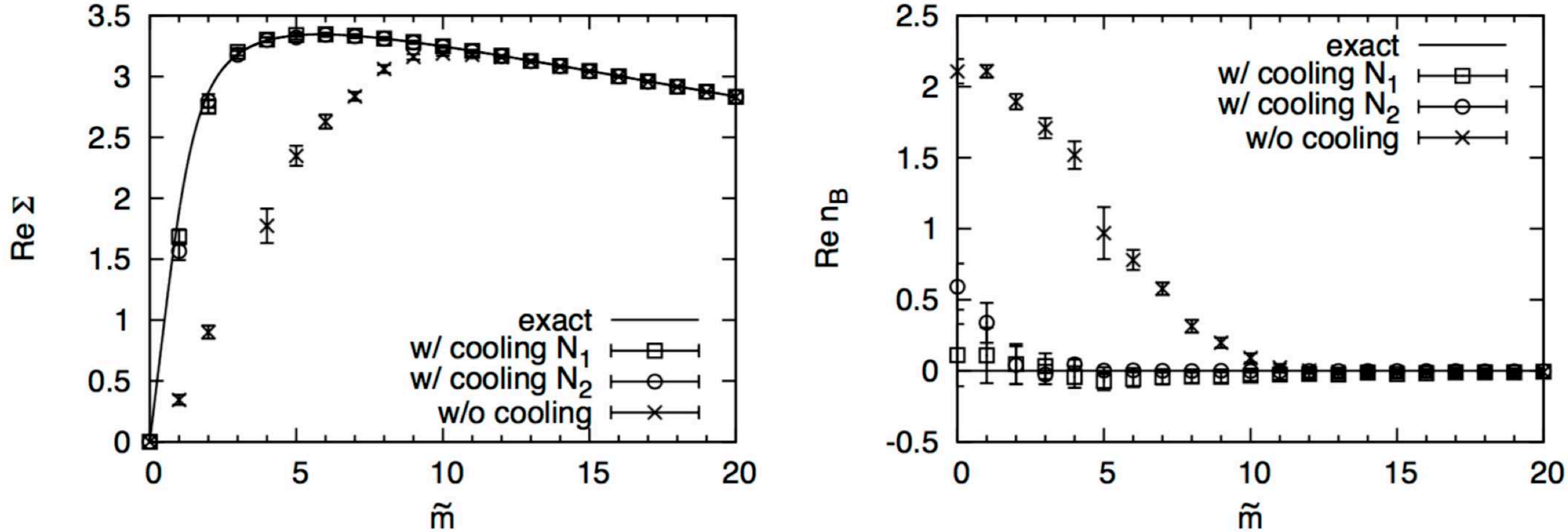
where Φ_1 and Φ_2 are complex $n \times n$ matrices.

Bloch-Brückmann-Kieburg-Splittorff-JV-2013

The Gaussian integral is only nonzero for terms that have an equal number of factors Φ_i and $\Phi_i^\dagger$ for $i = 1,\ 2$ so that the partition function does not depend on μ .

- $Z(m,\mu) = Z(m,0)$.
- $Z(m,\mu + \pi i/2) = Z(m,\mu)$.

Testing the Complex Langevin Algorithm



Test of the complex Langevin algorithm for the BBKSV Model

Nagata-Nishimura-Shimasaki-2016

Complex Langevin works after “gauge cooling” adapted for random matrix theory.

Why do Random Matrix Models Work?

- The random matrix theory has the global symmetries of QCD.
- The pattern of spontaneous symmetry breaking is the same as in QCD.
- In microscopic limit,

$$mV = \text{fixed}, \quad \lambda V = \text{fixed}, \quad \mu^2 V = \text{fixed} \qquad \text{for} \quad V \to \infty$$

  the above random matrix theories coincide with QCD.
- More precisely, in this limit random matrix theory coincides with the ϵ-domain of chiral perturbation theory.
- The mean field limit of the chiral Lagrangian in the p-counting scheme coincides with the ϵ -limit of QCD.

Microscopic Limit of QCD

The ϵ limit of the two flavor QCD partition function with chemical potentials μ_1 and μ_2 is given by

$$Z_\nu(m, \mu_1, \mu_2) = \int_{U \in U(2)} \det{}^\nu U e^{-\frac{1}{4}V(\mu_1-\mu_2)^2 F^2 Tr[U,\tau_3][U^\dagger,\tau_3] + mV\Sigma \mathrm{Tr}[U+U^{-1}]}.$$

This partition function can be derived from the microscopic limit of random matrix theory. Toublan-JV-2000

The partition function does not depend of the baryon chemical potential $\mu = (\mu_1 + \mu_2)/2$.

The chiral condensate is μ-independent for $\mu < \mu_c$.

Dirac Spectra

Why?

Dirac spectra at nonzero chemical potential

Chiral condensate

Dense matter, strangeness and waves in Dirac spectra

Dirac Spectra

Why are we interested in Dirac spectra?

$$
\begin{aligned}
Z(m,\mu) &= \langle \det(D+m+\mu\gamma_0)\rangle = \langle \prod_k (i\lambda_k + m)\rangle \\
&= \langle \det(\gamma_0(D+m)+\mu)\rangle = \langle \prod_k (\mu_k + \mu)\rangle
\end{aligned}
$$

- Chiral condensate

$$
\langle \bar\psi\psi\rangle = \frac{1}{V}\frac{d}{dm}\log Z = \frac{1}{V}\left\langle \sum_k \frac{1}{m+i\lambda_k}\right\rangle .
$$

- Baryon number density

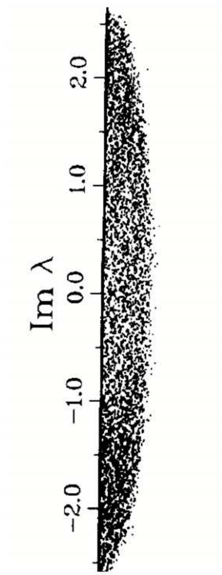
$$
\langle n_B\rangle = \frac{1}{V}\frac{d}{d\mu}\log Z = \frac{1}{V}\left\langle \sum_k \frac{1}{\mu+\mu_k}\right\rangle .
$$

Phase Factor and Dirac Eigenvalues

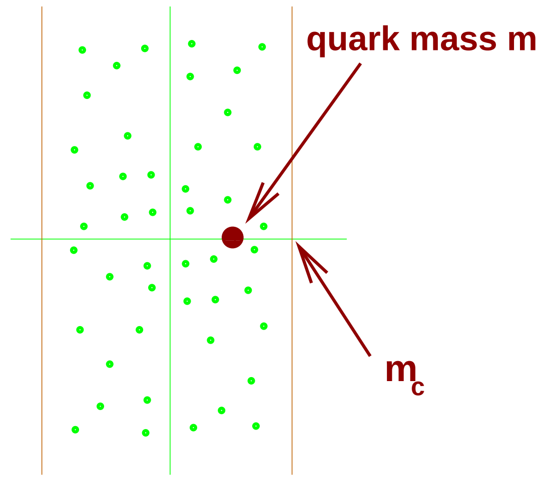
$$\det(D + m + \mu\gamma_0) = e^{i\theta} |\det(D + m + \mu\gamma_0)|$$

$\prod_k (i\lambda_k + m)$

phase factor



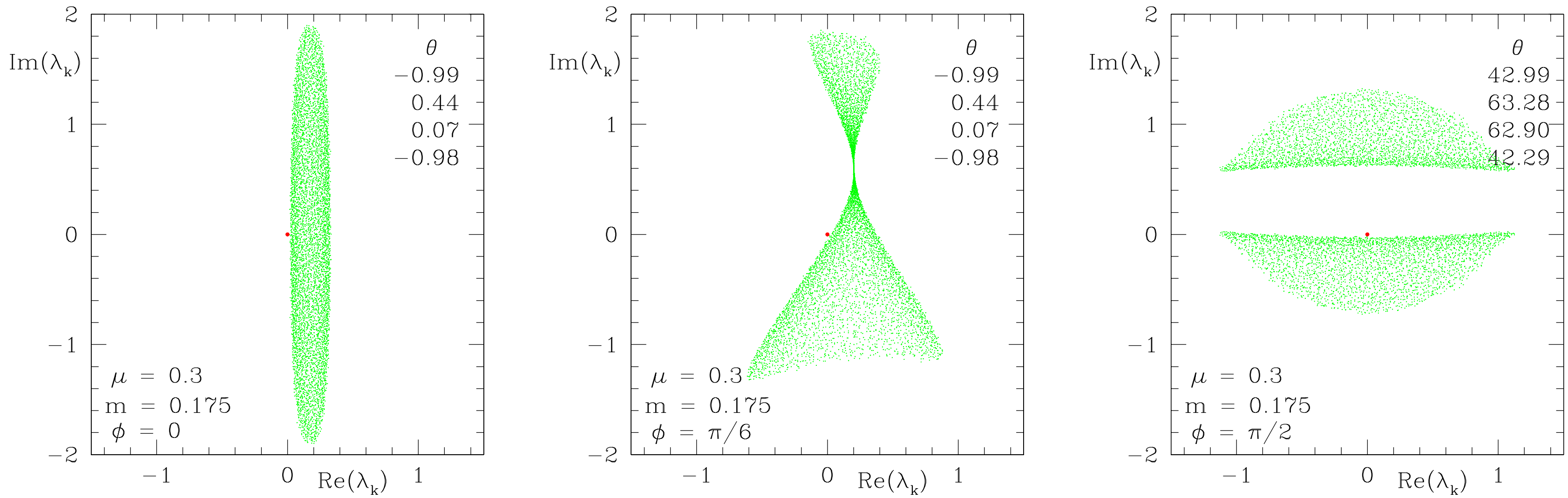
Barbour et al. 1986



Scatter plot of Dirac eigenvalues

$$m < m_c \qquad \text{then} \qquad \langle e^{i\theta} \rangle \sim 0$$

Dirac Spectra in Random Matrix Theory



Spectrum of $(\cos\phi + i\gamma_0 \sin\phi)(D + m + \mu\gamma_0)$

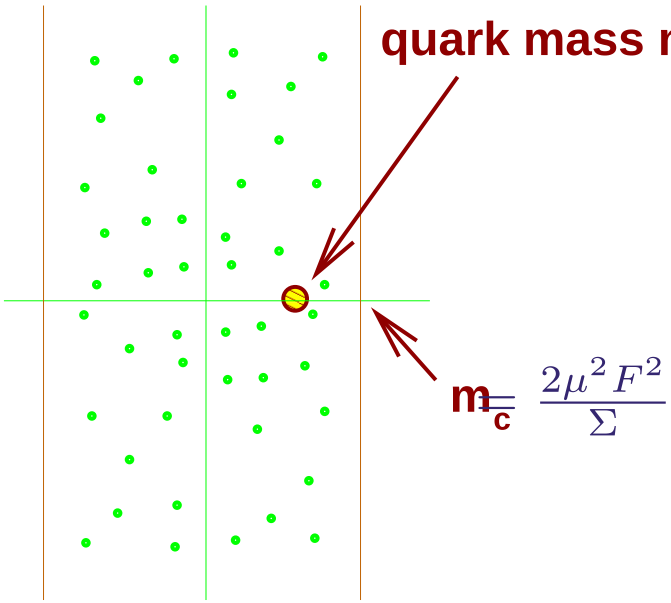
Lombardo-Splittorff-JV-2009

Dirac Spectra at Nonzero Chemical Potential

At nonzero chemical potential the Dirac operator is nonhermitian. Based on our experience from Random Matrix Theory, we expect the following:

- At small chemical potential we expect the eigenvalues to move perpendicular to the imaginary axis.
- The eigenvalues are distributed more or less homogeneously in a compact region.
- The topology of the region is trivial. A nontrivial topology can only occur after a phase transition.

Width of Quenched Dirac Spectrum



Scatter plot of Dirac eigenvalues

$$\mu^2 = \frac{1}{4} m_\pi^2 = \frac{m\Sigma}{2F^2},$$

can also be written as

$$m = \frac{2\mu^2 F^2}{\Sigma}.$$

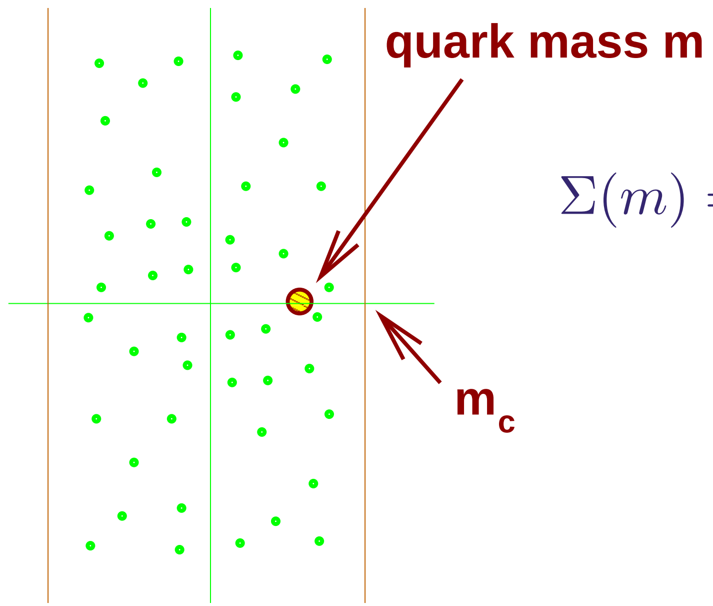
The width of the Dirac spectrum follows from a mean field analysis of the phase quenched partition function.

Gibbs-1986,Stephanov-1996, Toublan-JV-2000

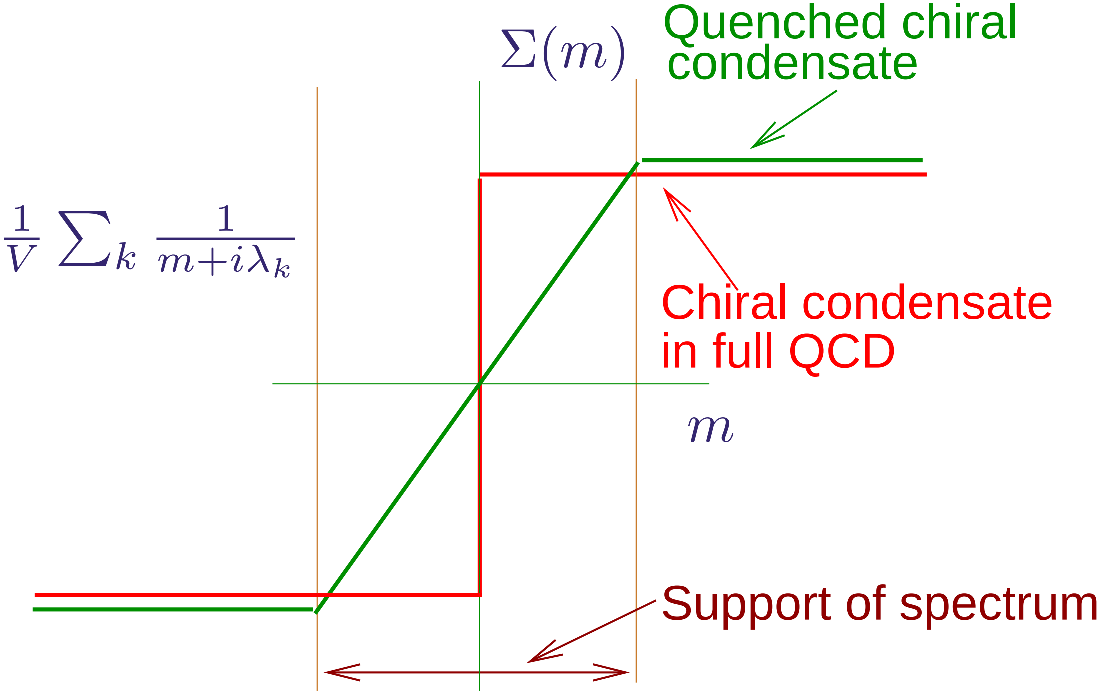
The sign problem becomes severe for $\mu > m_\pi/2$ when the quark mass is inside the support of the eigenvalues.

Splittorff-JV-2006

Dirac Spectrum and Chiral Condensate



$$\Sigma(m) = \frac{1}{V}\sum_k \frac{1}{m+i\lambda_k}$$



Scatter plot of Dirac eigenvalues

- For nonhermitean theories theories with a complex determinant, the support of the Dirac spectrum does not depend on the complex phase of the determinant.

- Exponential cancellations can wipe out the critical point and reveal a completely different physical system. This is the case of QCD at nonzero baryon density.

Spectral Density of Full QCD

The spectral density at nonzero chemical potential is defined by

$$\rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle.$$

- The average contains a complex determinant and therefore the spectral “density” is in general complex.
- At nonzero chemical potential the quenched or phase quenched chiral condensate vanishes even at low temperatures.
- The obtain a nonvanishing chiral condensate the spectral “density” in full QCD has to be drastically different.

‘

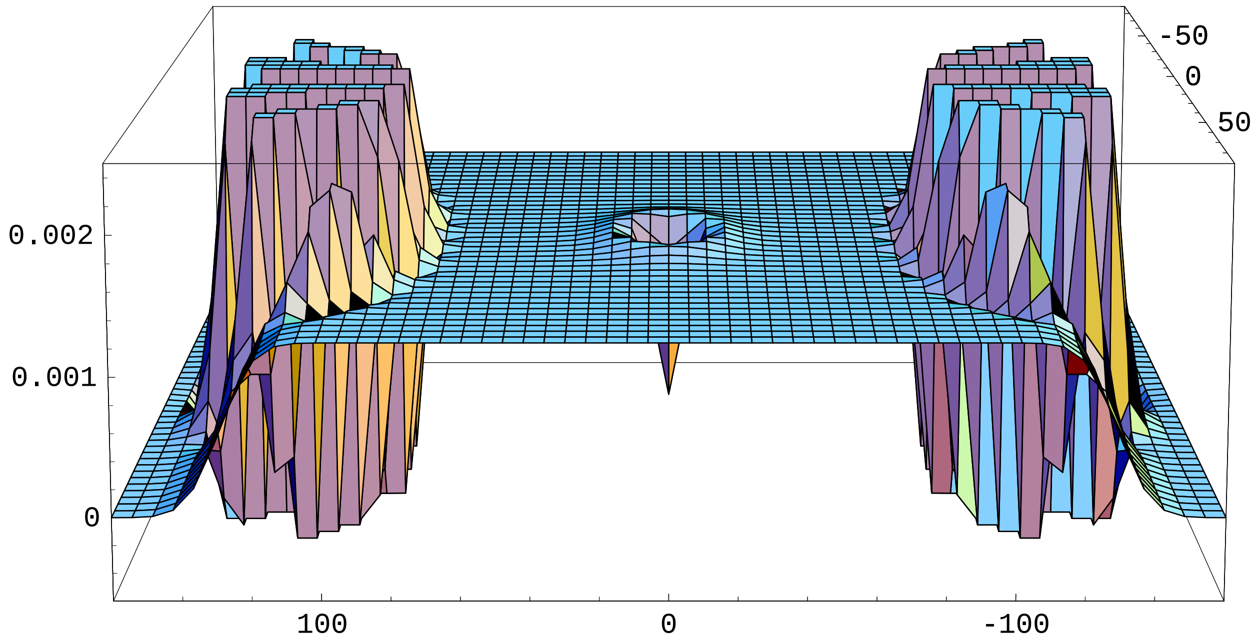
Spectral Density of Full QCD

- In the microscopic domain, the spectral density of full QCD is given by random matrix theory and can be evaluated analytically.

  Osborn-2004

- It turns out that the spectral density contains a region with strong oscillations with a period $\sim 1/V$ and an amplitude that grows exponentially with the volume.
- The oscillatory region extends over a finite part of the complex plane.

Dirac Spectrum QCD at $\mu \neq 0$



Real part of the spectral density for QCD with one flavor at nonzero chemical potential.

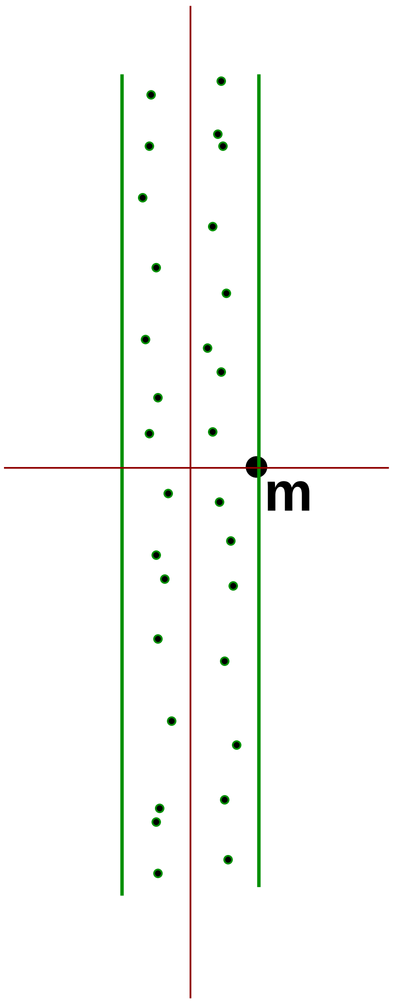
The oscillatory region is responsible for the discontinuity in the chiral condensate.

Osborn-Splittorf-JV-2005/2008, Osborn-2004

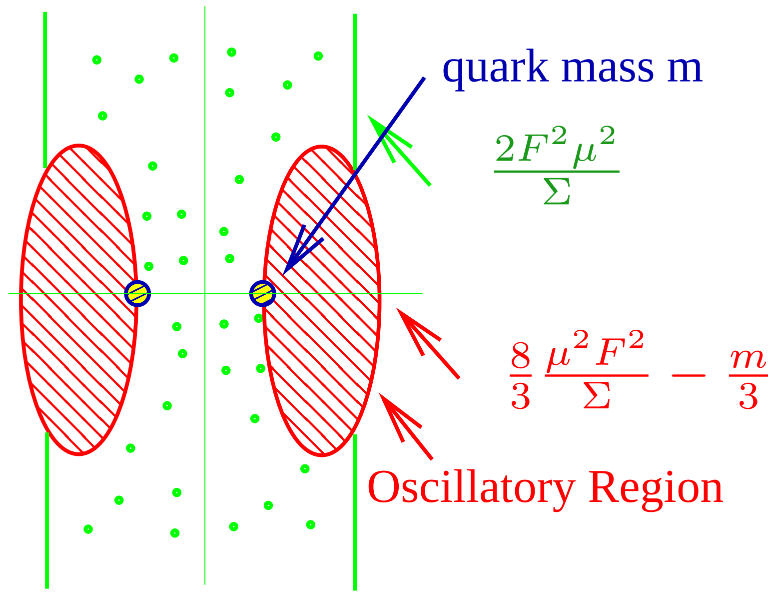
$\longrightarrow$

Mechanism for Discontinuity of the Chiral Condensate

- The Banks-Casher formula states that a discontinuity in the chiral condensate requires a dense Dirac spectrum on the imaginary axis.
- Using a random matrix theory at nonzero chemical potential we (OSV) showed that a discontinuity can be obtained from a strongly oscillating spectral density with a period $\sim 1/V$ and an amplitude that grows exponentially with the volume.
- This is a generic mechanism that occurs in nonhermitian theories with a sign problem. Osborn-Splittorff-JV-2006,Ravagli-JV-2007,Kanazawa-Wettig-2013,Wettig-JV-2014



Dense Matter, Strangeness and Waves in the Dirac Spectrum



Dirac spectrum for Full QCD.

The dotted region is in a pion condensed phase.
The dashed region is in a kaon condensed phase.
The remainder of the complex phase is in the normal phase.
Osborn-Splittorff-JV-2005/2008, Osborn-2004

Generating function for Dirac spectrum

$$Z = \langle \det(D + \mu\gamma_0 + m) | \det(D + \mu\gamma_0 + z)|^2 \rangle.$$

Three flavor partition function with an isospin and strangeness chemical potential.

Comments

- The oscillatory region is responsible for the discontinuity in the chiral condensate.
- This implies that the oscillatory region has to vanish for $T > T_c$
- The oscillatory region is absent for $\mu < m_\pi/2$.
- For QCD with dynamical quarks we have two independent mechanisms to restore chiral symmetry for $\mu > m_\pi/2$:
  - ⋆ A gap develops in the Dirac spectrum.
  - ⋆ The oscillatory region shrinks to zero.

VII. Conclusions

- It is one of the great triumphs of Random Matrix Theory that it can describe the low-energy limit of a fundamental quantum field theory.

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- Without random matrix theory the cancellation mechanism that leads to the Silver Blaze property of the chiral condensate could not have been understood.

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