Towards high momentum transfer in lattice QCD

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CSSM & CoEPP
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ECT* Workshop:
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18–22 April 2016
Trento, Italy

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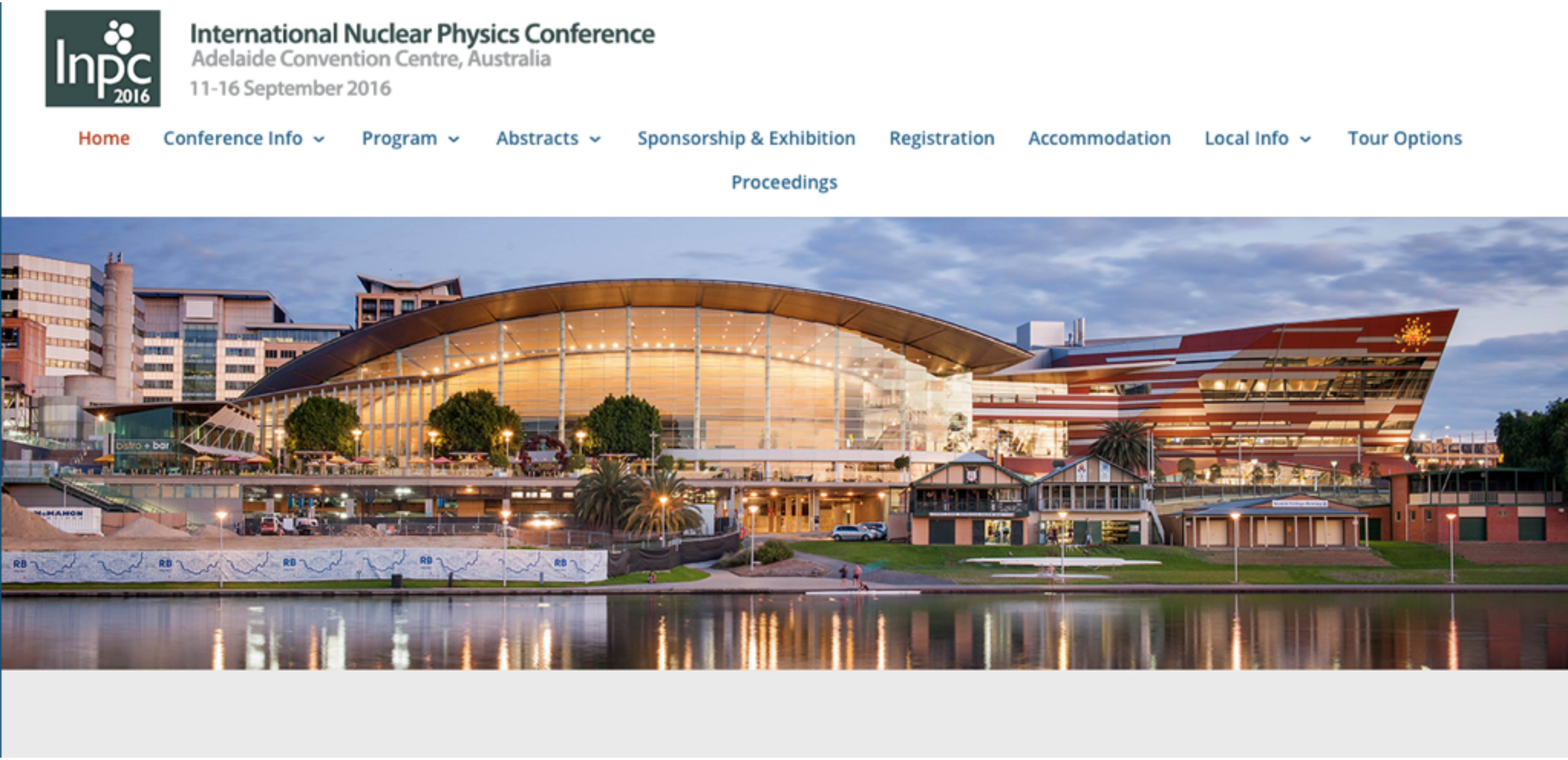
International Nuclear Physics Conference [INPC 2016]

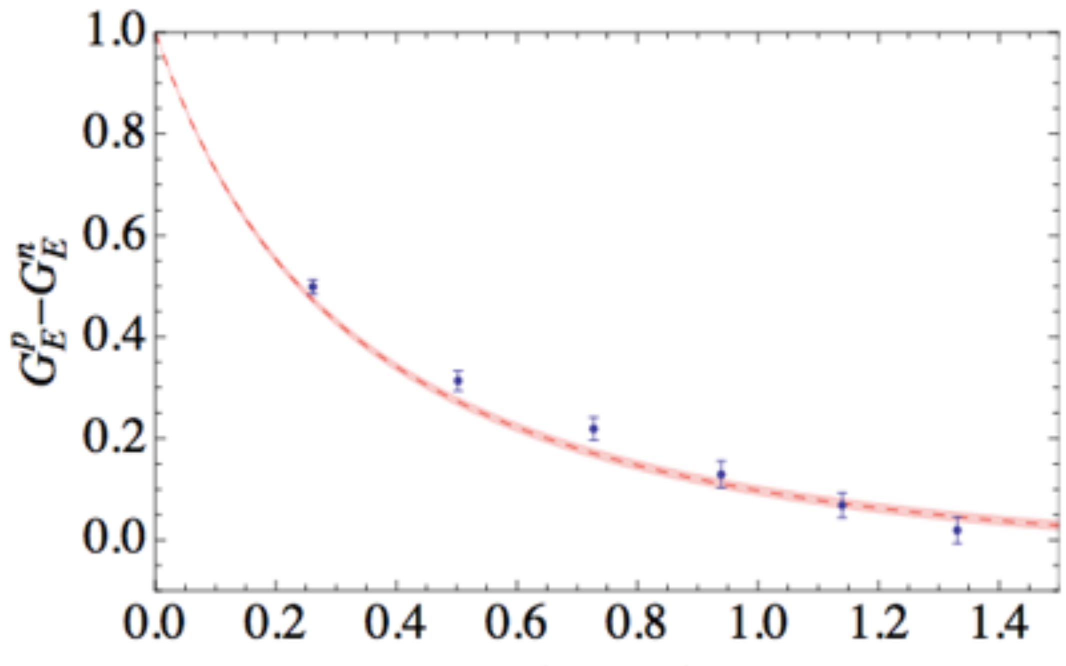


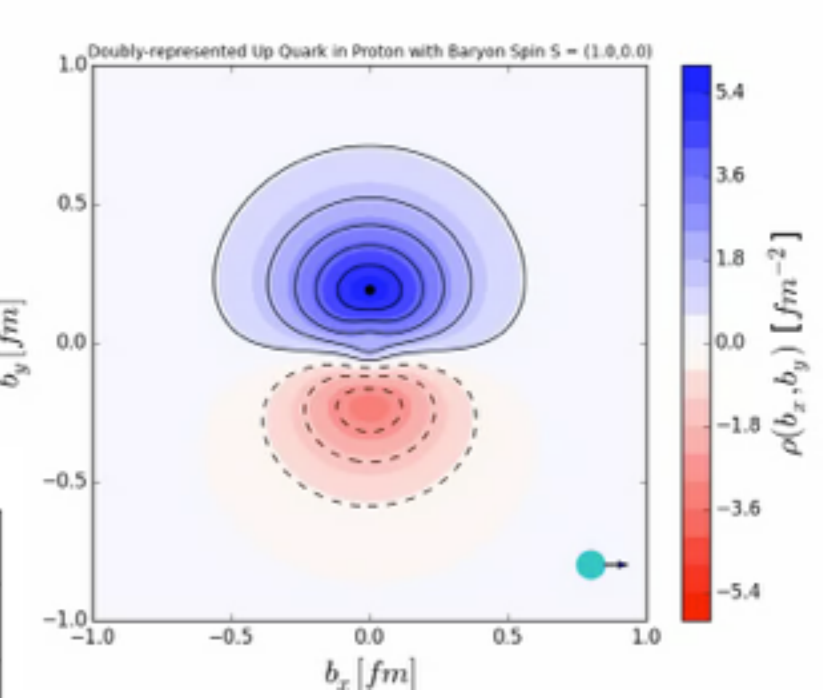
International Nuclear Physics Conference
Adelaide Convention Centre, Australia
11-16 September 2016

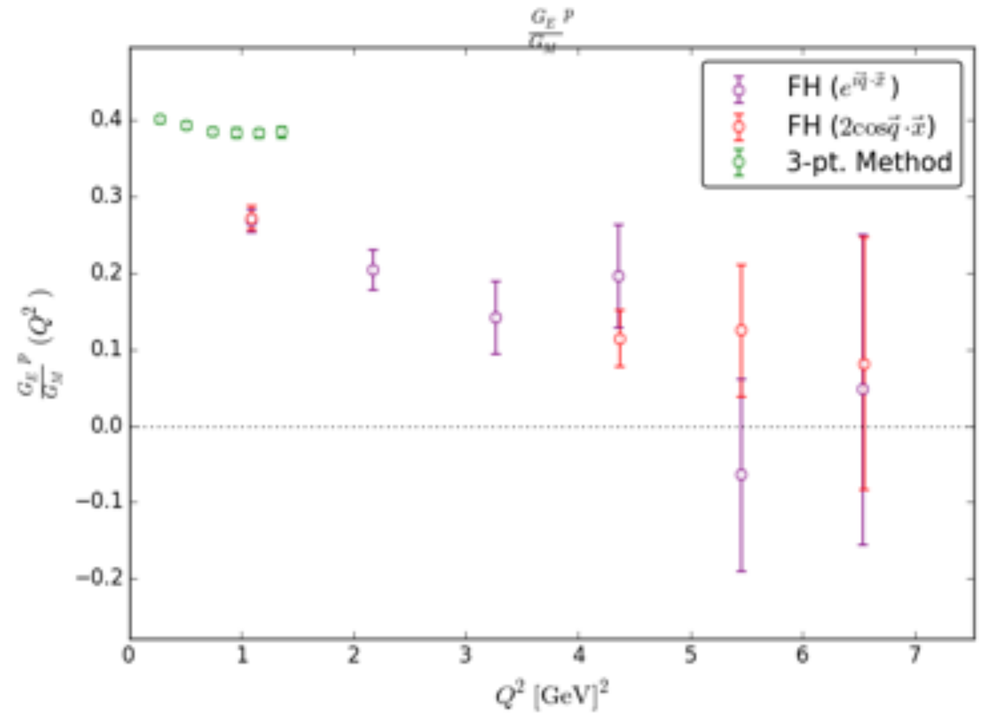
www.inpc2016.com

Outline

- Vector form factors
- Transverse spin densities
- Lattice QCD
  - 3-pt functions
  - “Feynman–Hellmann”
- Form factors at large momentum transfer







Shape of the proton

Shape of the proton

Shape of the proton

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Shape of the proton

Vector form factors: In brief

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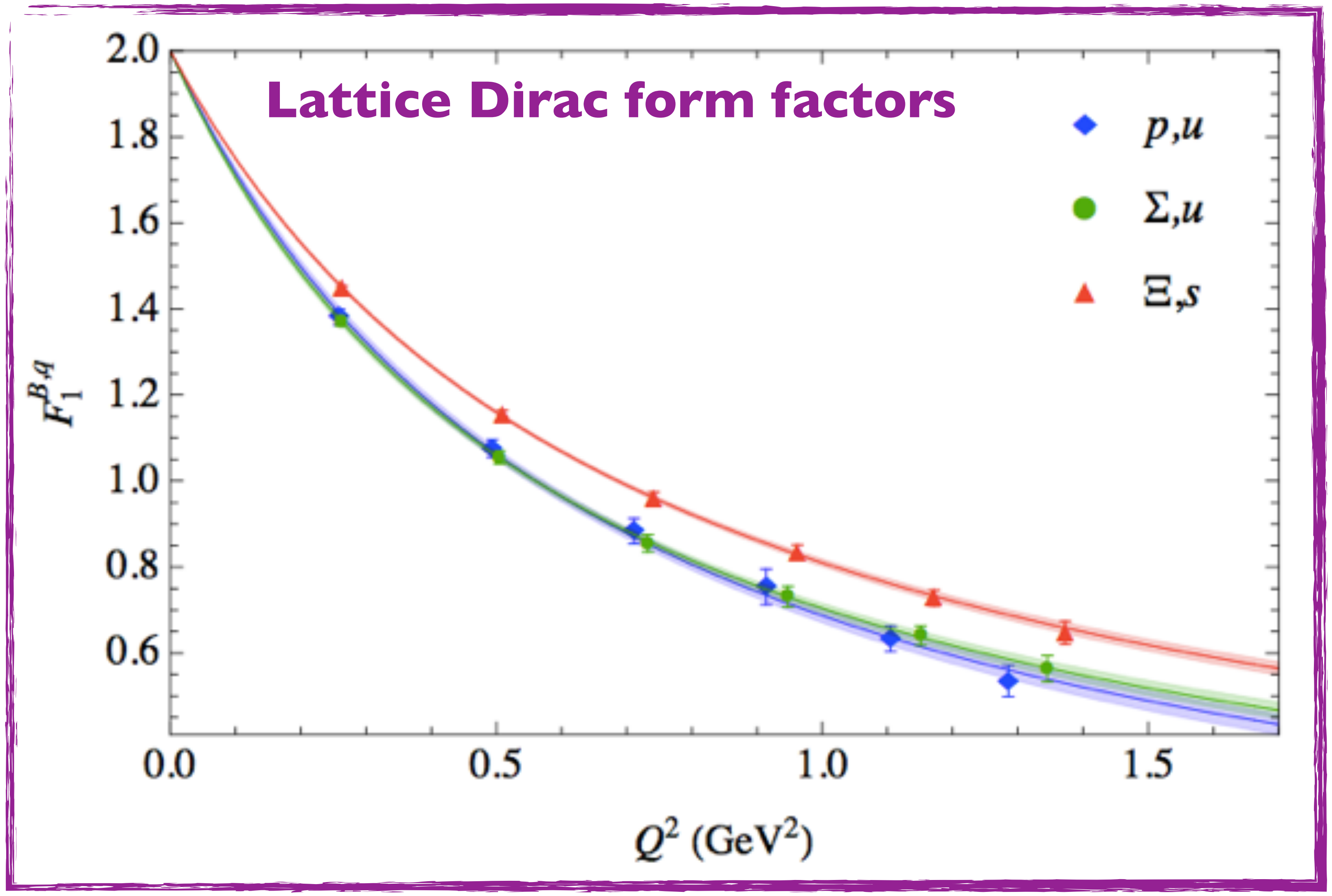
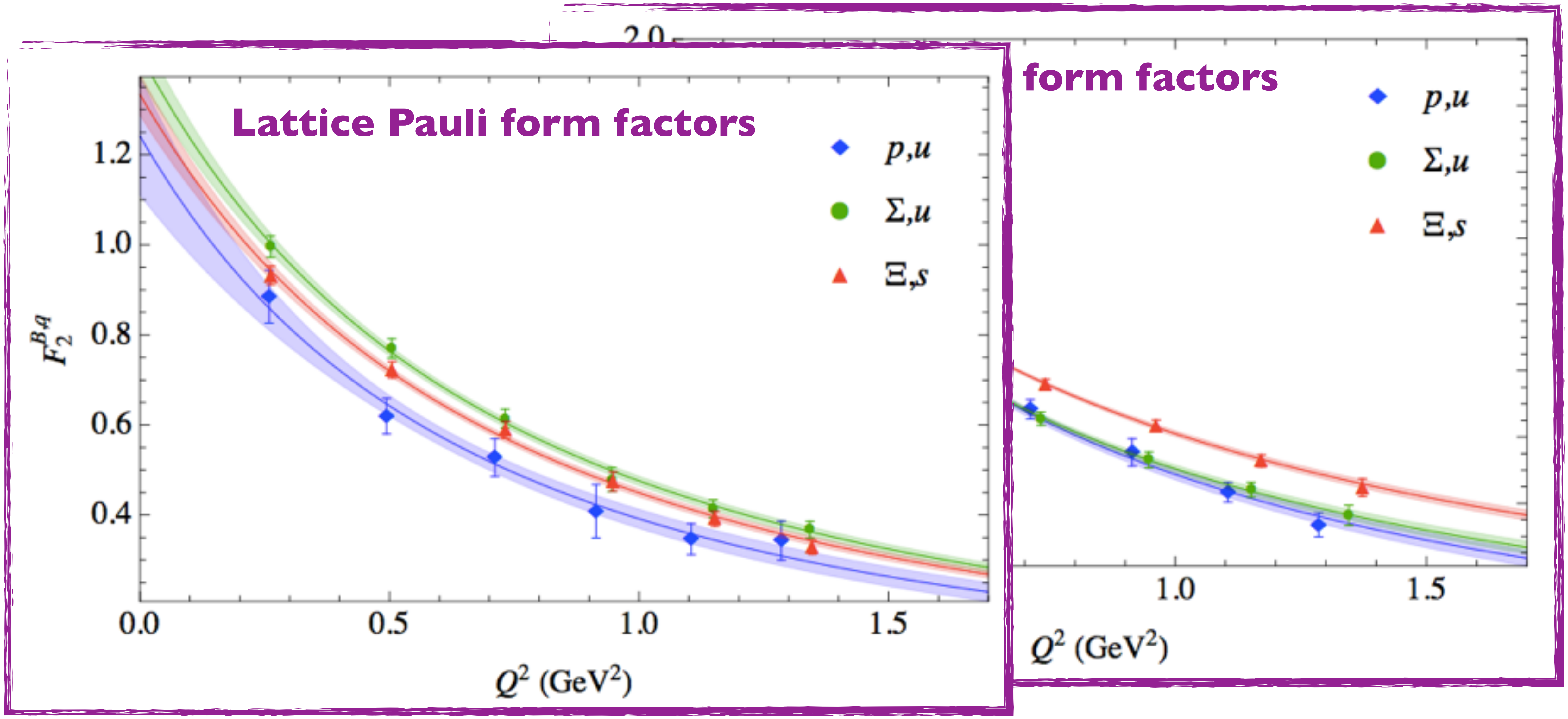


Vector form factors: In brief



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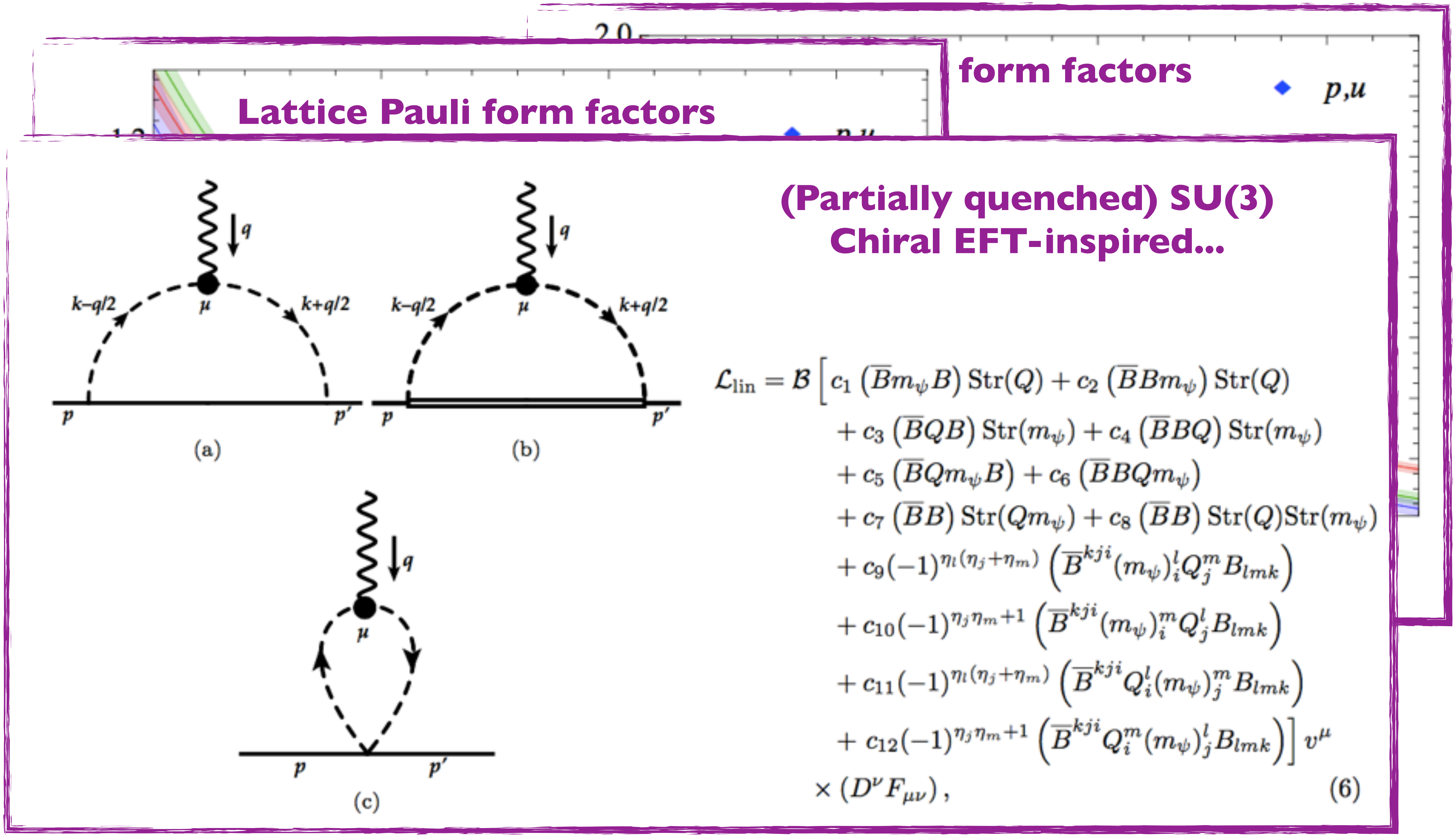


form factors

Lattice Pauli form factors

**(Partially quenched) SU(3)
Chiral EFT-inspired...**

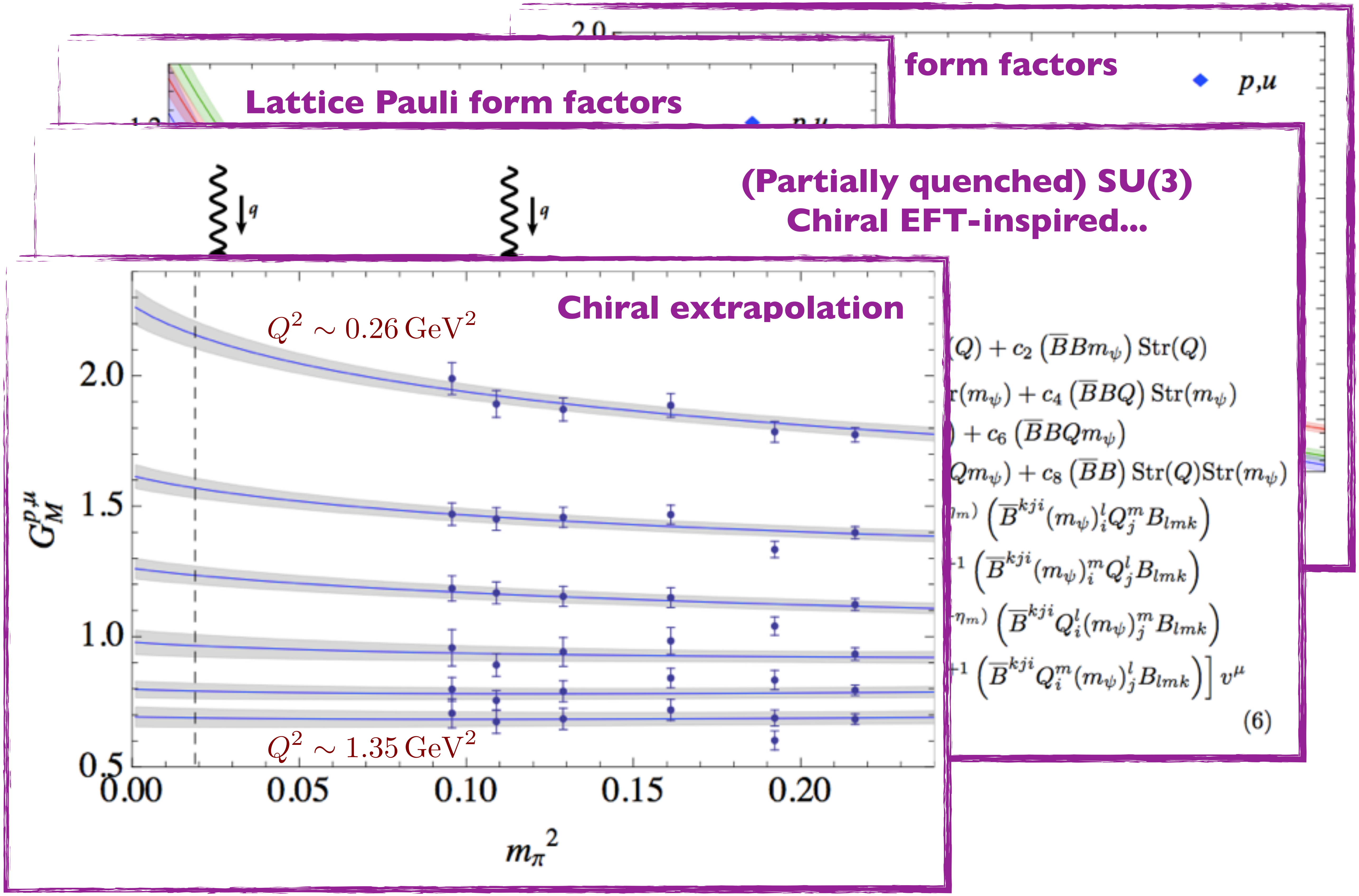
(a)

(b)

(c)

$$
\begin{aligned}
\mathcal{L}_{\rm lin} = \mathcal{B}\Big[& c_1 \left(\overline{B} m_\psi B\right) \mathrm{Str}(Q) + c_2 \left(\overline{B} B m_\psi\right) \mathrm{Str}(Q) \\
& + c_3 \left(\overline{B} Q B\right) \mathrm{Str}(m_\psi) + c_4 \left(\overline{B} B Q\right) \mathrm{Str}(m_\psi) \\
& + c_5 \left(\overline{B} Q m_\psi B\right) + c_6 \left(\overline{B} B Q m_\psi\right) \\
& + c_7 \left(\overline{B} B\right) \mathrm{Str}(Q m_\psi) + c_8 \left(\overline{B} B\right) \mathrm{Str}(Q) \mathrm{Str}(m_\psi) \\
& + c_9 (-1)^{\eta_l(\eta_j+\eta_m)} \left(\overline{B}^{kji} (m_\psi)^l_i Q^m_j B_{lmk}\right) \\
& + c_{10} (-1)^{\eta_j \eta_m + 1} \left(\overline{B}^{kji} (m_\psi)^m_i Q^l_j B_{lmk}\right) \\
& + c_{11} (-1)^{\eta_l(\eta_j+\eta_m)} \left(\overline{B}^{kji} Q^l_i (m_\psi)^m_j B_{lmk}\right) \\
& + c_{12} (-1)^{\eta_j \eta_m + 1} \left(\overline{B}^{kji} Q^m_i (m_\psi)^l_j B_{lmk}\right)\Big] v^\mu \\
\times & \left(D^\nu F_{\mu\nu}\right), \qquad (6)
\end{aligned}
$$

Vector form factors: In brief



form factors

Lattice Pauli form factors

(Partially quenched) SU(3)
Chiral EFT-inspired...

Chiral extrapolation

$(Q) + c_2\left(\overline{B}Bm_\psi\right)\mathrm{Str}(Q)$

$\mathrm{r}(m_\psi) + c_4\left(\overline{B}BQ\right)\mathrm{Str}(m_\psi)$

$) + c_6\left(\overline{B}BQm_\psi\right)$

$Qm_\psi) + c_8\left(\overline{B}B\right)\mathrm{Str}(Q)\mathrm{Str}(m_\psi)$

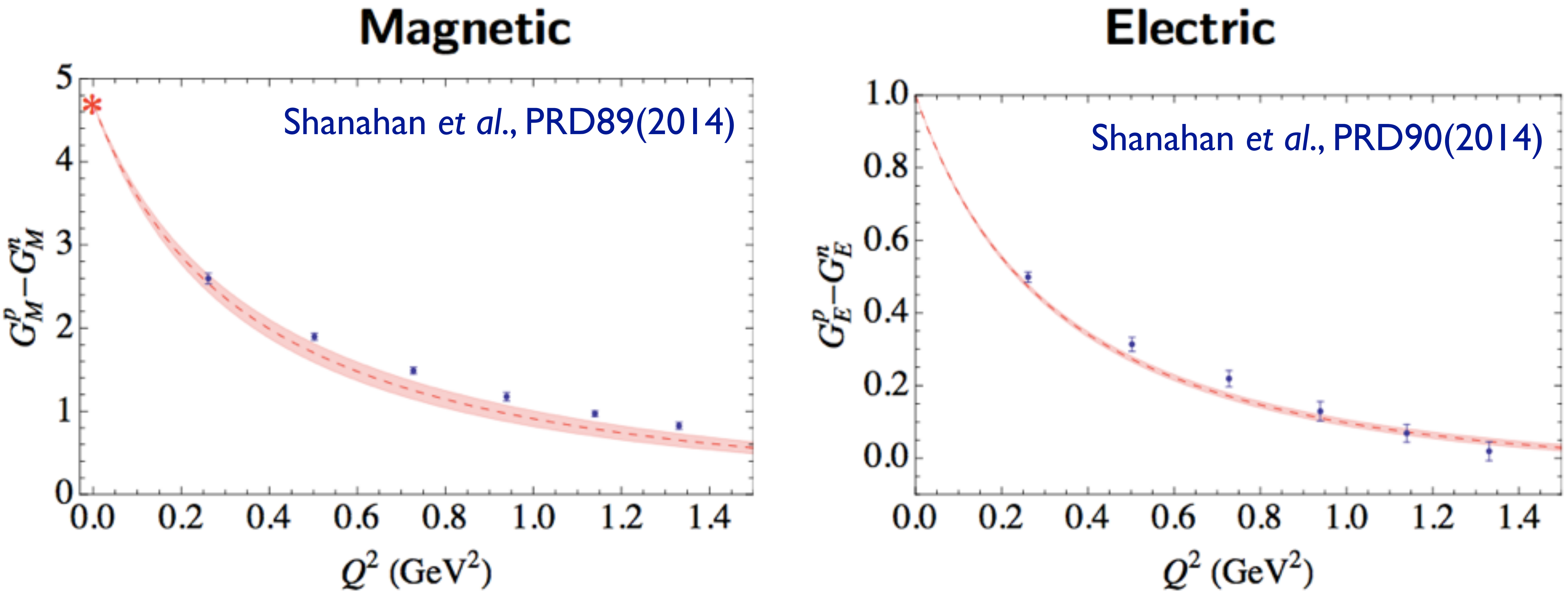
$\eta_m)\left(\overline{B}^{kji}(m_\psi)_i^l Q_j^m B_{lmk}\right)$

$+1\left(\overline{B}^{kji}(m_\psi)_i^m Q_j^l B_{lmk}\right)$

$\eta_m)\left(\overline{B}^{kji}Q_i^l(m_\psi)_j^m B_{lmk}\right)$

$+1\left(\overline{B}^{kji}Q_i^m(m_\psi)_j^l B_{lmk}\right)\Big]\, v^\mu$

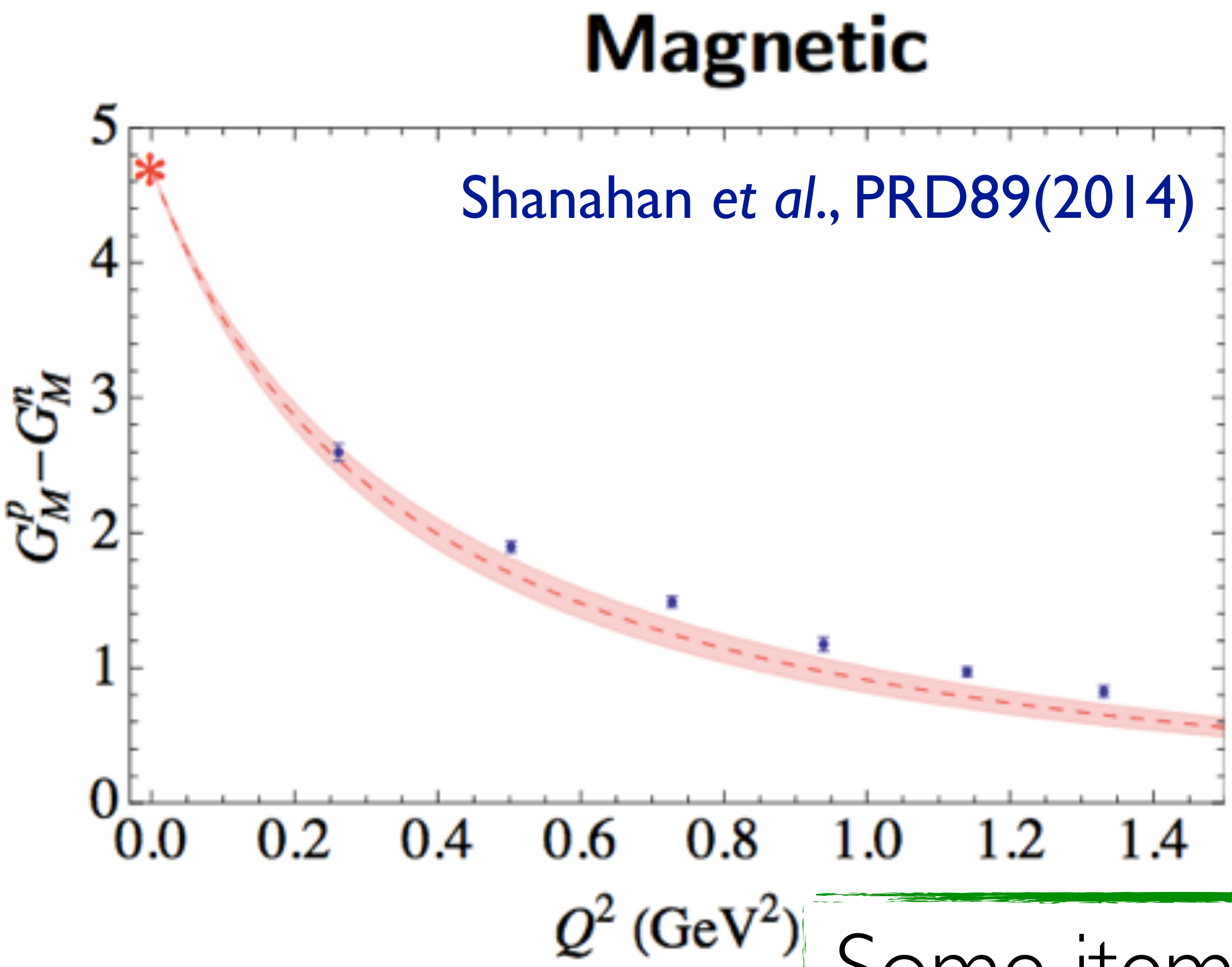
(6)

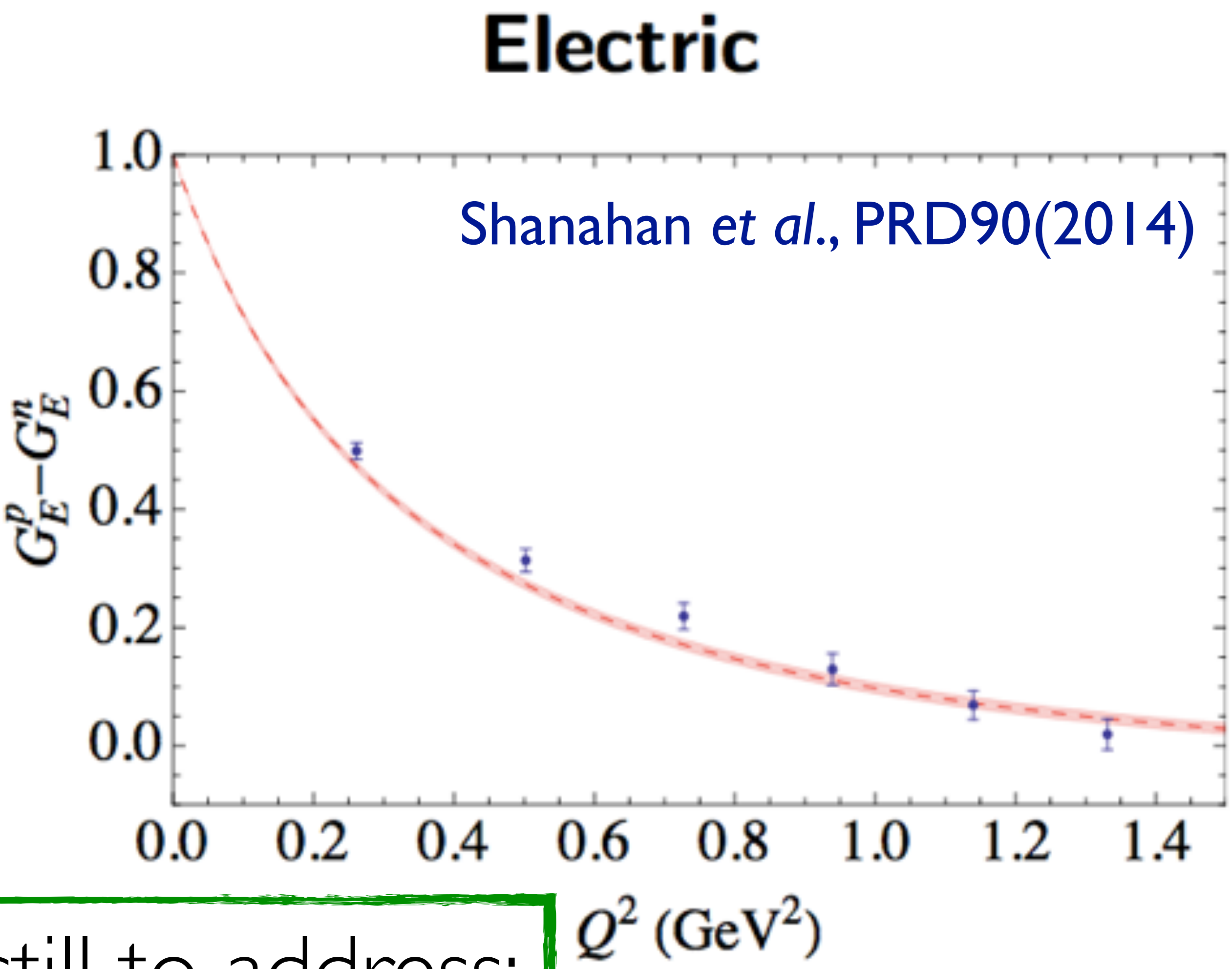


Isovector nucleon form factors

Red curves:
Kelly parameterisation

Isovector nucleon form factors





Some items still to address:

* Finite "a" extrapolation

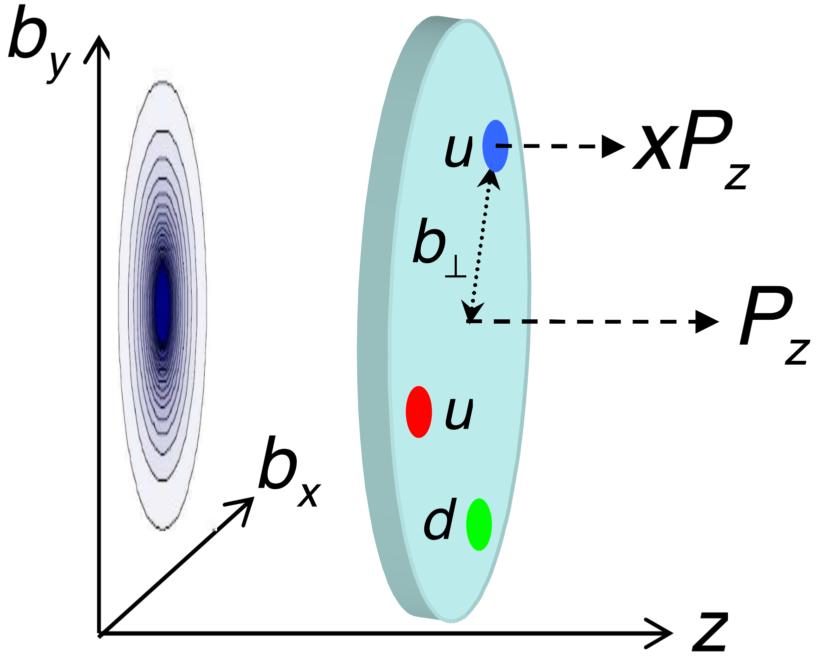
* Excited state contamination

Red curves:
Kelly parameterisation

Transverse densities

- Generalised parton distributions
  - Have a density interpretation with respect to transverse displacement

$$\begin{aligned}\rho(x,b_\perp,s_\perp,S_\perp) &= \langle N_\perp| \int_{-\infty}^{\infty} e^{i\eta x}\overline{q}\left(-\frac{\eta}{2},b_\perp\right)\frac{1}{2}\left[\gamma^+ - s^j_\perp i\sigma^{+j}\gamma_5\right] q\left(\frac{\eta}{2},b_\perp\right)|N_\perp\rangle \\ &= \frac{1}{2}\left\{H(x,b_\perp^2) + s^i_\perp S^i_\perp\left(H_T(x,b_\perp^2) - \frac{1}{4m_N^2}\Delta_{b_\perp}\tilde{H}_T(x,b_\perp^2)\right)\right. \\ &\left.+\frac{b^j_\perp \epsilon^{ji}}{m_N}\left(S^i_\perp E'(x,b_\perp^2) + s^i_\perp \overline{E}'_T(x,b_\perp^2)\right) + s^i_\perp \frac{(2b^i_\perp b^j_\perp - b^2_\perp\delta^{ij})}{m_N^2} S^j_\perp \tilde{H}''_T(x,b_\perp^2)\right\}\end{aligned}$$



[Hägler, …]

Lattice: Moments of longitudinal momenta

- Generalised form factors
- Today: just the most trivial moment

$$\rho(b_\perp, s_\perp, S_\perp) = \int_{-1}^{1} dx\ \rho(x, b_\perp, s_\perp, S_\perp)$$

$$= \frac{1}{2}\{A_{10}(b_\perp^2) + s_\perp^i S_\perp^i \left(A_{T10}(b_\perp^2) - \frac{1}{4m^2} \triangle_{b_\perp} \tilde{A}_{T10}(b_\perp^2) \right)$$

$$+ \frac{b_\perp^j \epsilon^{ji}}{m} \left(S_\perp^i B_{10}'(b_\perp^2) + s_\perp^i \overline{B}_{T10}'(b_\perp^2) \right)$$

$$+ s_\perp^i \left(2 b_\perp^i b_\perp^j - b_\perp^2 \delta^{ij} \right) S_\perp^j \frac{1}{m^2} \tilde{A}_{T10}''(b_\perp^2)\}$$

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$$
\begin{aligned}
\rho(b_\perp, s_\perp, S_\perp) &= \int_{-1}^{1} dx\ \rho(x, b_\perp, s_\perp, S_\perp) \\
&= \boxed{\frac{1}{2}\{A_{10}(b_\perp^2) +} s_\perp^i S_\perp^i \left(A_{T10}(b_\perp^2) - \frac{1}{4m^2} \triangle_{b_\perp} \tilde{A}_{T10}(b_\perp^2) \right) \\
&+ \frac{b_\perp^j \epsilon^{ji}}{m} \left(S_\perp^i B_{10}'(b_\perp^2) + s_\perp^i \overline{B}_{T10}'(b_\perp^2) \right) \\
&+ s_\perp^i \left(2 b_\perp^i b_\perp^j - b_\perp^2 \delta^{ij} \right) S_\perp^j \frac{1}{m^2} \tilde{A}_{T10}''(b_\perp^2) \}
\end{aligned}
$$

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Lattice: Moments of longitudinal momenta

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$m_\pi \simeq 330\,\mathrm{MeV}$

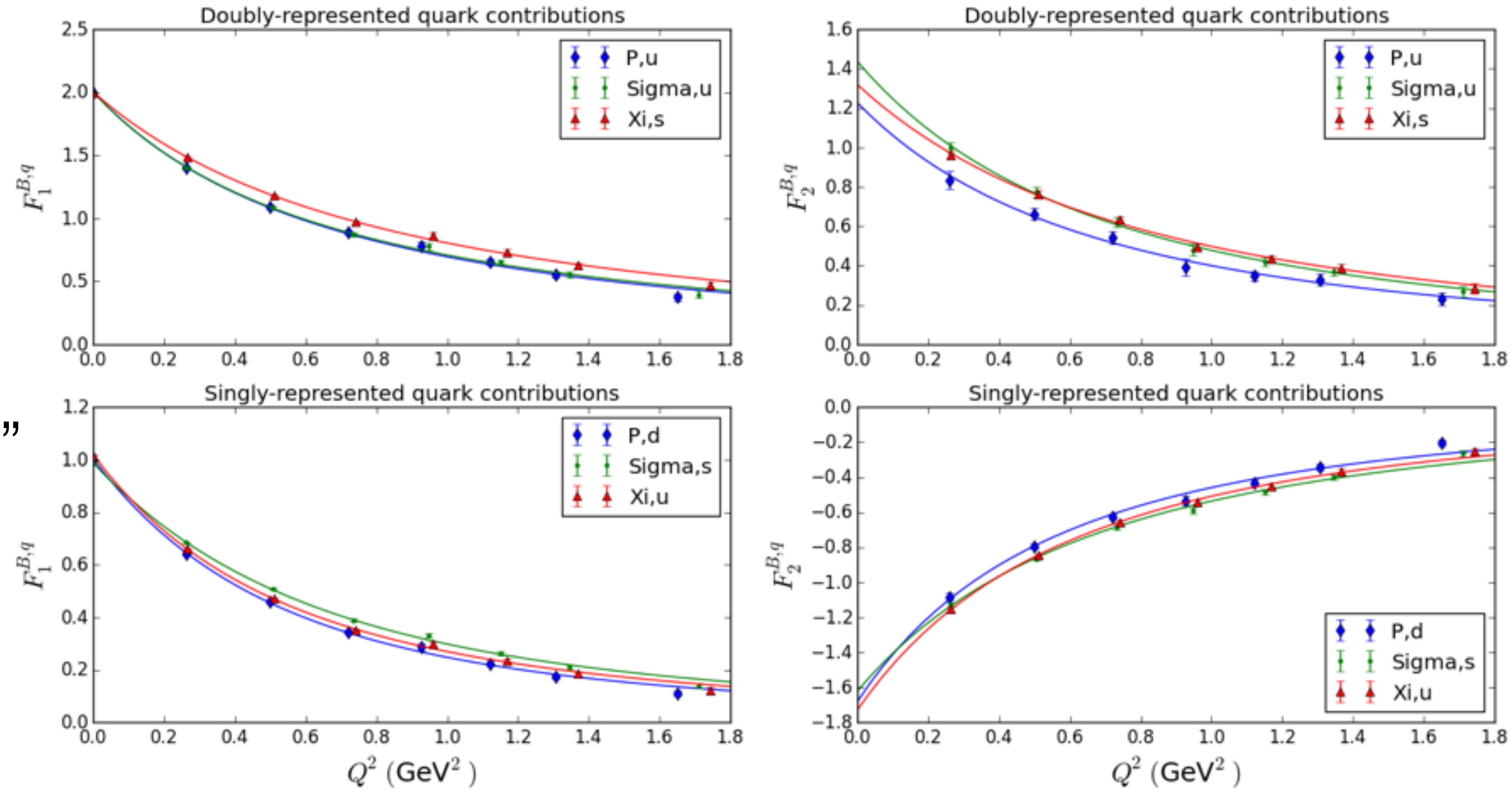
$m_K \simeq 435\,\mathrm{MeV}$

F1

F2

“up”

“down”



Vector form factors

Nucleon, Sigma & Cascade

$m_\pi \simeq 330\,\mathrm{MeV}$

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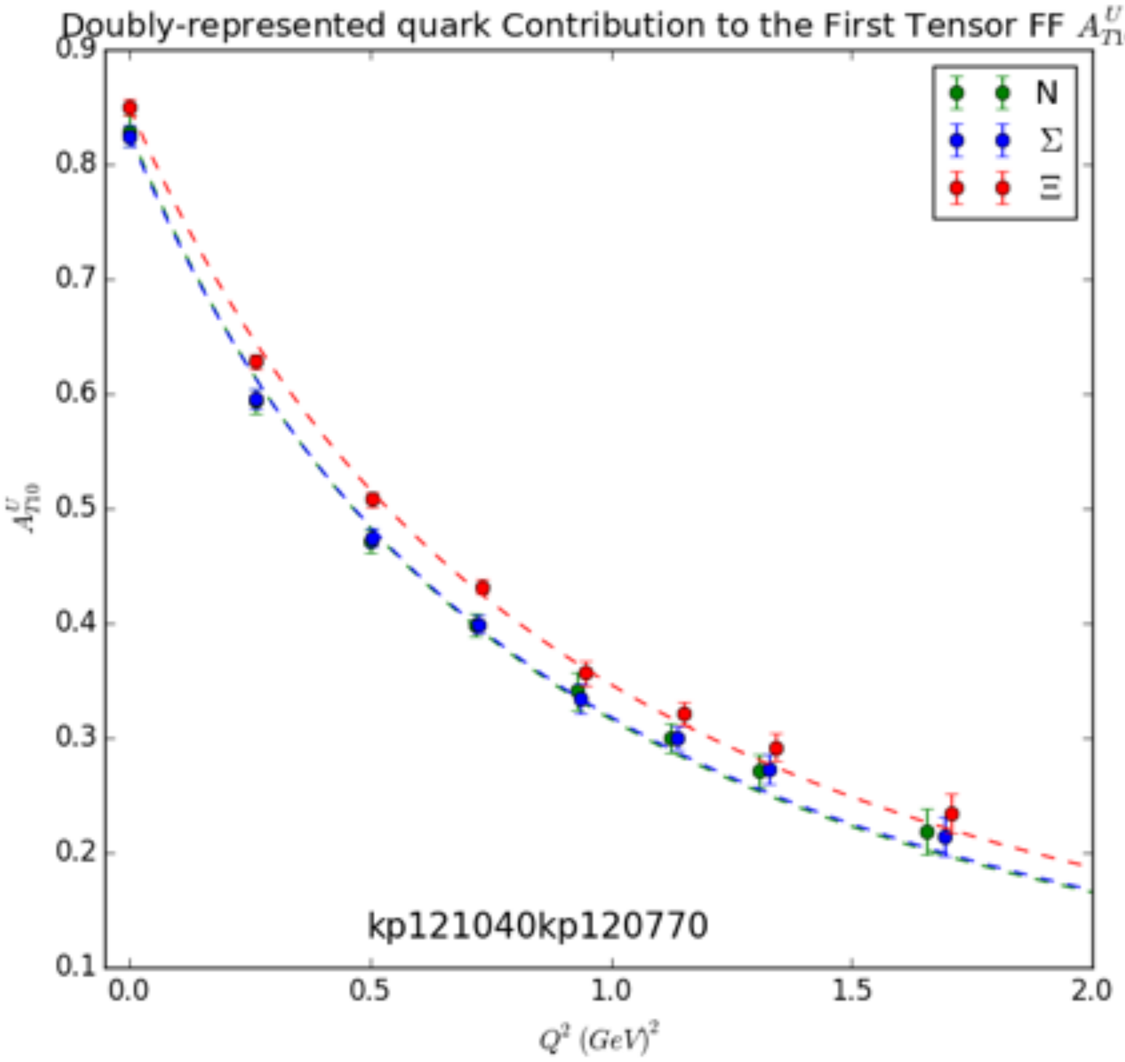
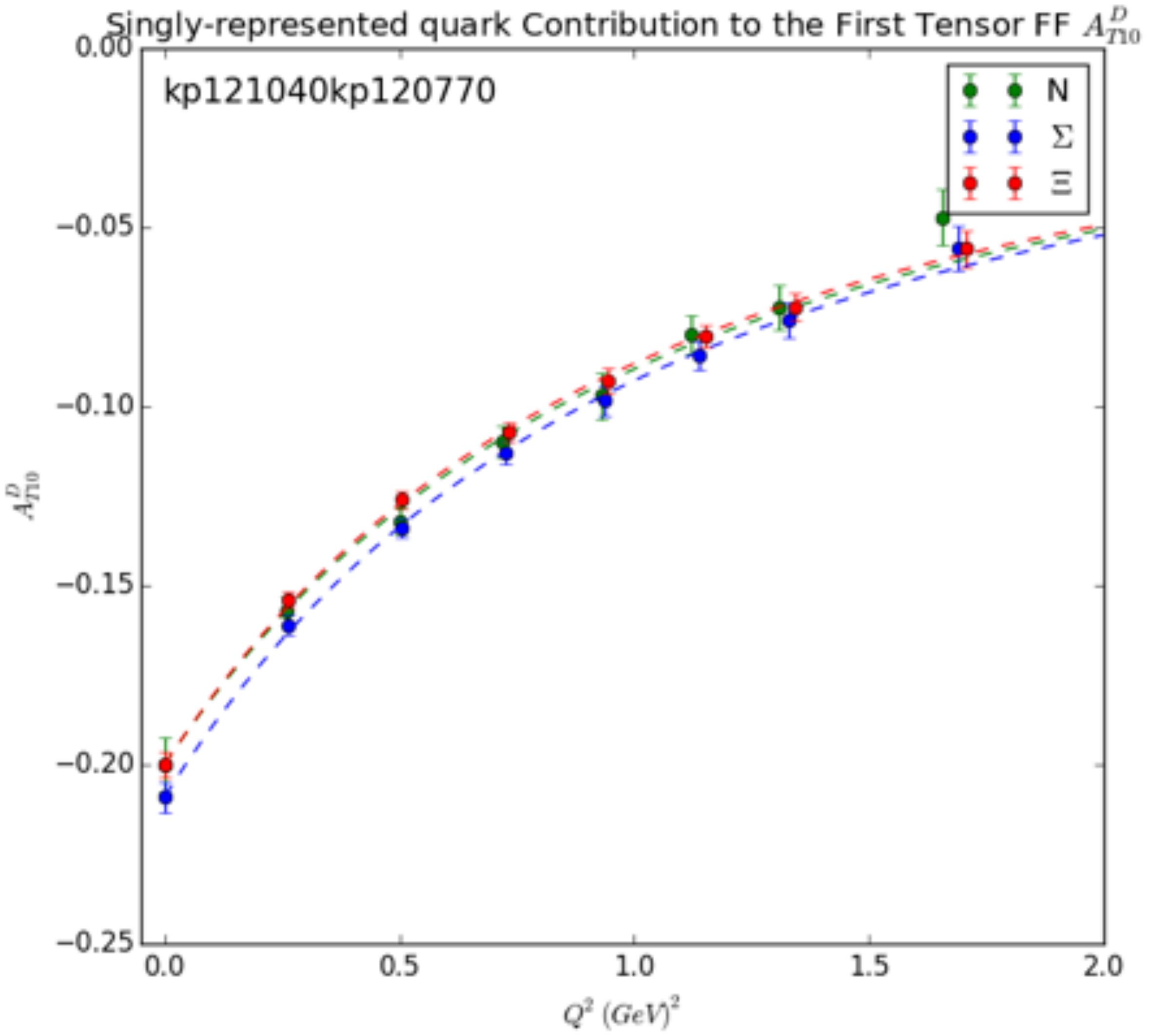
“up”



“down”



A_{T10}

$$\langle P'\Lambda' \mid \overline{\psi}(0) i\sigma^{\mu\nu}\psi(0) \mid P\Lambda\rangle = \overline{u}(P',\Lambda')\{i\sigma^{\mu\nu}A_{T10}(t) + \frac{\overline{P}^{[\mu}\Delta^{\nu]}}{m^2}\tilde{A}_{T10}(t) + \frac{\gamma^{[\mu}\overline{P}^{\nu]}}{2m}B_{T10}(t)\}u(P,\Lambda)$$

$m_\pi \simeq 330\,\mathrm{MeV}$
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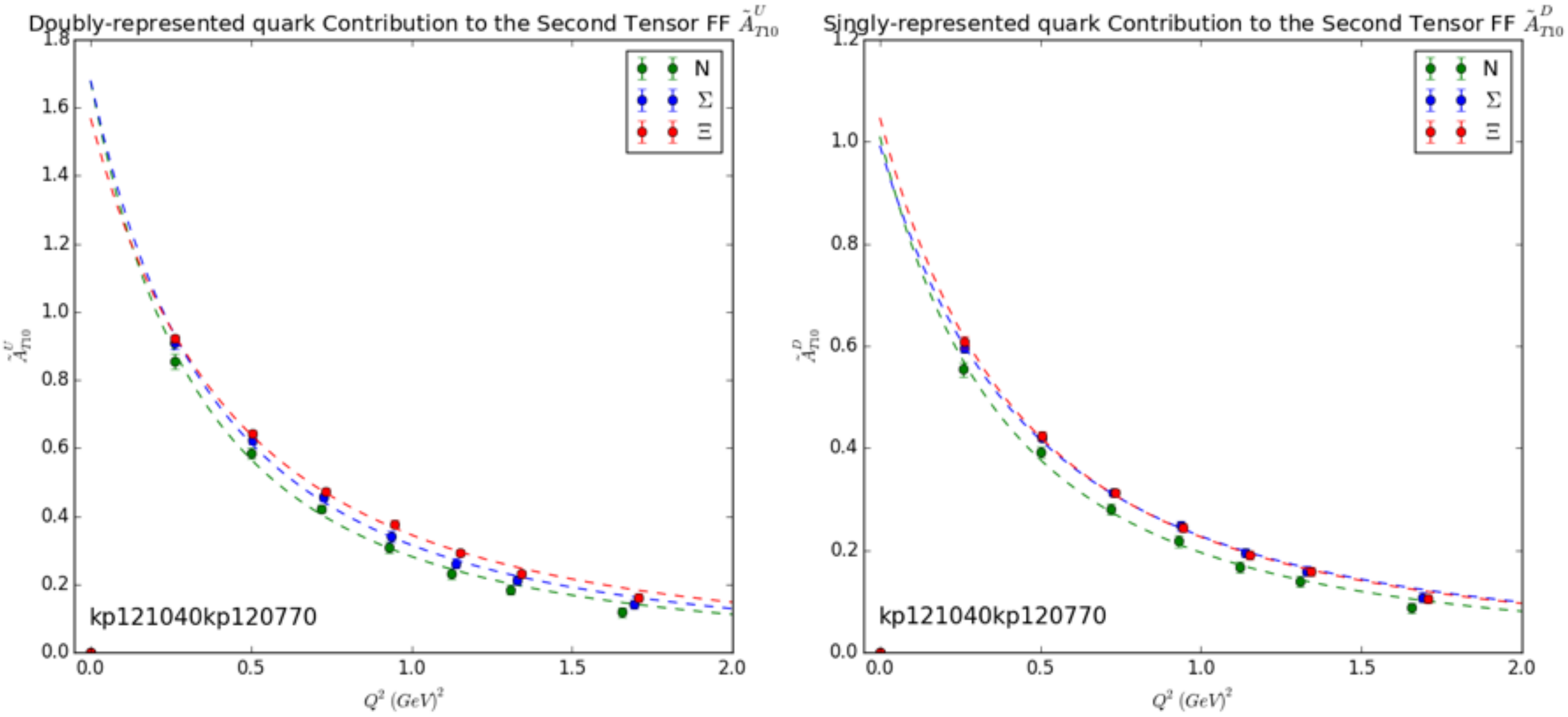
"up" "down"



$$\tilde{A}_{T10}$$

$$\langle P'\Lambda' \mid \overline{\psi}(0) i\sigma^{\mu\nu}\psi(0) \mid P\Lambda\rangle = \overline{u}(P',\Lambda')\{i\sigma^{\mu\nu}A_{T10}(t) + \frac{\overline{P}^{[\mu}\Delta^{\nu]}}{m^2}\tilde{A}_{T10}(t) + \frac{\gamma^{[\mu}\overline{P}^{\nu]}}{2m}B_{T10}(t)\}u(P,\Lambda)$$

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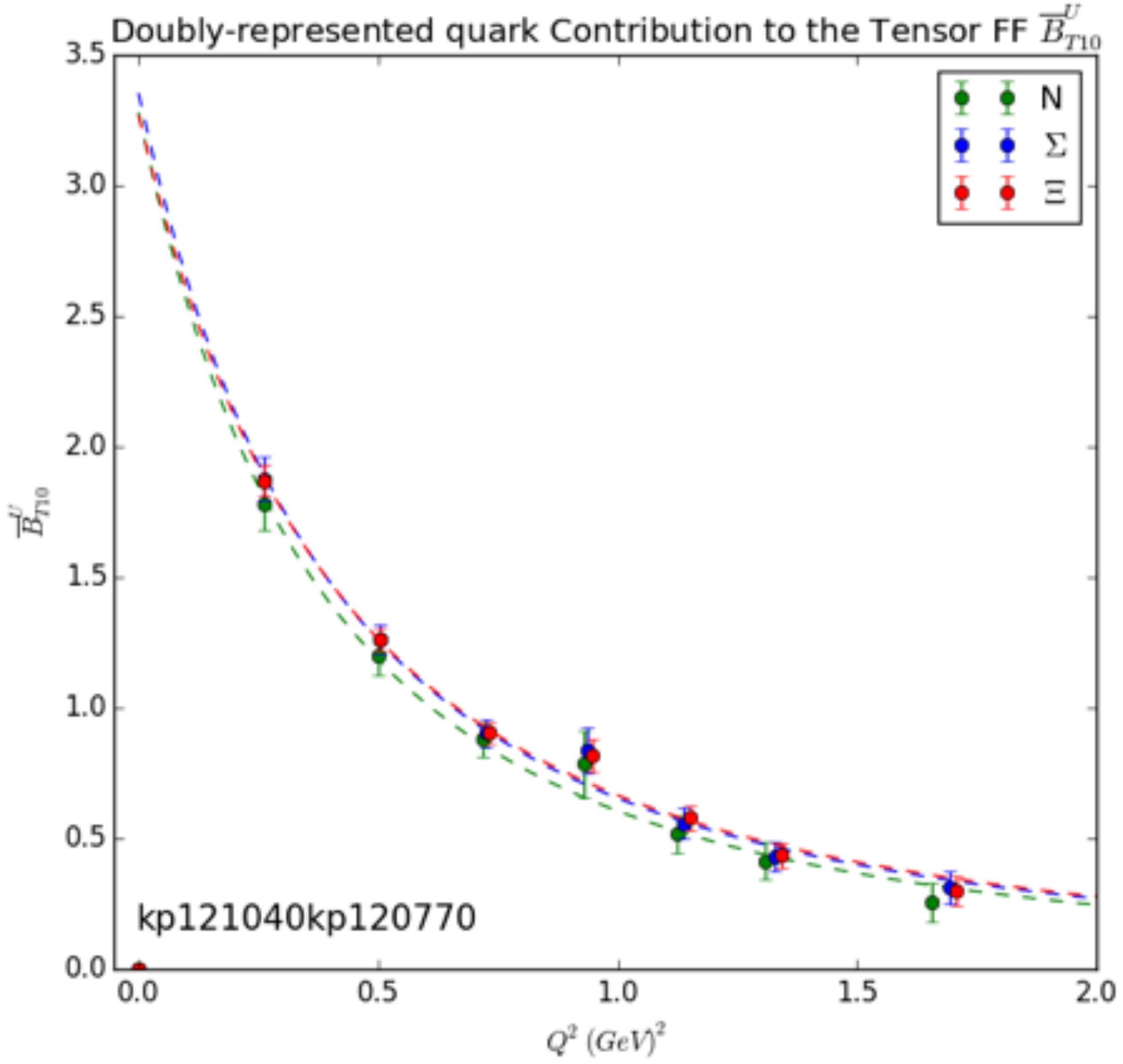
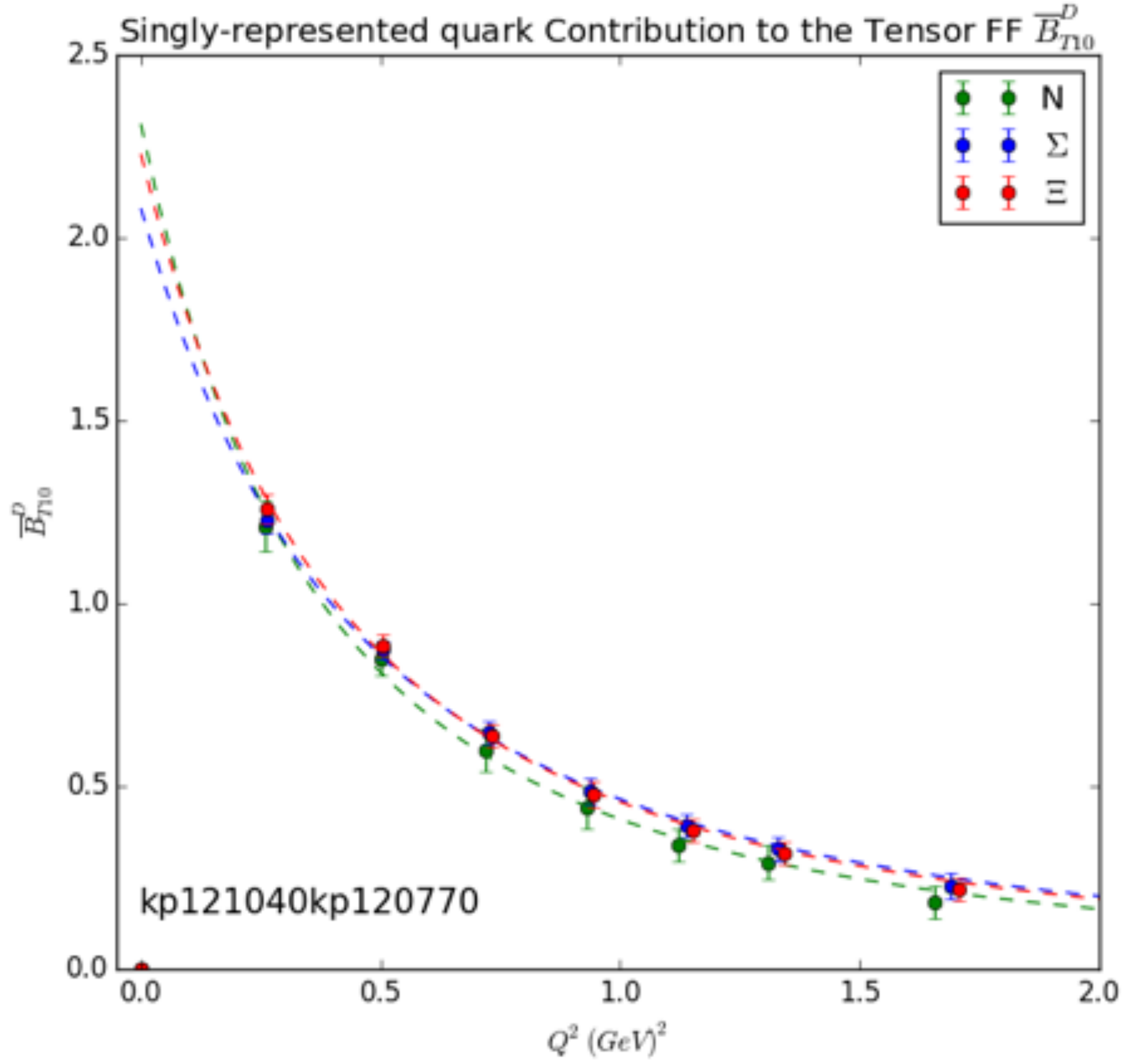


Doubly-represented quark Contribution to the Tensor FF $\overline{B}^{U}_{T10}$

“down”

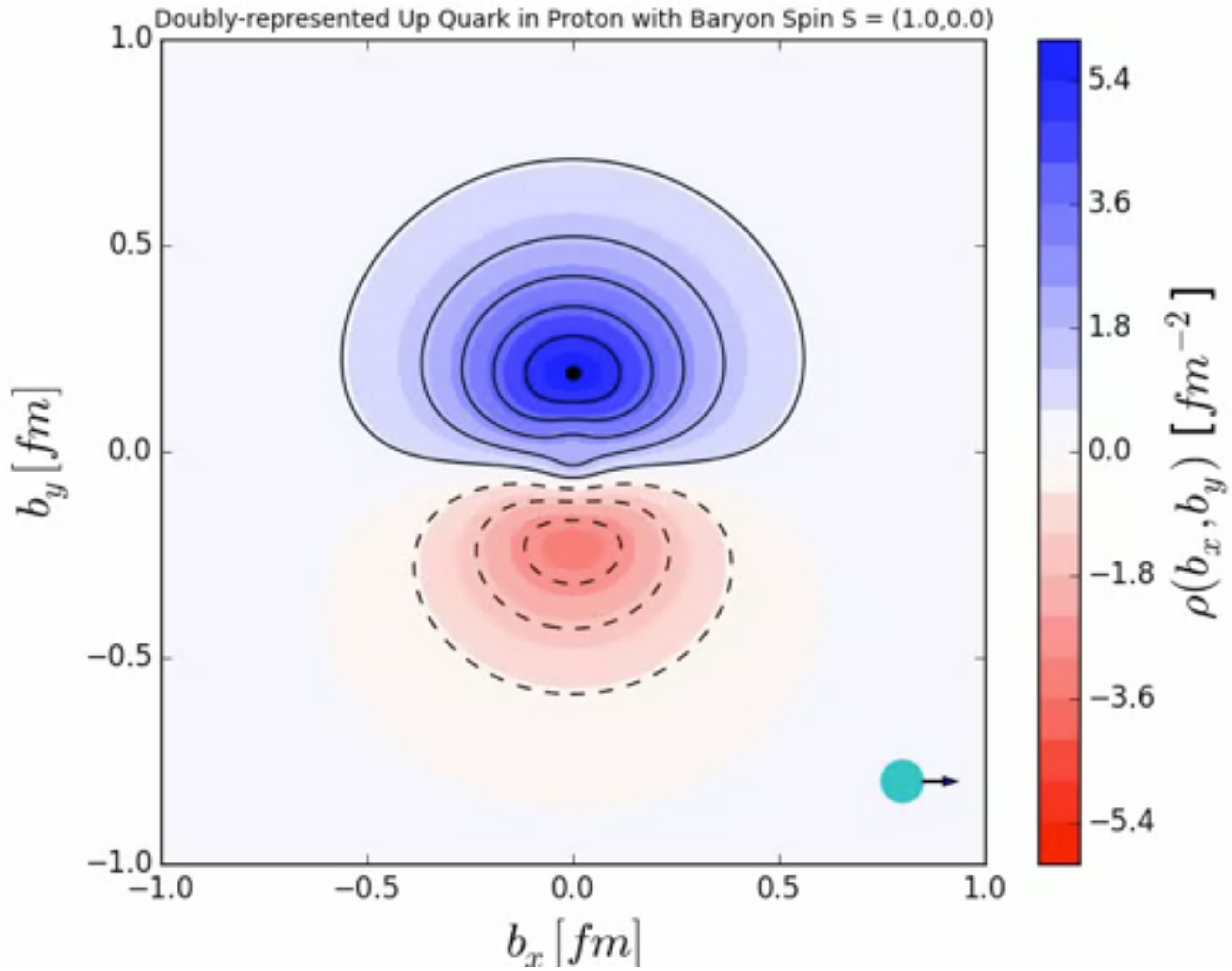


Singly-represented quark Contribution to the Tensor FF $\overline{B}^{D}_{T10}$

$$\overline{B}_{T10} = 2\tilde{A}_{T10} + B_{T10}$$

$$\begin{aligned}\langle P'\Lambda' \mid \overline{\psi}(0) i\sigma^{\mu\nu}\psi(0) \mid P\Lambda\rangle =& \overline{u}(P',\Lambda')\{i\sigma^{\mu\nu}A_{T10}(t) \\ &+ \frac{\overline{P}^{[\mu}\Delta^{\nu]}}{m^2}\tilde{A}_{T10}(t) + \frac{\gamma^{[\mu}\overline{P}^{\nu]}}{2m}B_{T10}(t)\}u(P,\Lambda)\end{aligned}$$

preliminary

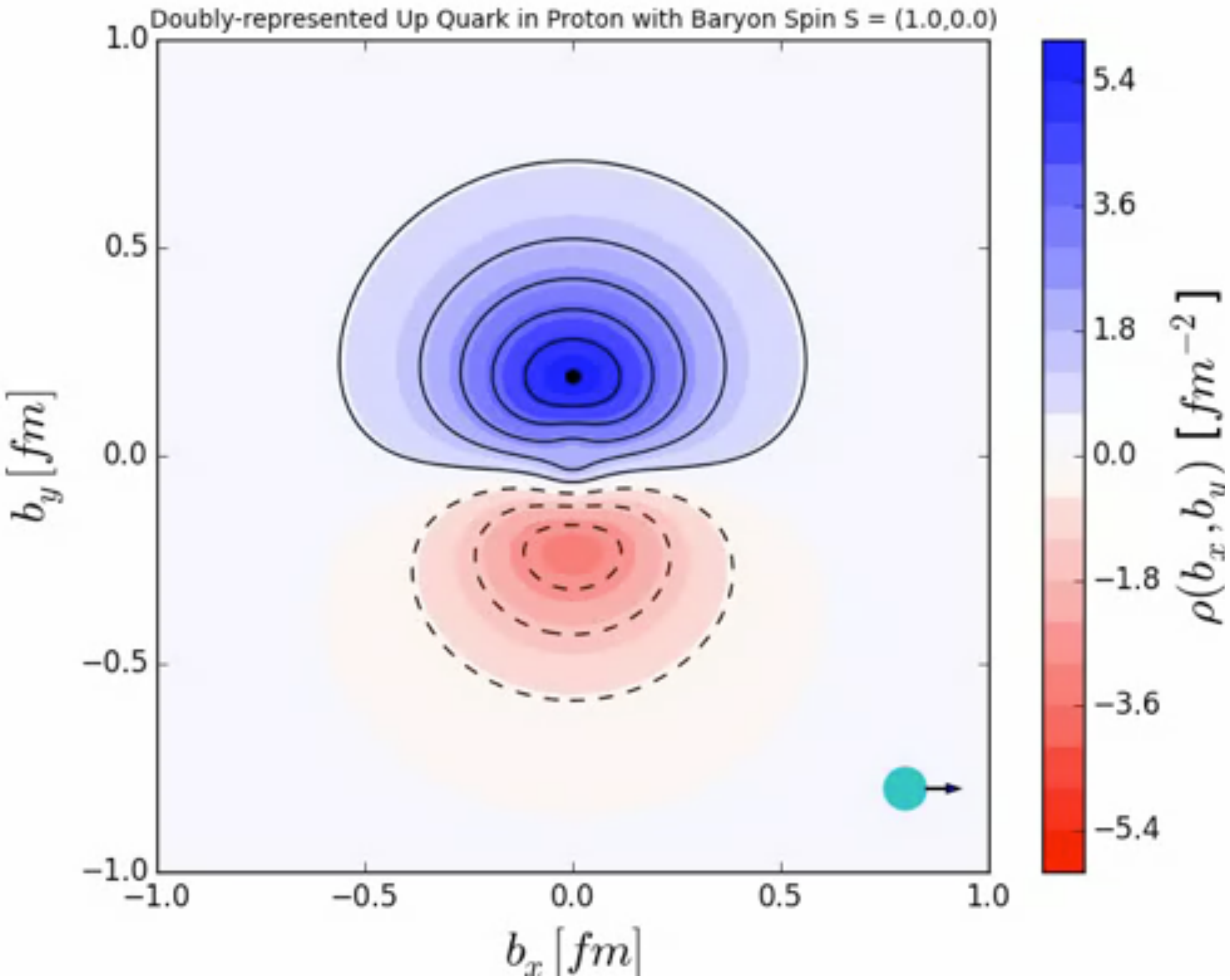


unpolarised quarks in transverse polarised nucleon

up-quark transverse densities

“chirally extrapolated”

preliminary



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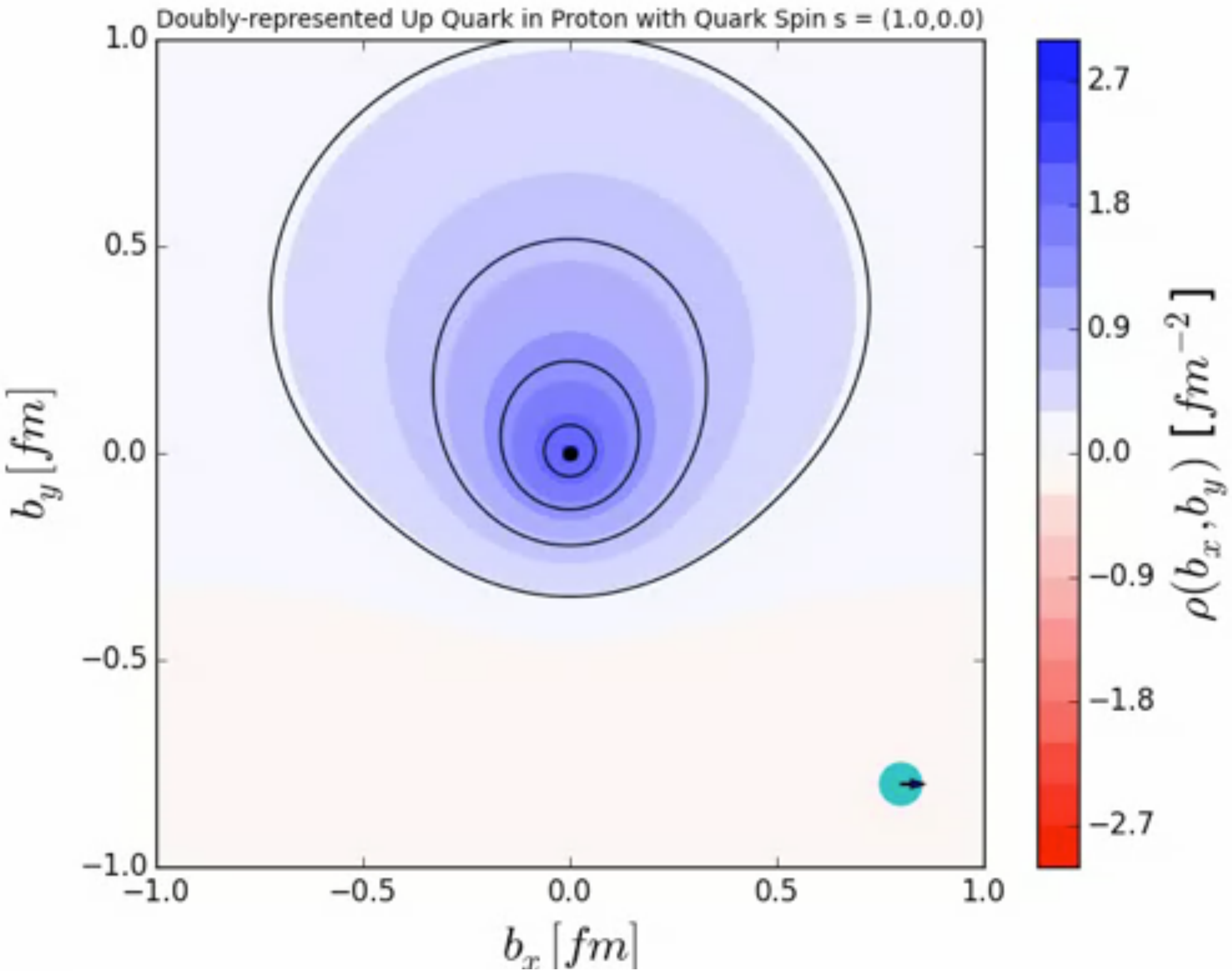

transverse polarised quarks in unpolarised nucleon

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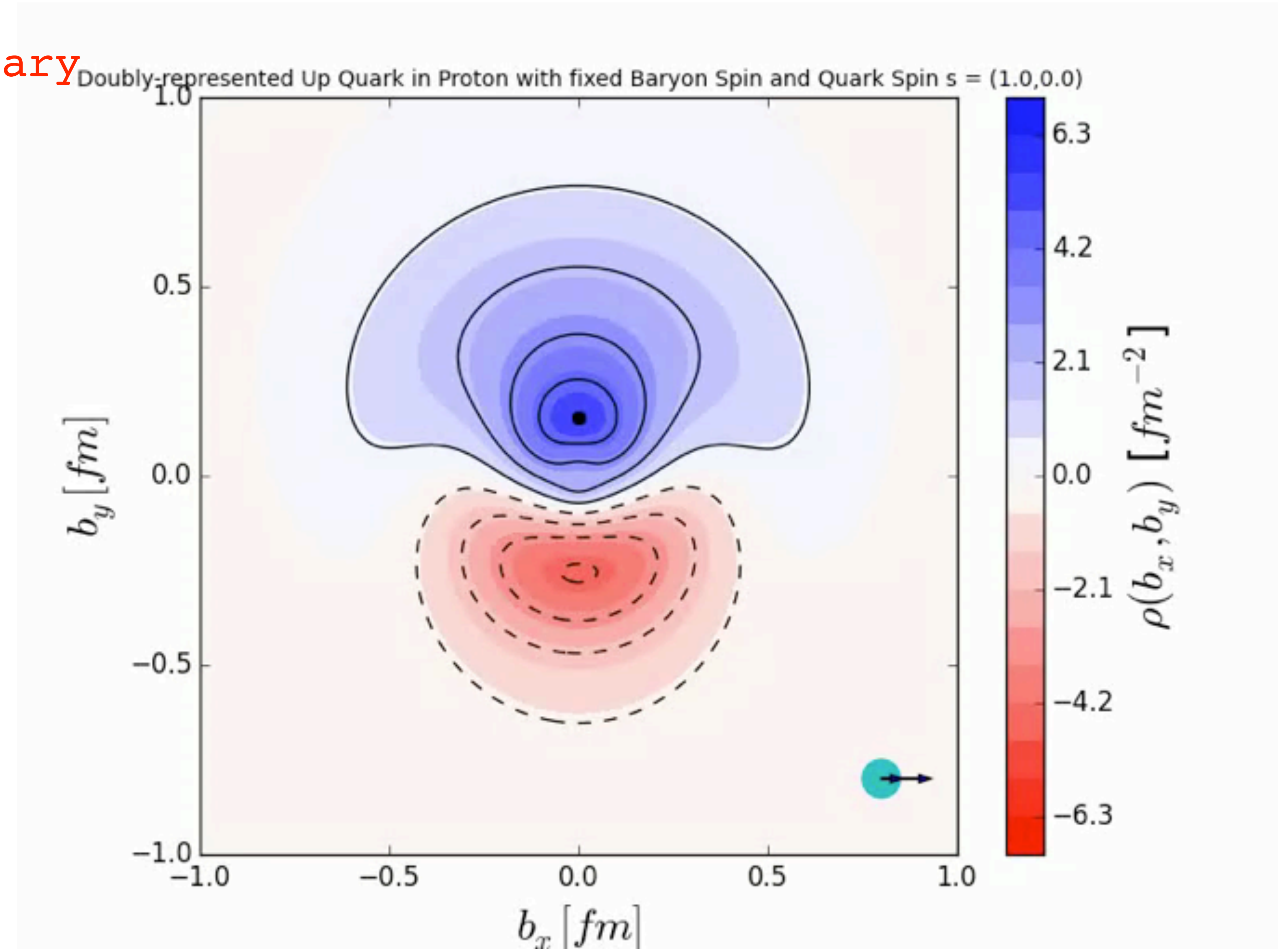
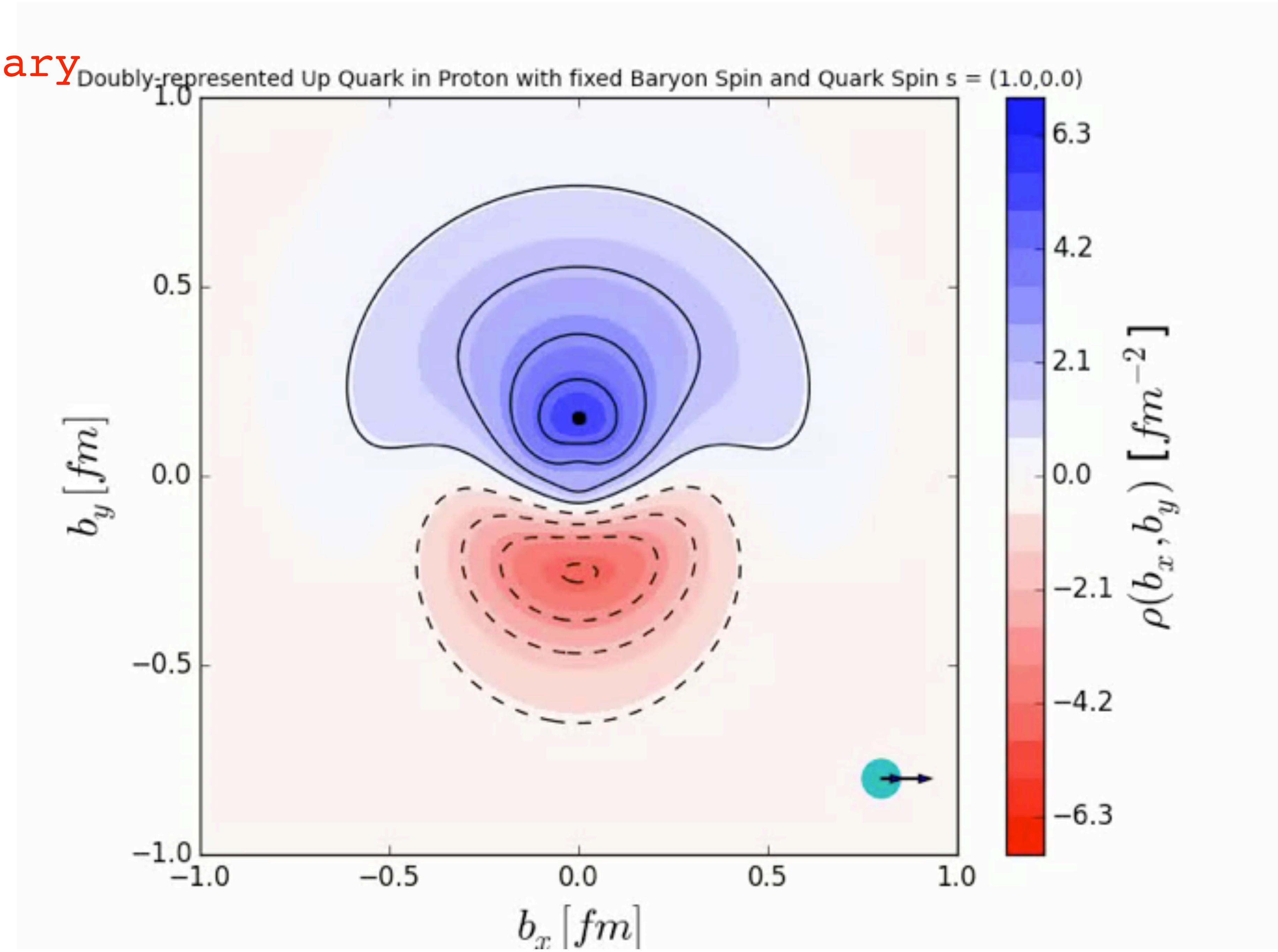


s-S-b correlations

Nucleon

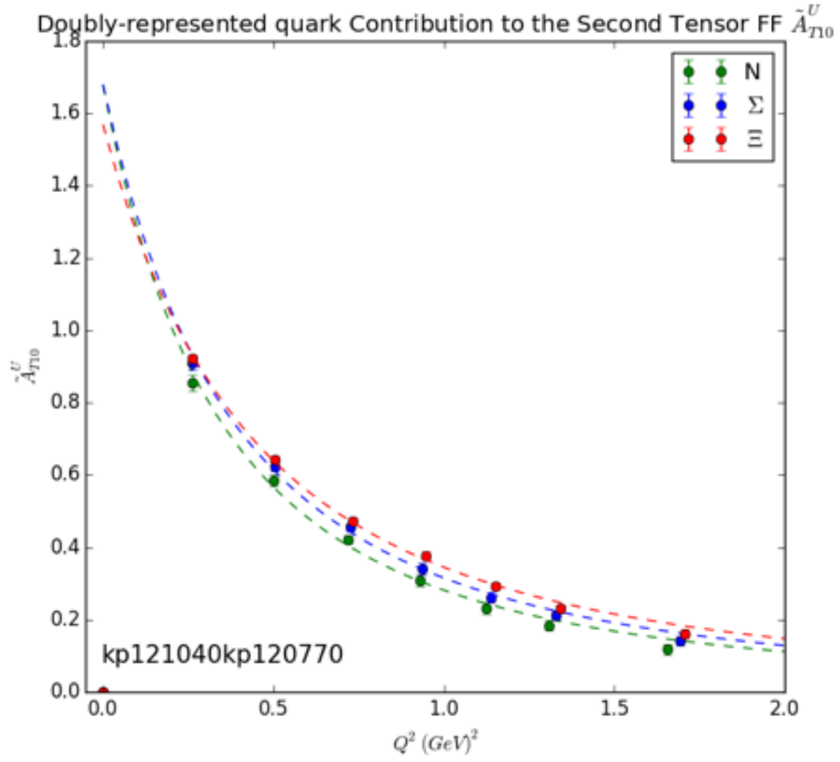
preliminary


Doubly-represented Up Quark in Proton with fixed Baryon Spin and Quark Spin s = (1.0,0.0)

s-S-b correlations

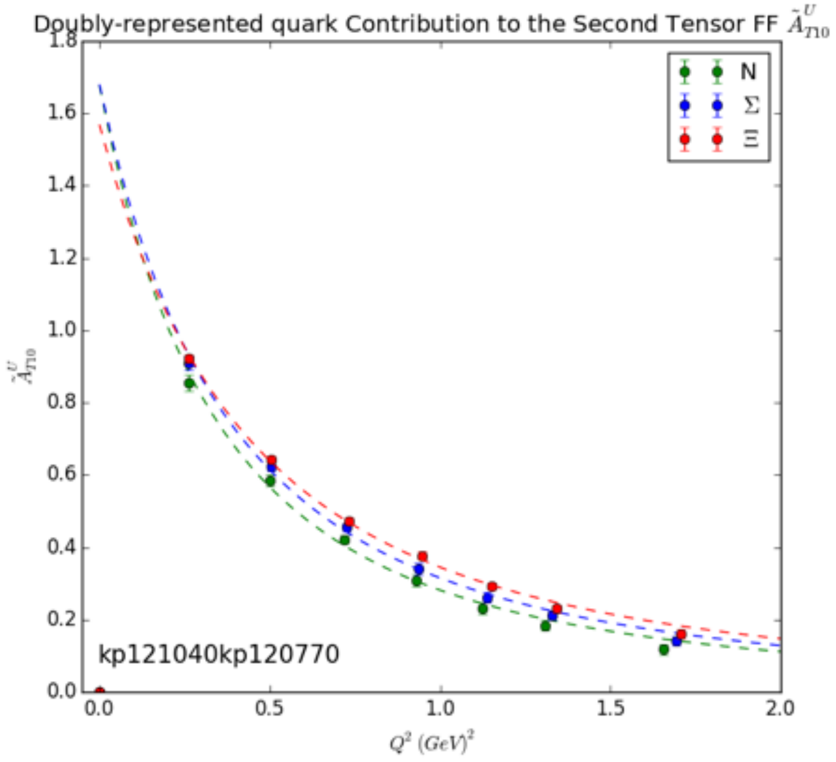
Nucleon

Large momentum transfer?



Large momentum transfer?

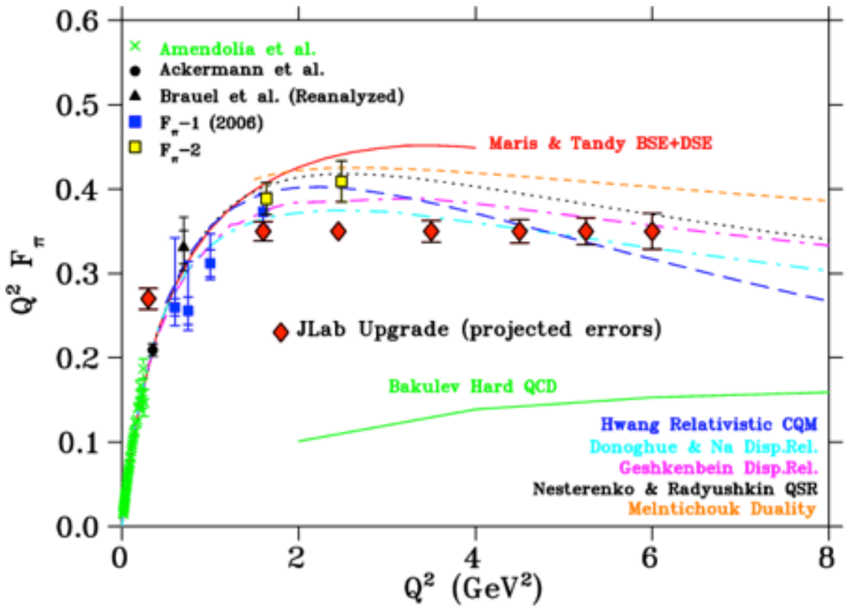
- Fourier transforms sensitive to “model” of large Q^2 behaviour

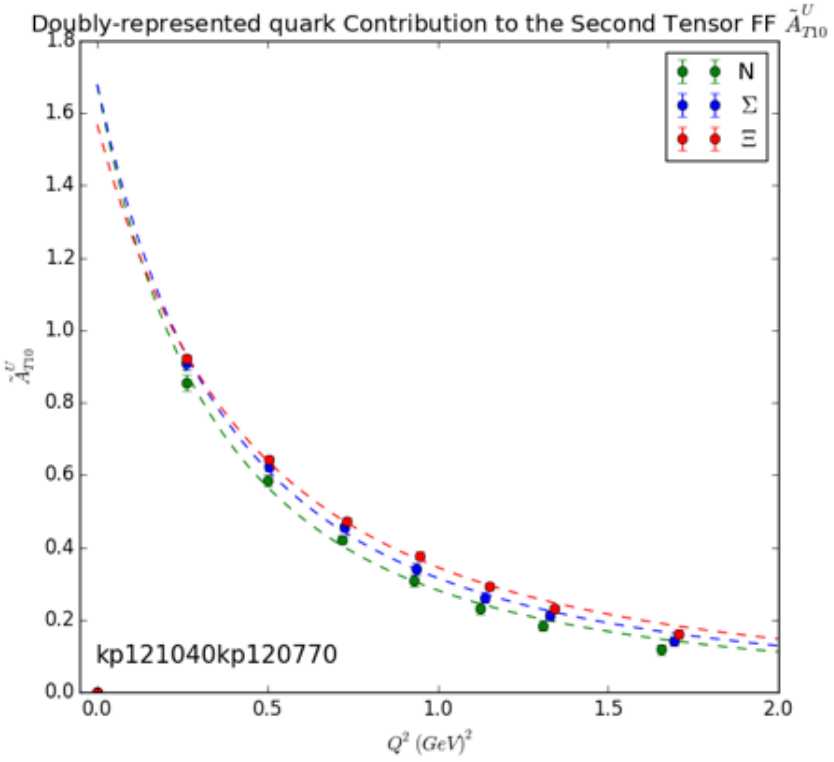


Large momentum transfer?

- Fourier transforms sensitive to "model" of large Q^2 behaviour

- Can lattice help inform us about the pion form factor?





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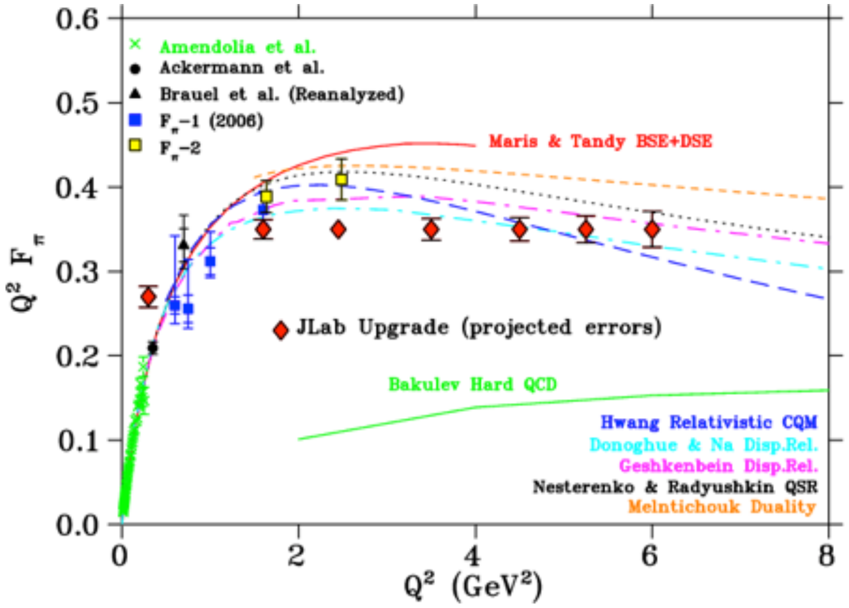
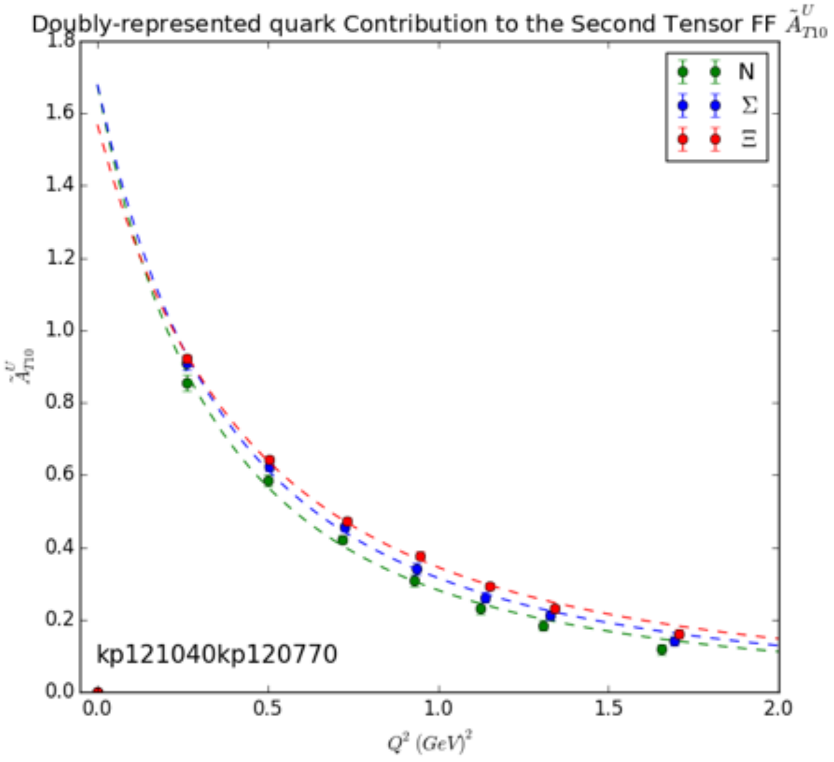
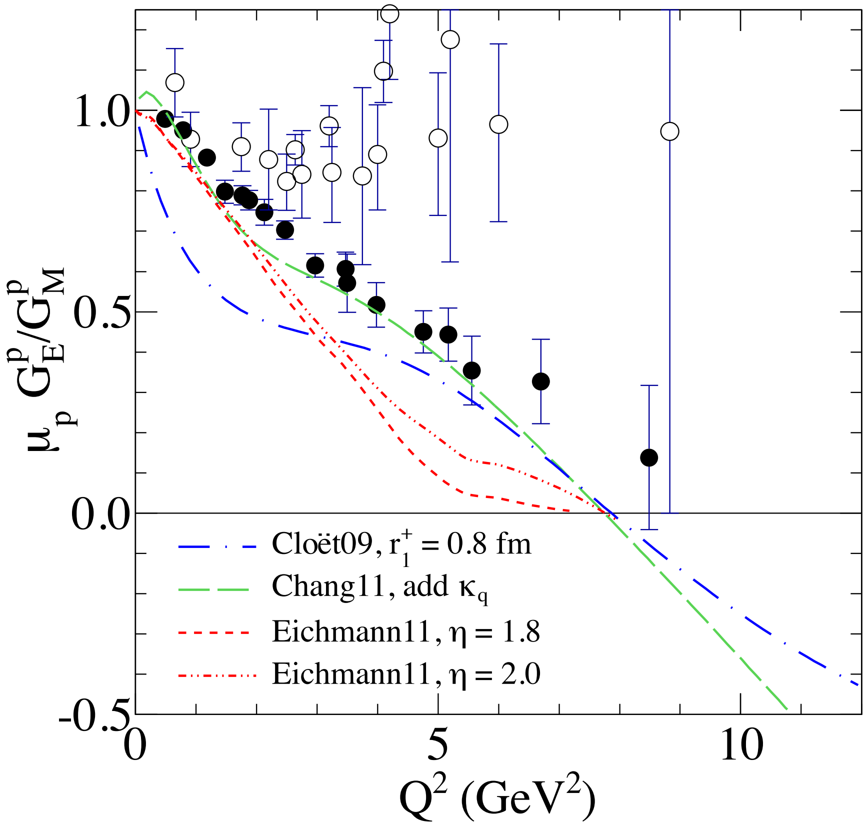


- What about the proton & GE/GM?





Form factors from lattice QCD

Lattice QCD: 2-pt function ⇒ energy eigenstates

- QCD path integral: discretise Euclidean spacetime; derivatives to finite difference; gluon field encoded in gauge links; fermion actions & chiral symmetry; etc.
- Stochastic sampling of vacuum gluon configurations, weighted by QCD action

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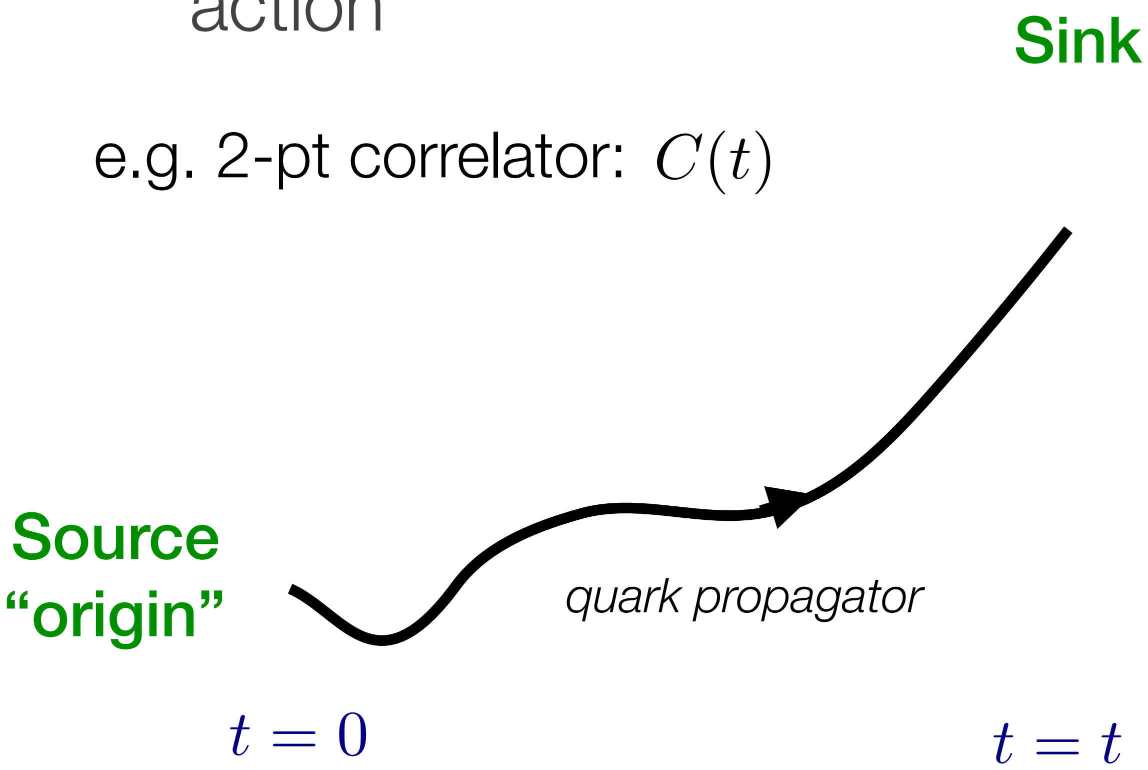
e.g. 2-pt correlator: $C(t)$



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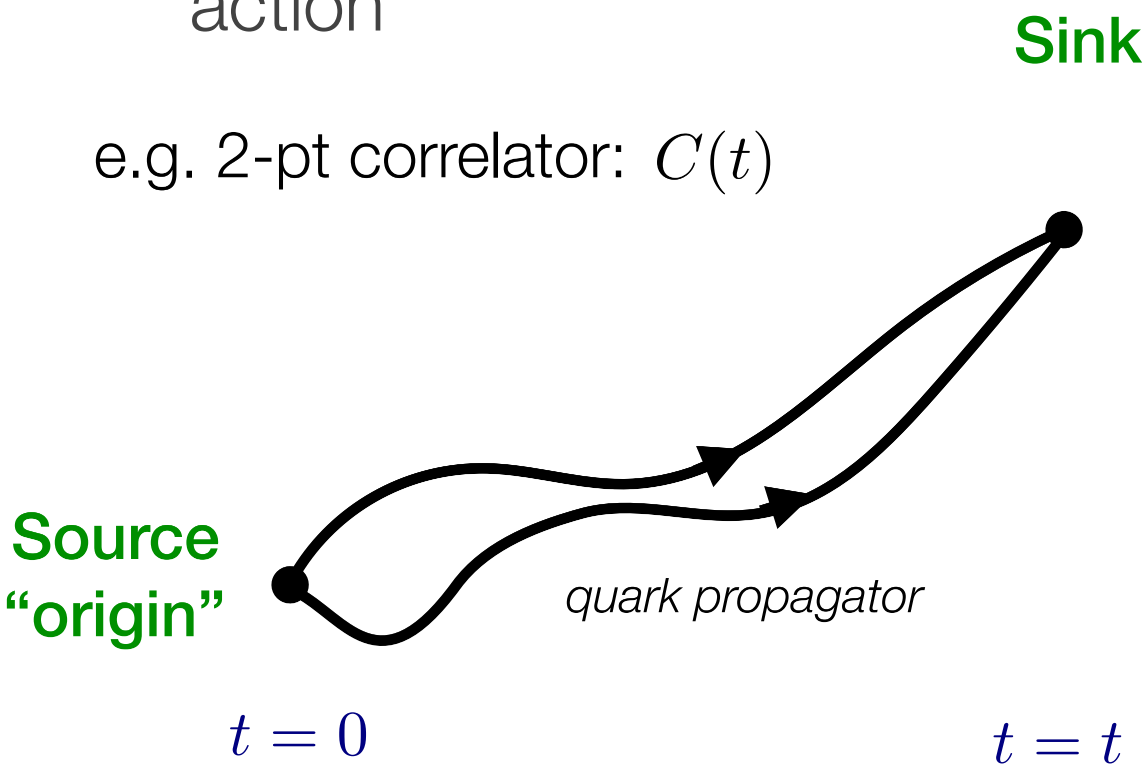

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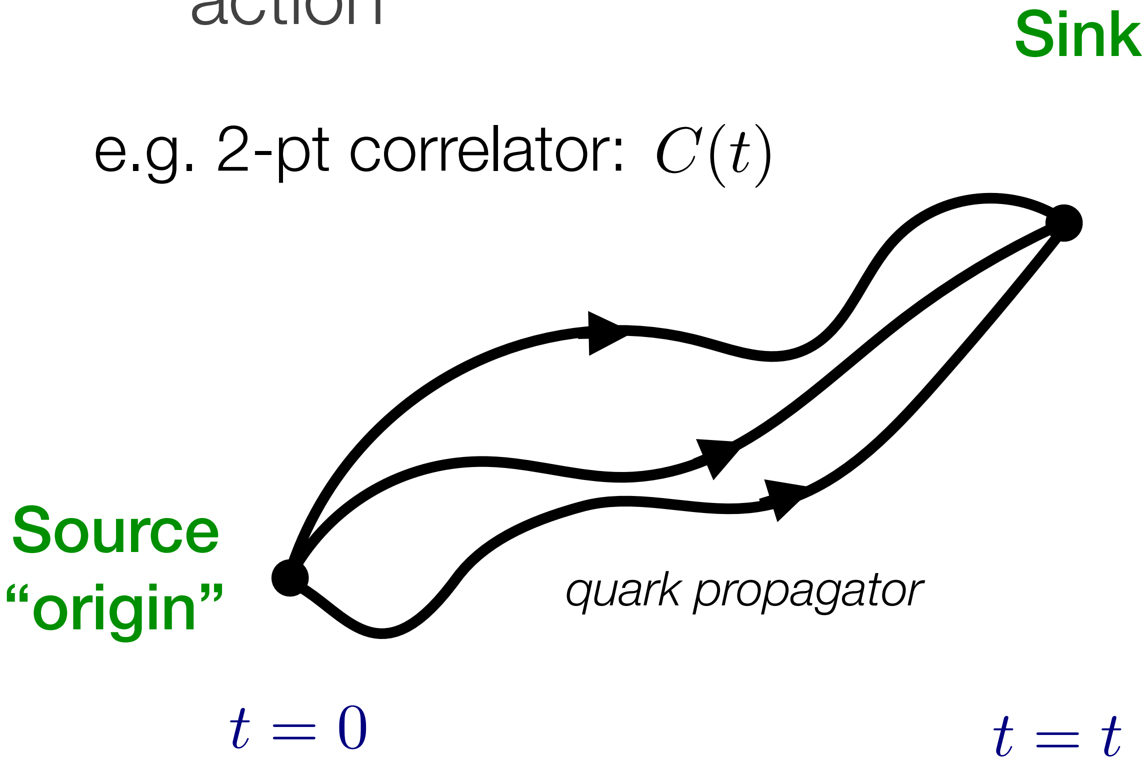

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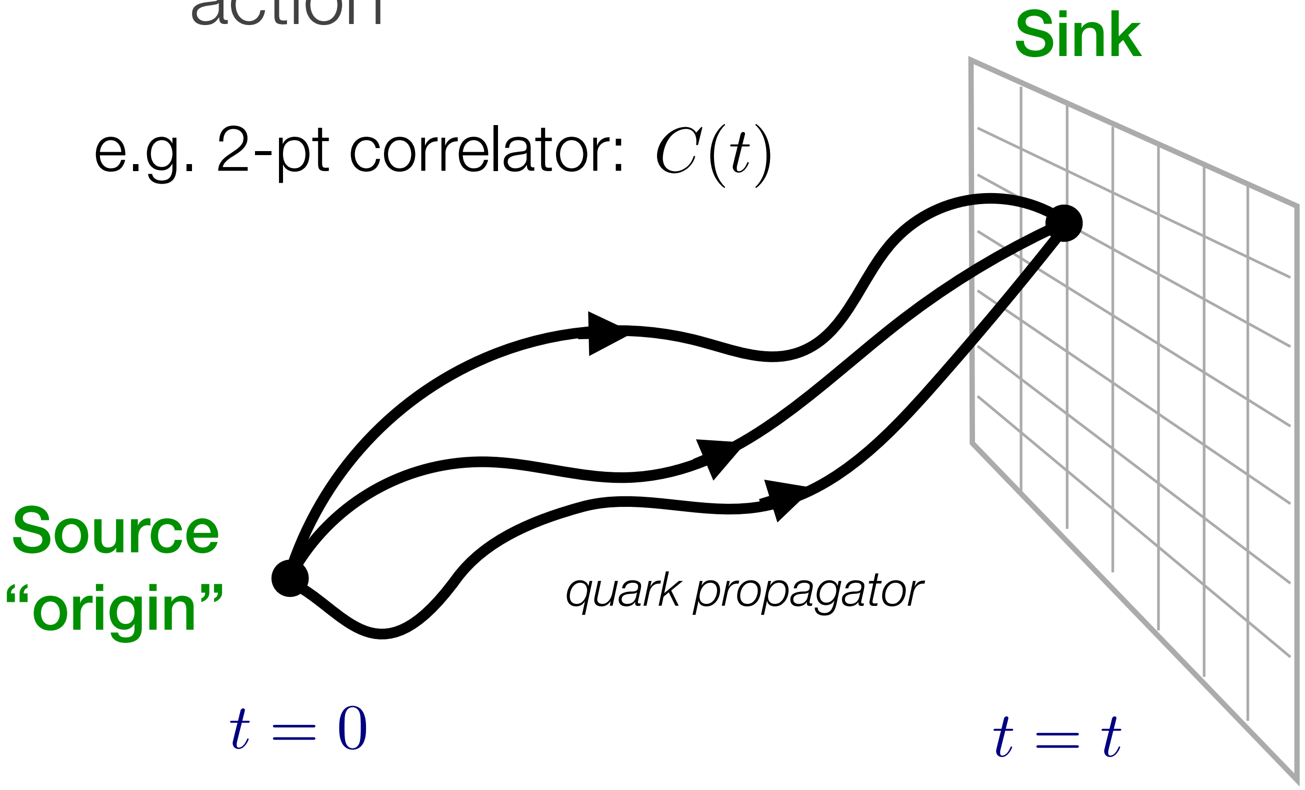

Fourier project onto 3-volume at sink
⇒ definite 3-momentum; e.g. $\mathbf{p}' = 0$

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- Stochastic sampling of vacuum gluon configurations, weighted by QCD action

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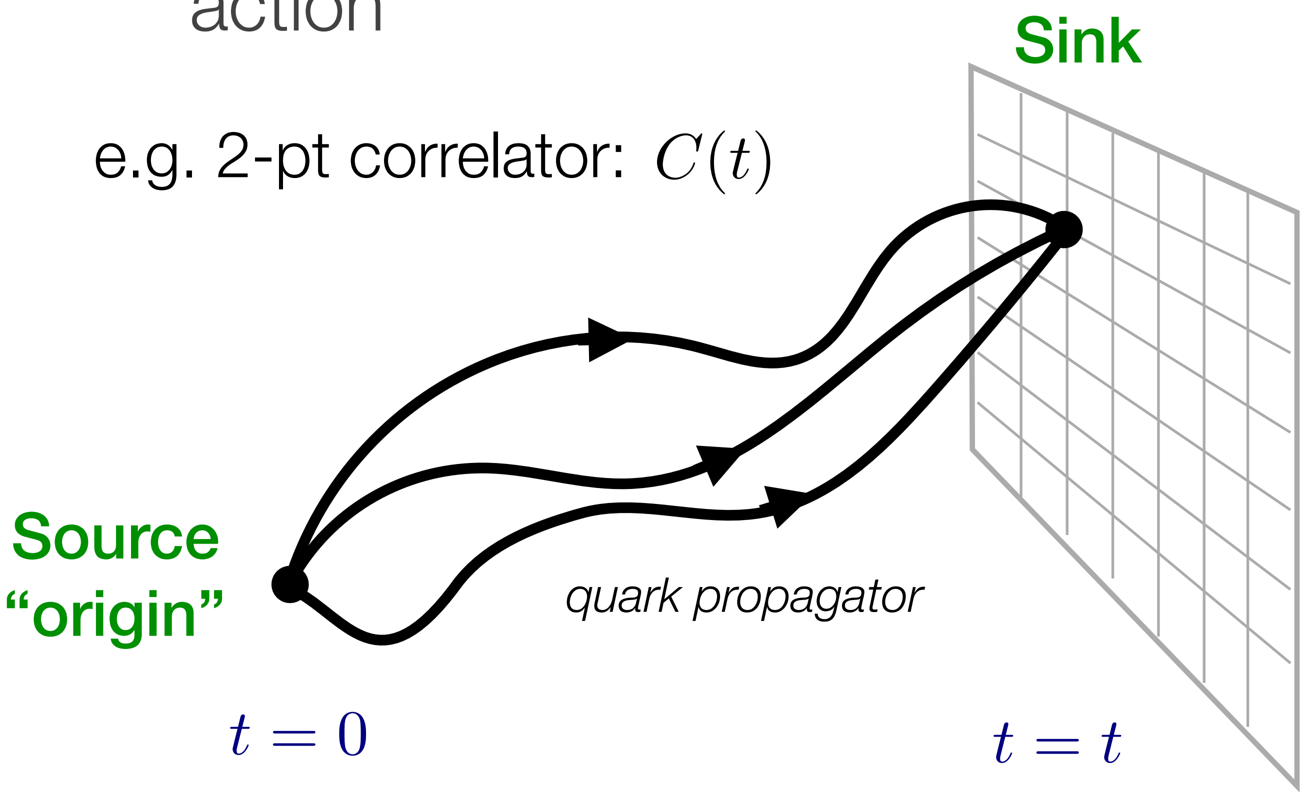


Fourier project onto 3-volume at sink
⇒ definite 3-momentum; e.g. $\mathbf{p}' = 0$

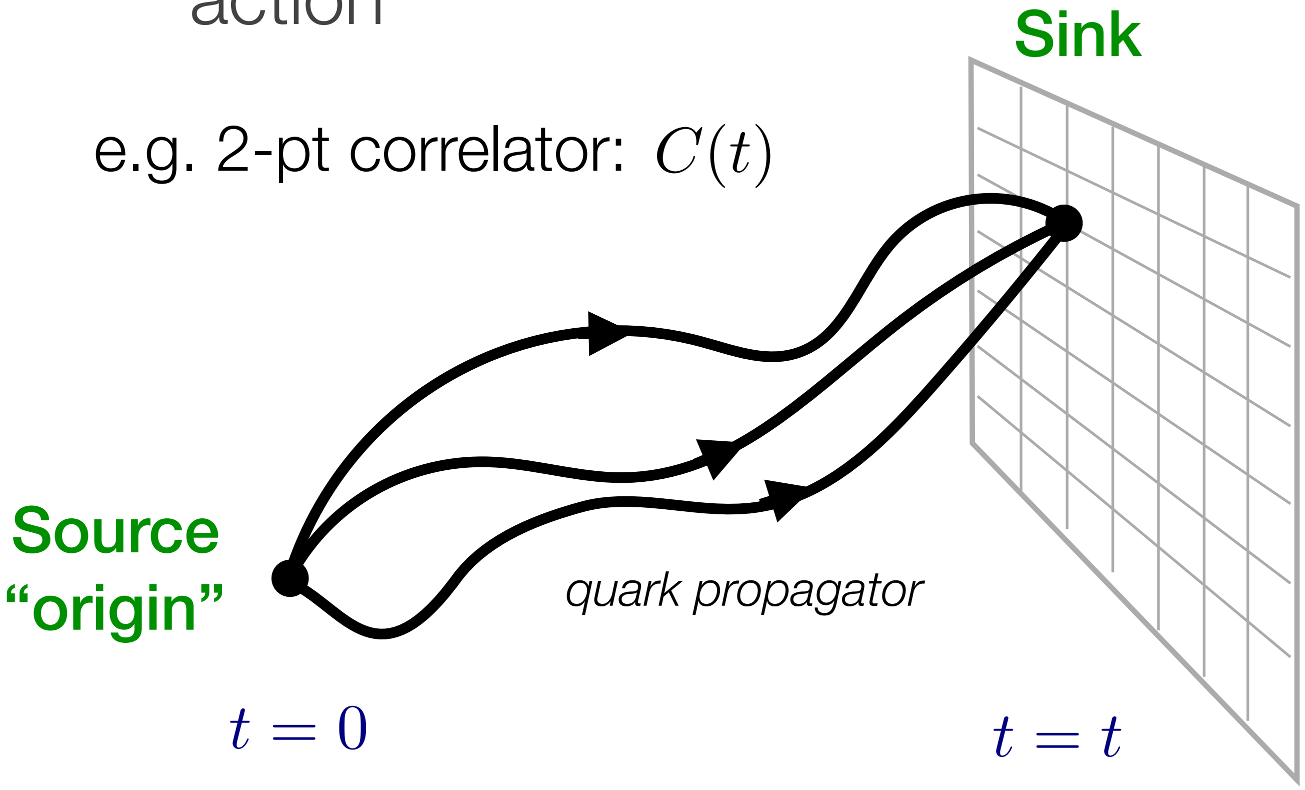
Euclidean time evolution: $\exp(-Ht)$

lowest energy state dominates at large *t*

Lattice QCD: 2-pt function ⇒ energy eigenstates

- QCD path integral: discretise Euclidean spacetime; derivatives to finite difference; gluon field encoded in gauge links; fermion actions & chiral symmetry; etc.

- Stochastic sampling of vacuum gluon configurations, weighted by QCD action



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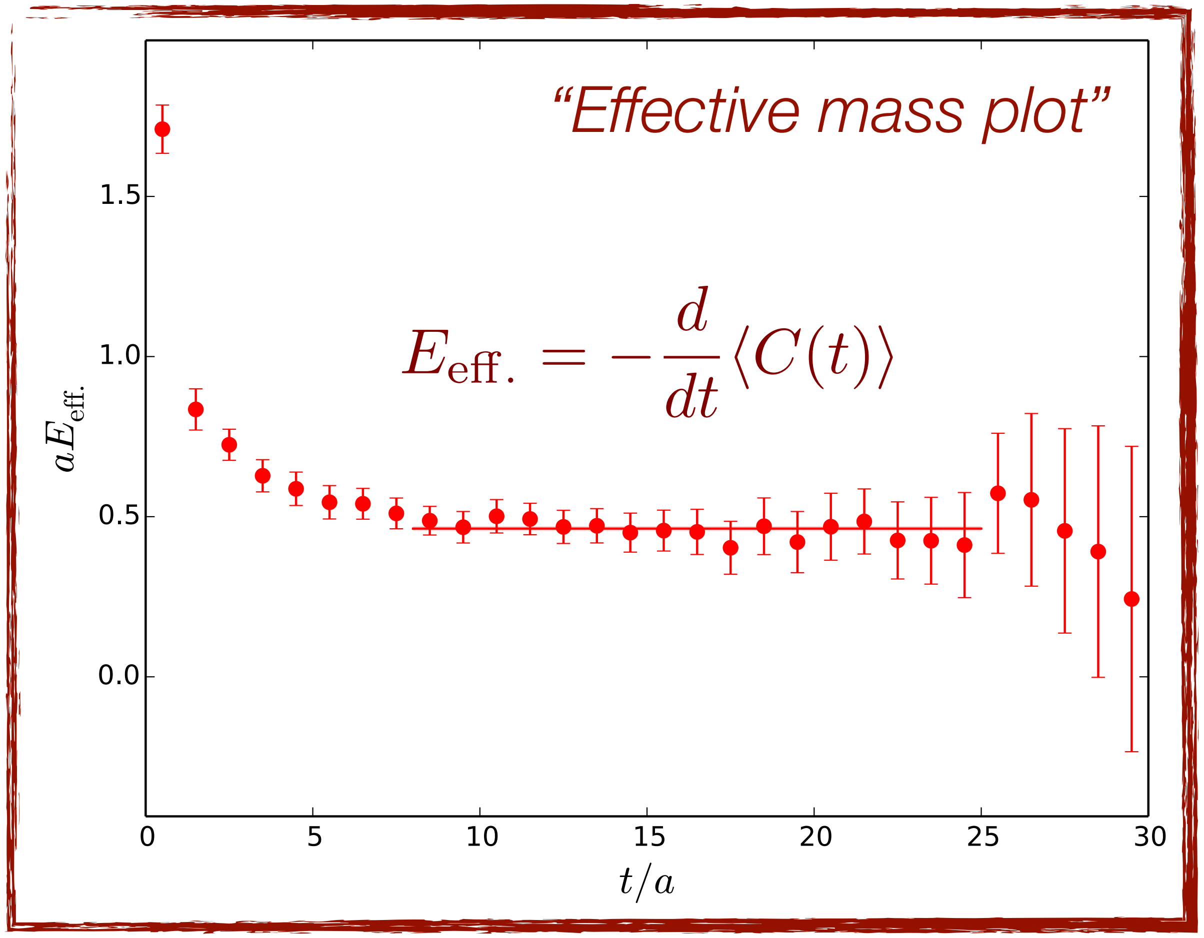

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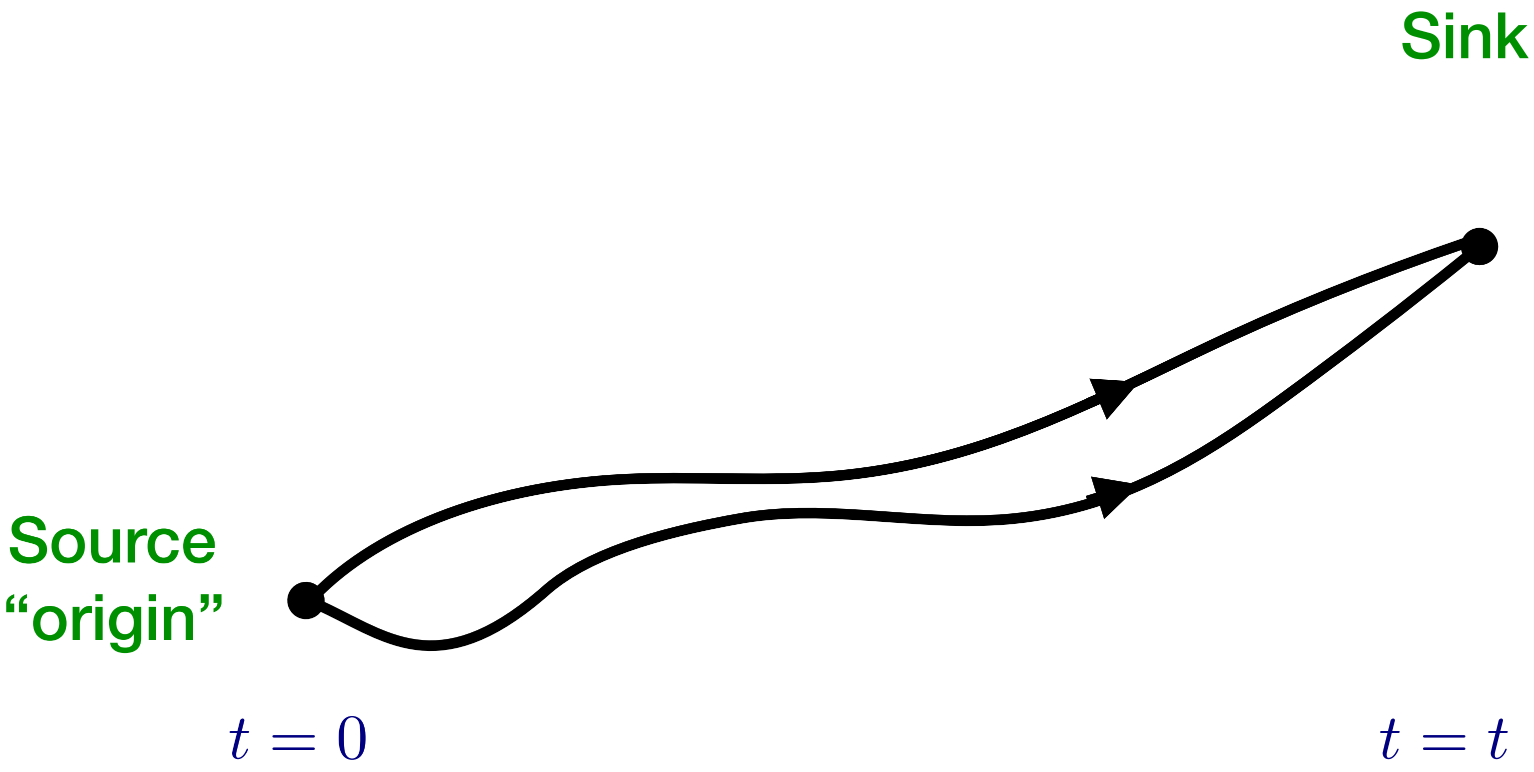
- 3-pt correlator



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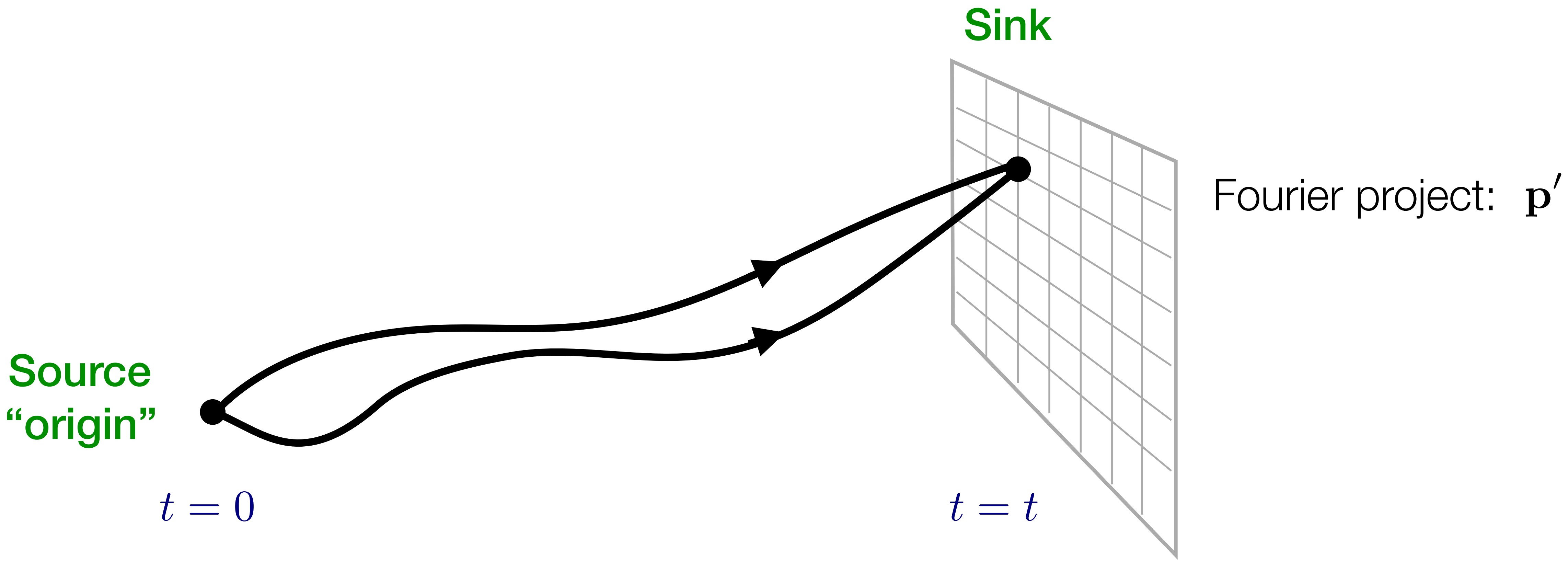

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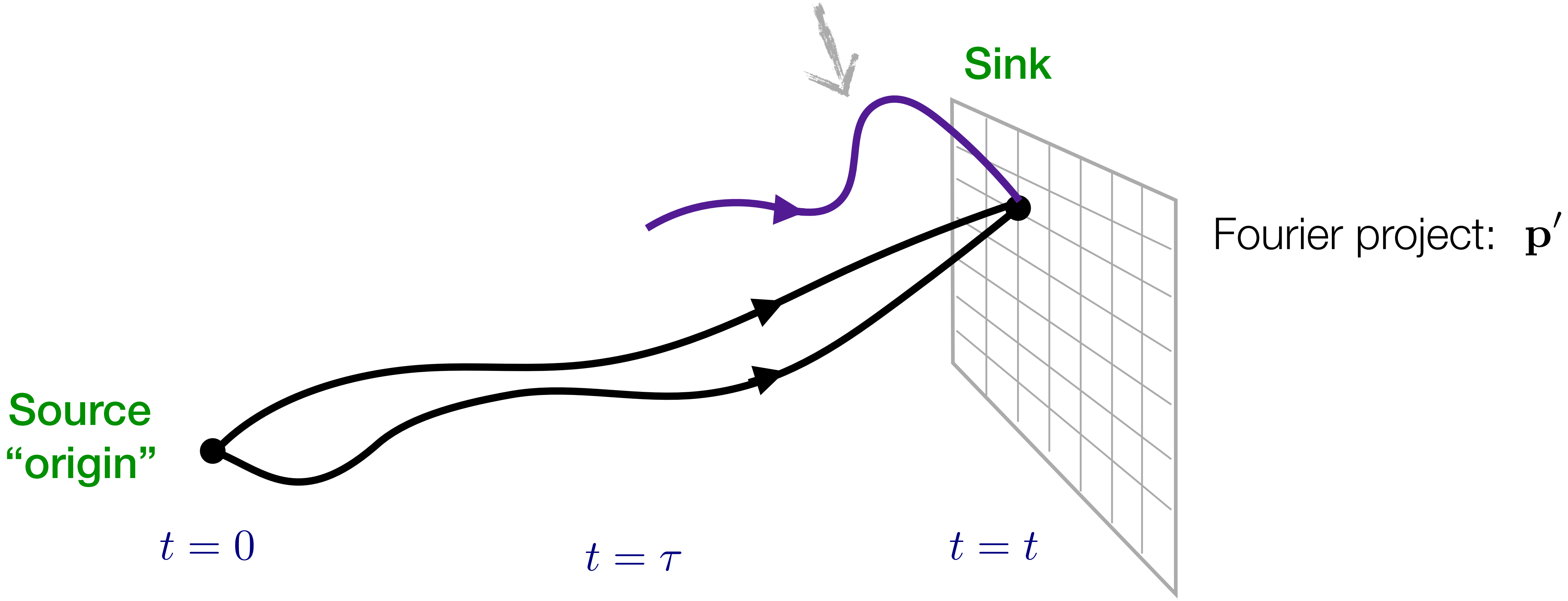

New quark propagator solution:
inverted from “di-quark" at sink

Sink

Fourier project: $\mathbf{p}'$

Source
“origin”

$t = 0$

$t = \tau$

$t = t$

Lattice QCD: 3-pt function ⇒ matrix elements

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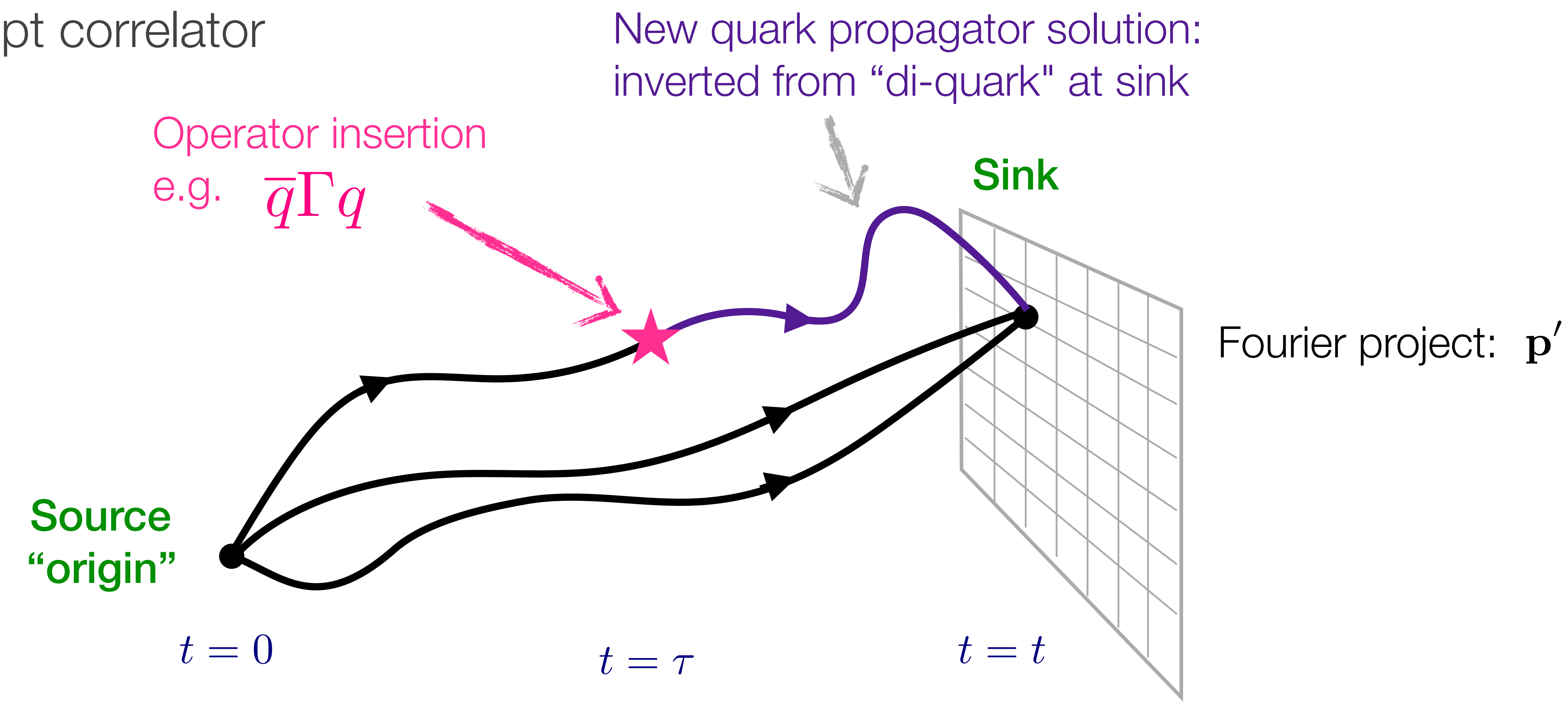

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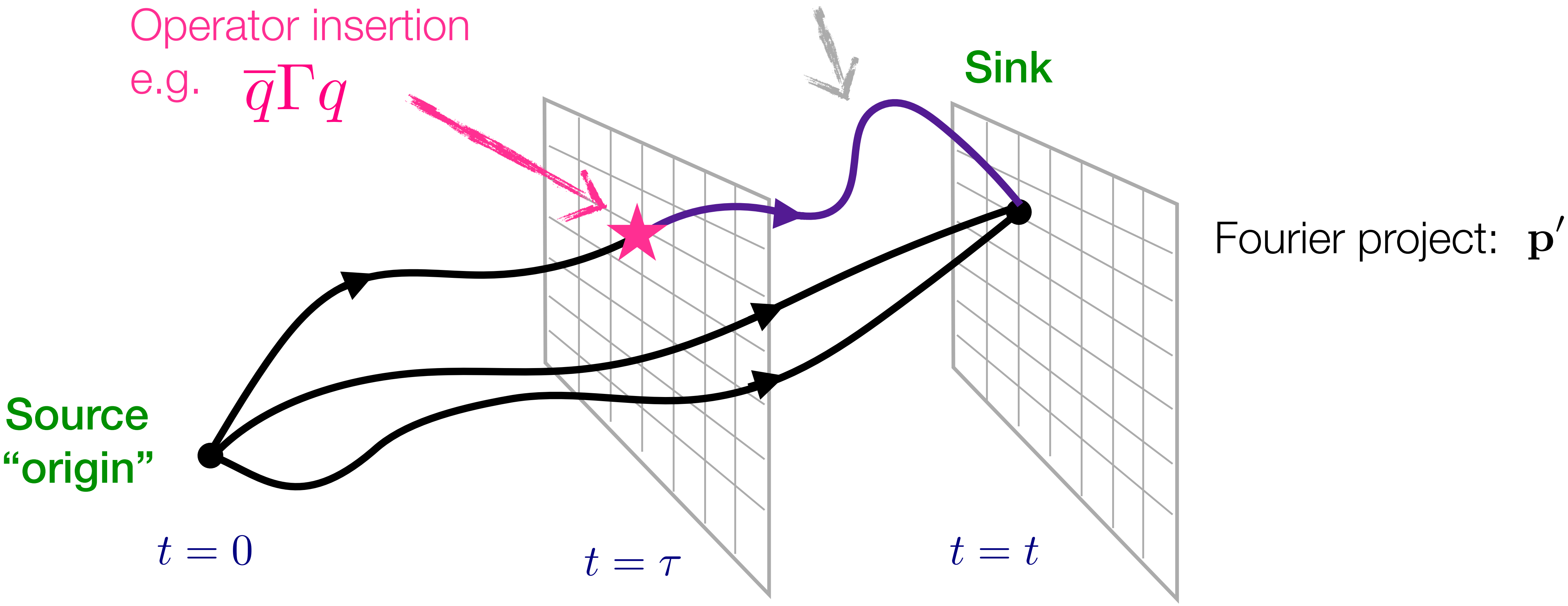

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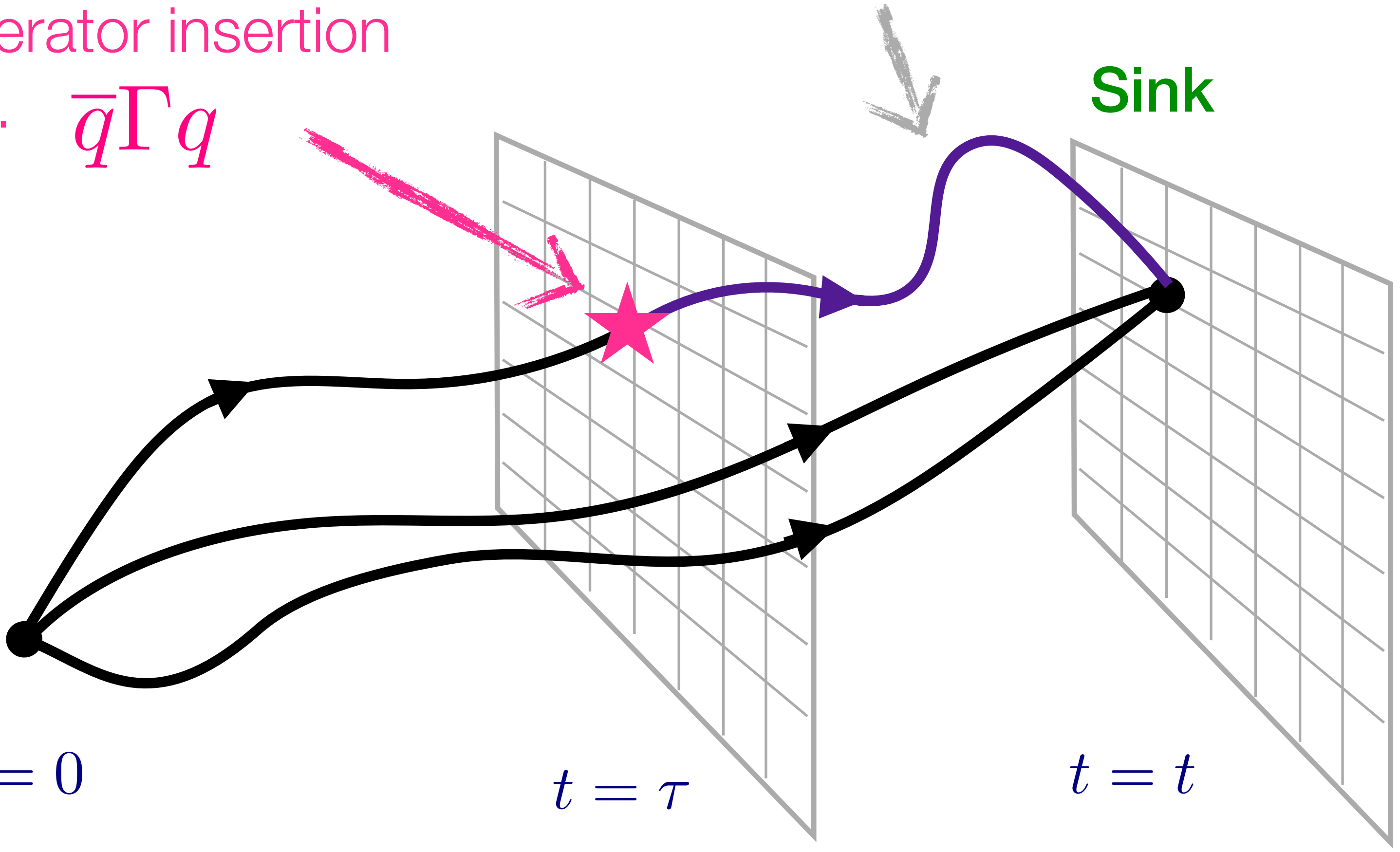

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Source momentum fixed
by momentum conservation:
$\mathbf{p} = \mathbf{p}' - \mathbf{q}$

Fourier project
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Lattice QCD: 3-pt function ⇒ matrix elements

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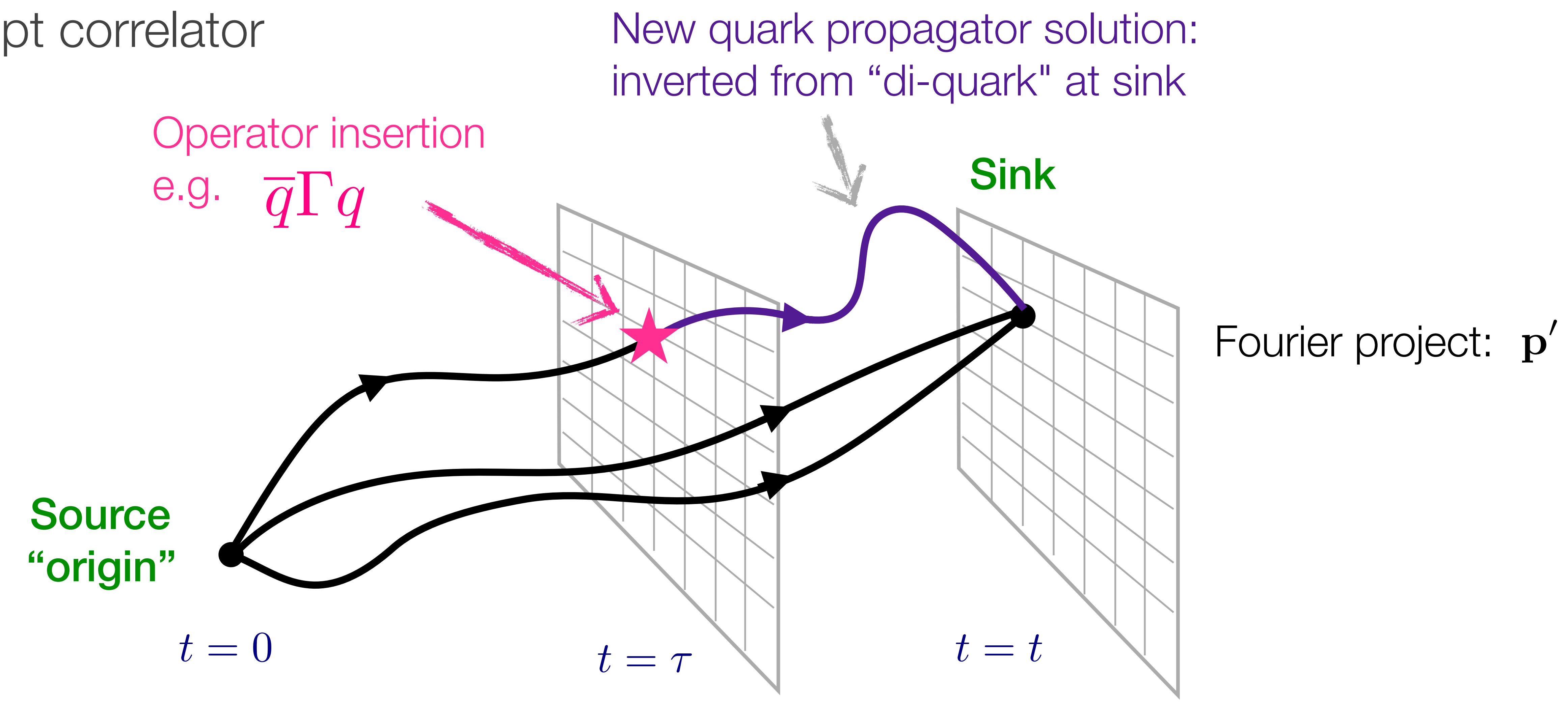

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Source “origin”

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Fourier project momentum insertion: $\mathbf{q}$

Remove time dependence by dividing out 2-pt correlators

$$\frac{\langle C_3(t,\tau;\mathbf{p}',\mathbf{q})\rangle}{\langle C_2(t-\tau,\mathbf{p}')\rangle\langle C_2(\tau,\mathbf{p})\rangle} \sim \langle N(\mathbf{p}')|\mathcal{O}(\mathbf{q})|N(\mathbf{p})\rangle$$

Lattice QCD: matrix elements ⇒ form factors

- Determine matrix element from ratio of lattice 3-pt and 2-pt correlators

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- Extraction of corresponding form factors from usual tensor decomposition
  - e.g. vector form factors:

$$\langle N(\mathbf{p}')|\overline{q}\gamma^\mu q|N(\mathbf{p})\rangle = \overline{u}(p',s')\left[\gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_N}F_2(Q^2)\right]u(p,s)$$

$$Q^2 = \mathbf{q}^2 - (E_{\mathbf{p}'} - E_{\mathbf{p}})^2$$

  - choose variety of current components; spin alignments; momentum transfer directions. Generally an over-constrained set of equations to isolate form factors

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ratios

3-pt & 2-pt functions

matrix elements

form factors

ground-state isolation

tensor decomposition

Alternative technique:
Feynman–Hellmann for lattice matrix elements

Matrix elements from “Feynman–Hellmann”

- Feynman–Hellmann in quantum mechanics:

$$\frac{dE_n}{d\lambda} = \langle n|\frac{\partial H}{\partial\lambda}|n\rangle$$

  - matrix elements of the derivative of the Hamiltonian determined by derivative of corresponding energy eigenstates

- Lattice QCD: evaluate energy shifts with respect to weak external fields

- Analogous to considering the energy of a fermion in a weak uniform magnetic field:

$$E(\mathbf{B}) = m - \boldsymbol{\mu}.\mathbf{B} + \frac{|e\mathbf{B}|}{2m} - 2\pi\beta_M|\mathbf{B}|^2 + \mathcal{O}(\mathbf{B}^3)$$

Feynman–Hellmann Theorem

- A method for determining hadronic matrix elements from energy shifts
- Suppose we want $\langle H|\mathcal{O}|H\rangle$
- Proceed by $S \to S + \lambda \int \mathrm{d}^4x\, \mathcal{O}(x)$

real parameter

local operator, e.g. $\bar{q}(x)\gamma_5\gamma_3 q(x)$

- FH tells us

$$\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)}\left\langle H\left|\frac{\partial S(\lambda)}{\partial \lambda}\right|H\right\rangle$$

- Calculation of matrix element $\equiv$ hadron spectroscopy [2-pt functions only]

$$\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)}\langle H|\mathcal{O}|H\rangle$$

Feynman–Hellmann: Hadron spin

- To access hadron spin fractions, we modify the action to include the axial current

$$S \to S + \lambda \sum_x \bar{q}(x) i\gamma_5\gamma_3 q(x)$$

- FH Theorem then gives

$$\left.\frac{\partial E_H(\lambda)}{\partial\lambda}\right|_{\lambda=0} = \frac{1}{2M_H}\langle H|\bar{q}i\gamma_5\gamma_3 q|H\rangle$$

- but for a spin-*J* hadron with polarisation *m* in the *z*-direction

$$\langle H, Jm|\bar{q}i\gamma_5\gamma_3 q|H, Jm\rangle = 2M_H\Delta q^{Jm}$$

$$\Rightarrow \quad \Delta q = \left.\frac{\partial E_H(\lambda)}{\partial\lambda}\right|_{\lambda=0}$$

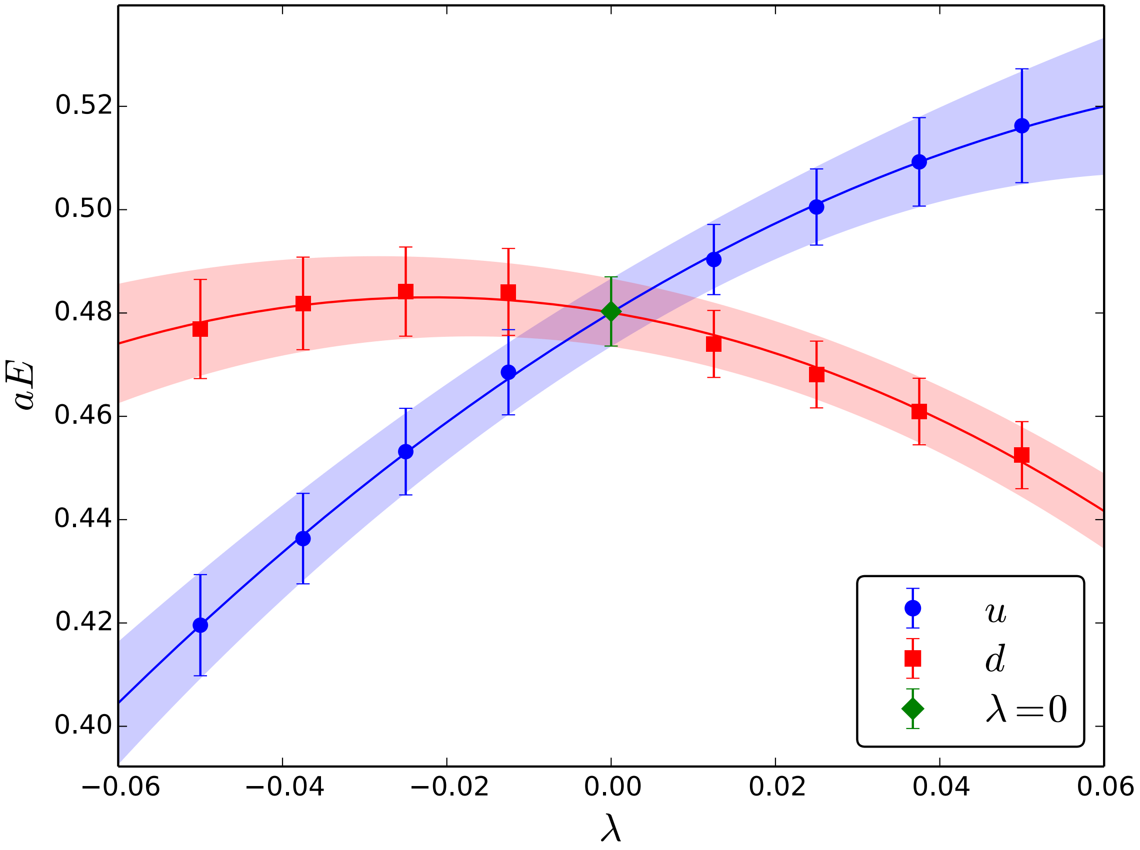
- Also note: reversing hadron polarisation $\equiv$ changing sign of λ

Connected Spin Contributions

[Chambers *et al*. PRD(2014)]

- Start with nucleon mass vs. field strength λ

SU(3) symmetric point $m_\pi = m_K \simeq 470 MeV$



Fit: quadratic in λ

➡ linear terms give Δu and Δd

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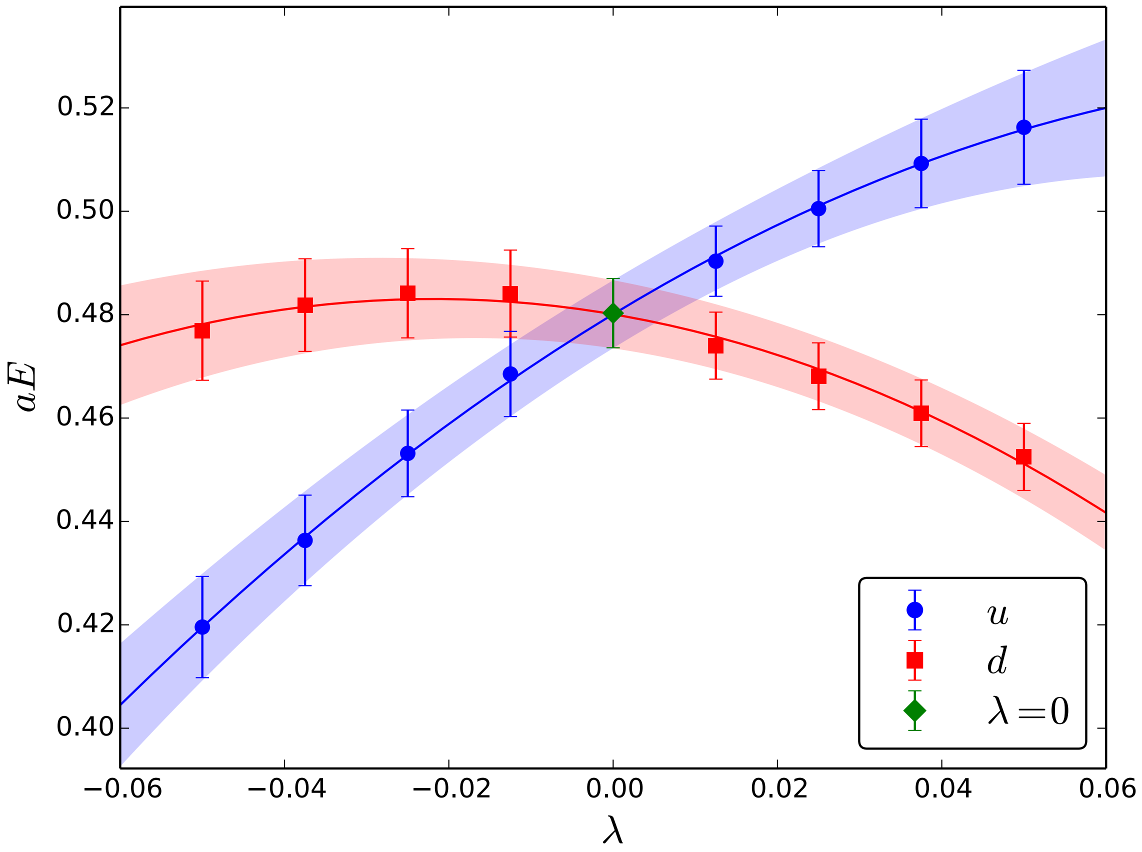
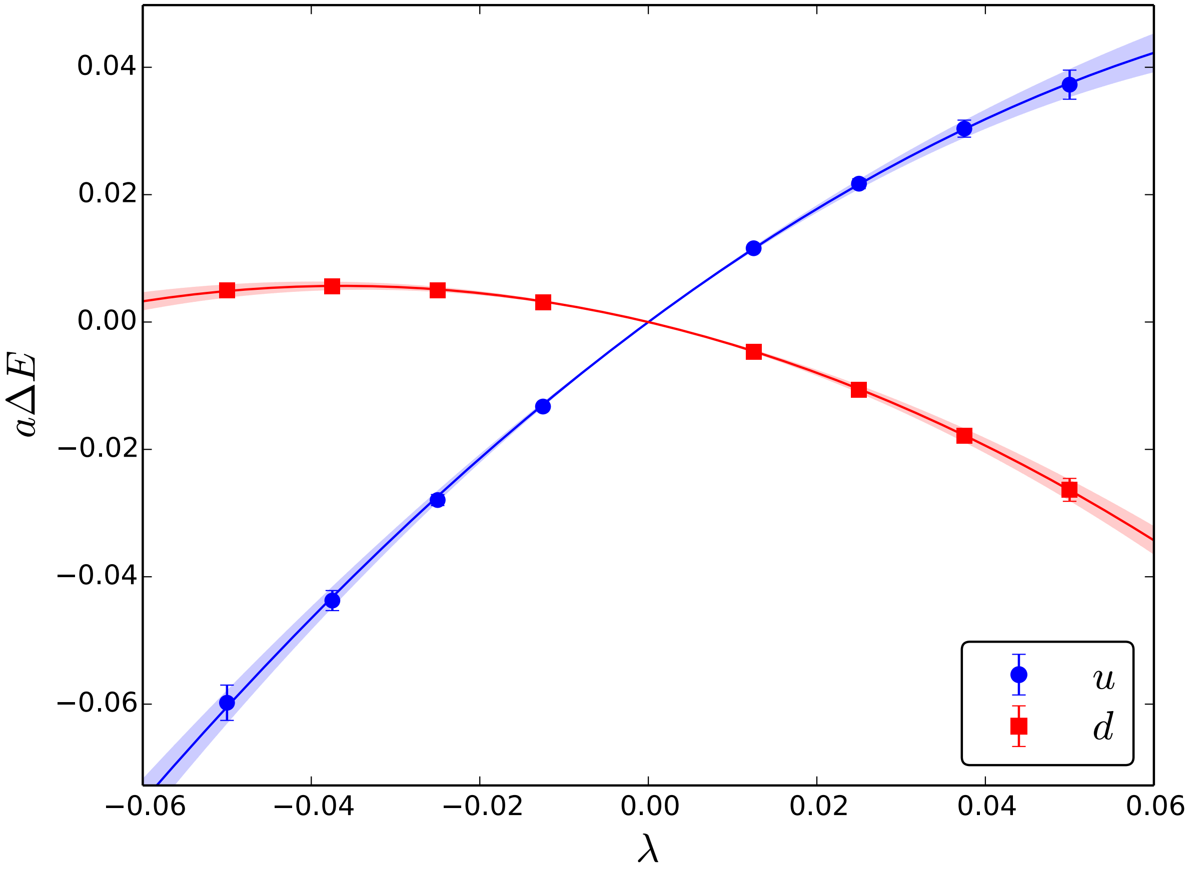


Fit energy differences



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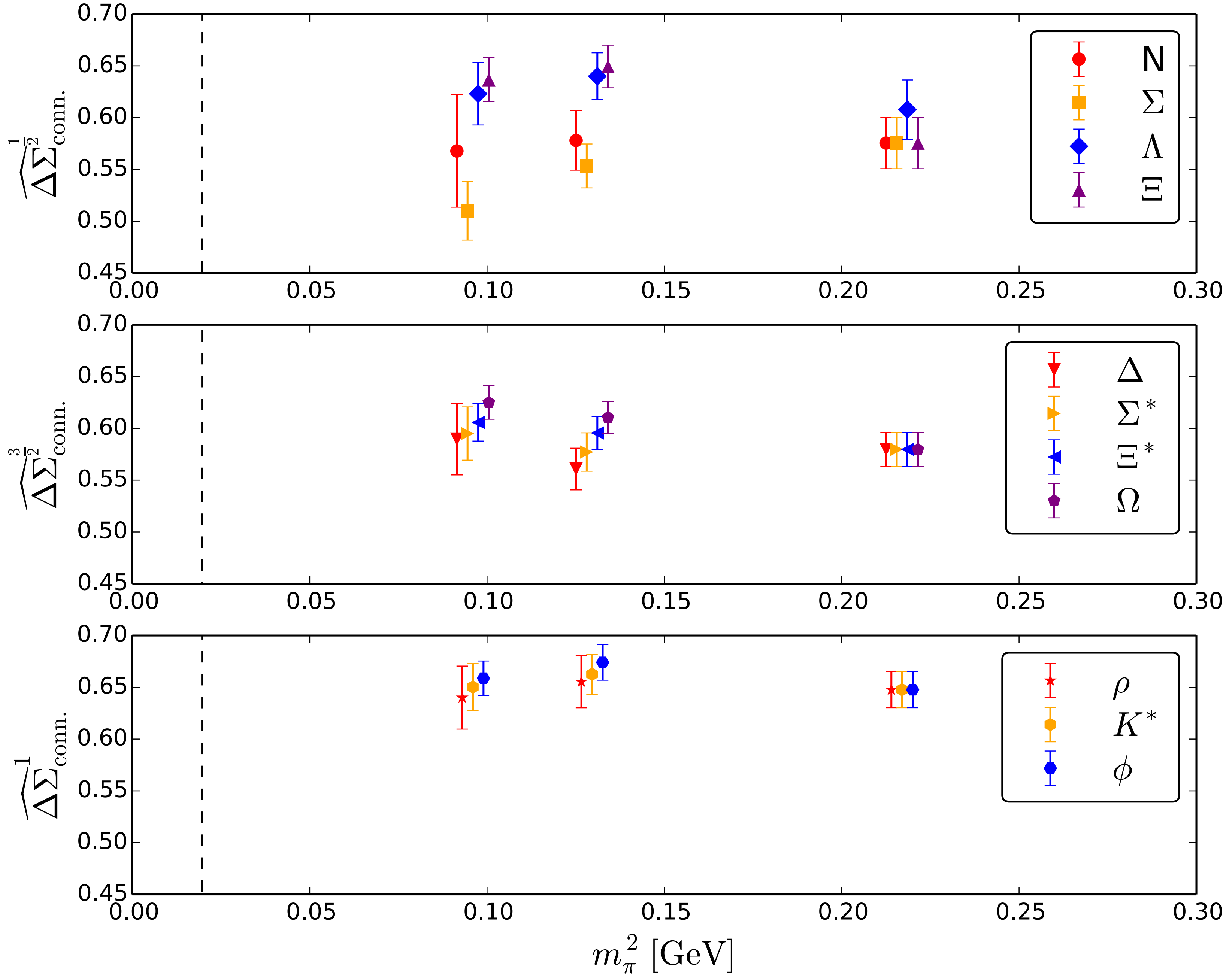
Connected Spin Contributions

[Chambers *et al*. PRD(2014)]

- Connected spin factions in various hadrons

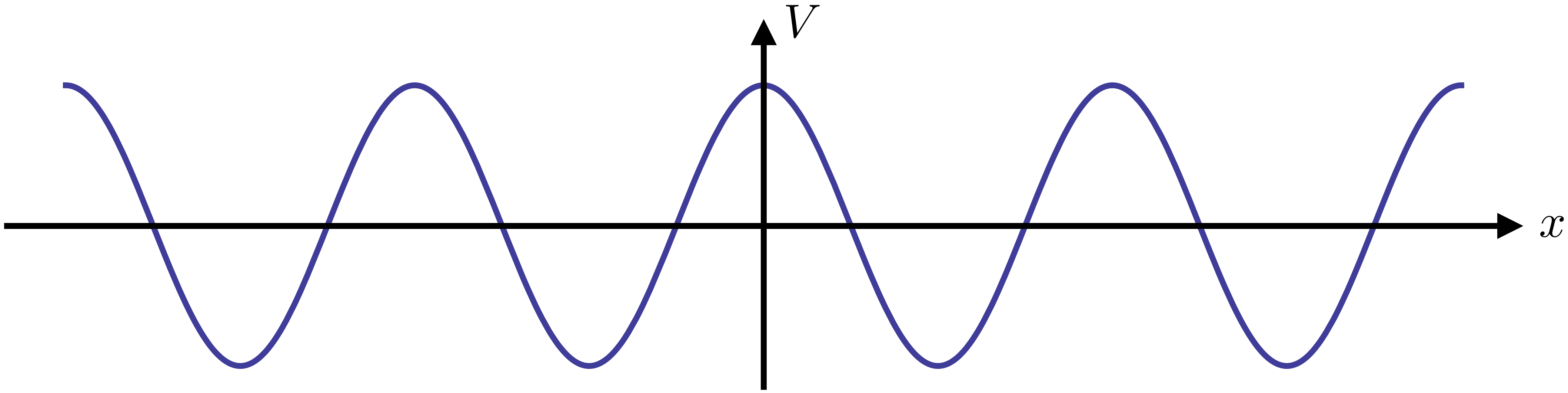


(Connected) Spin Fraction Universal ~60%

Non-forward matrix elements:
Momentum transfer from external field

Warm up: Periodic potential, 1-D QM

- Almost free particle $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$
- Subject to weak external periodic potential $V(x) = 2\lambda V_0 \cos(qx)$



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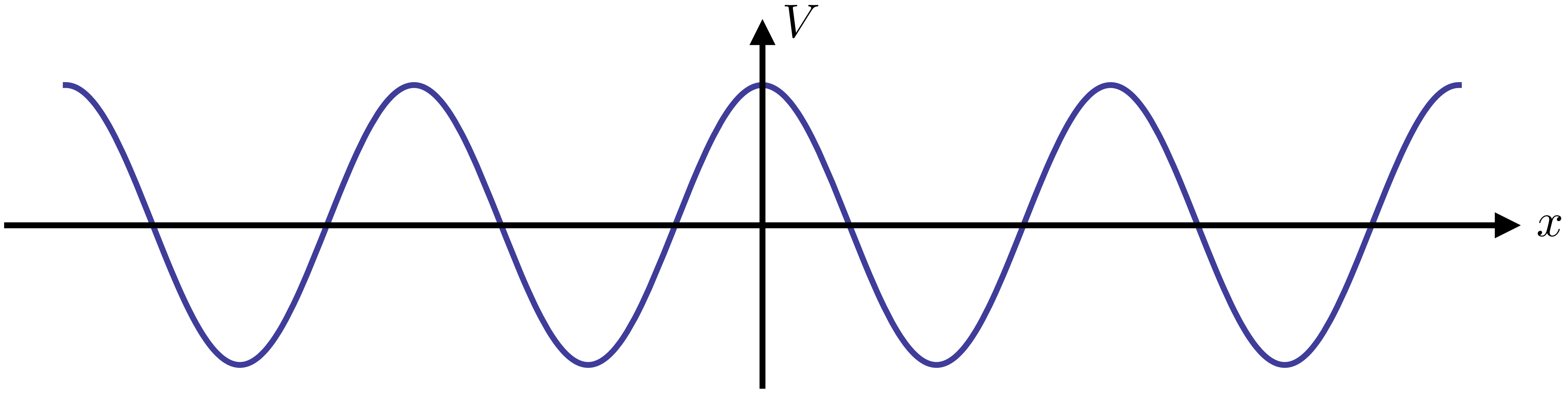
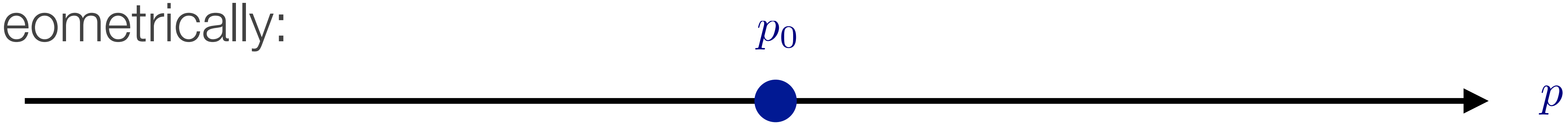


$$\hat{V}|p\rangle = \lambda V_0 |p+q\rangle + \lambda V_0 |p-q\rangle$$

Warm up: Periodic potential, 1-D QM

- Geometrically:



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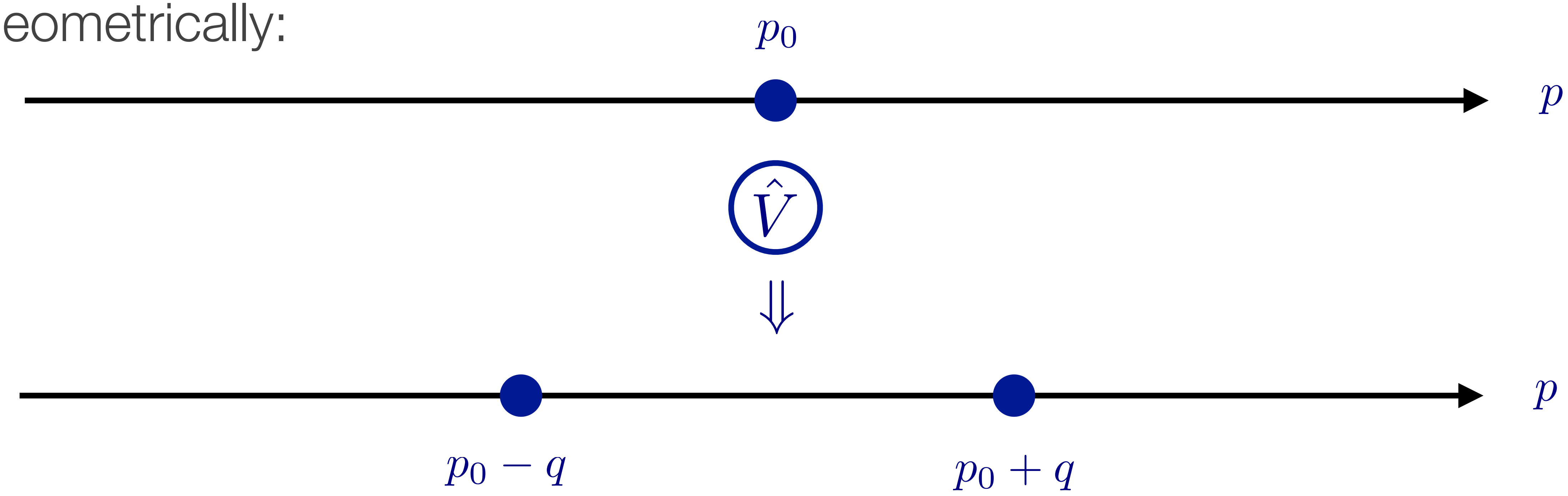

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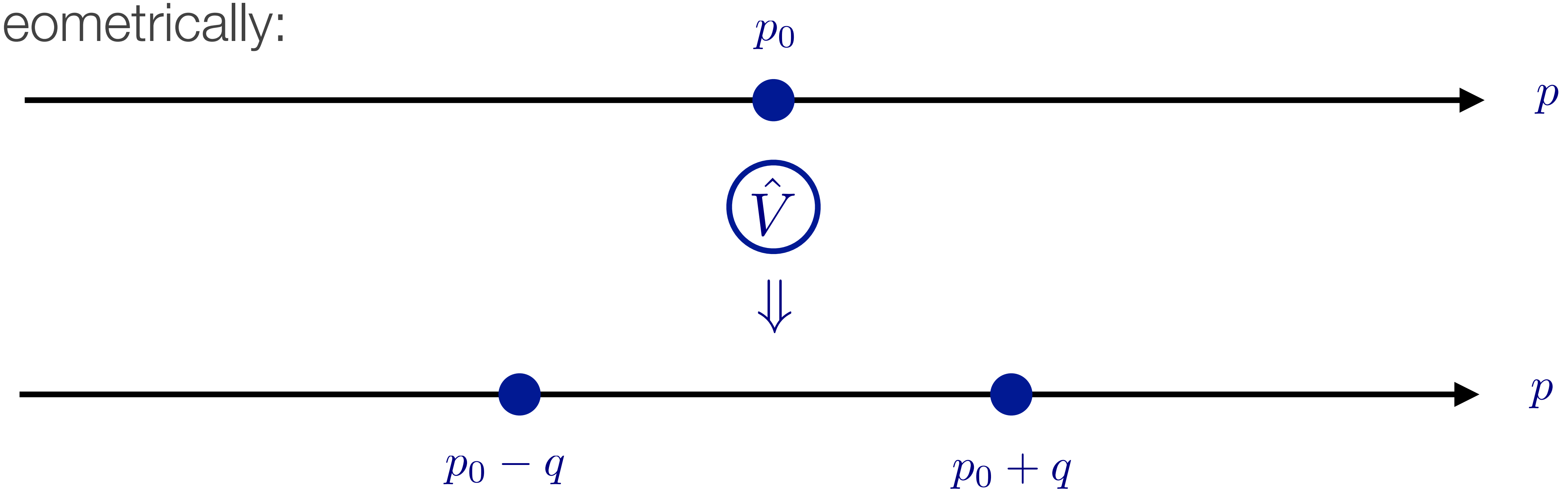

$$\Rightarrow \langle p|\hat{V}|p\rangle = 0$$

No first order energy shifts?

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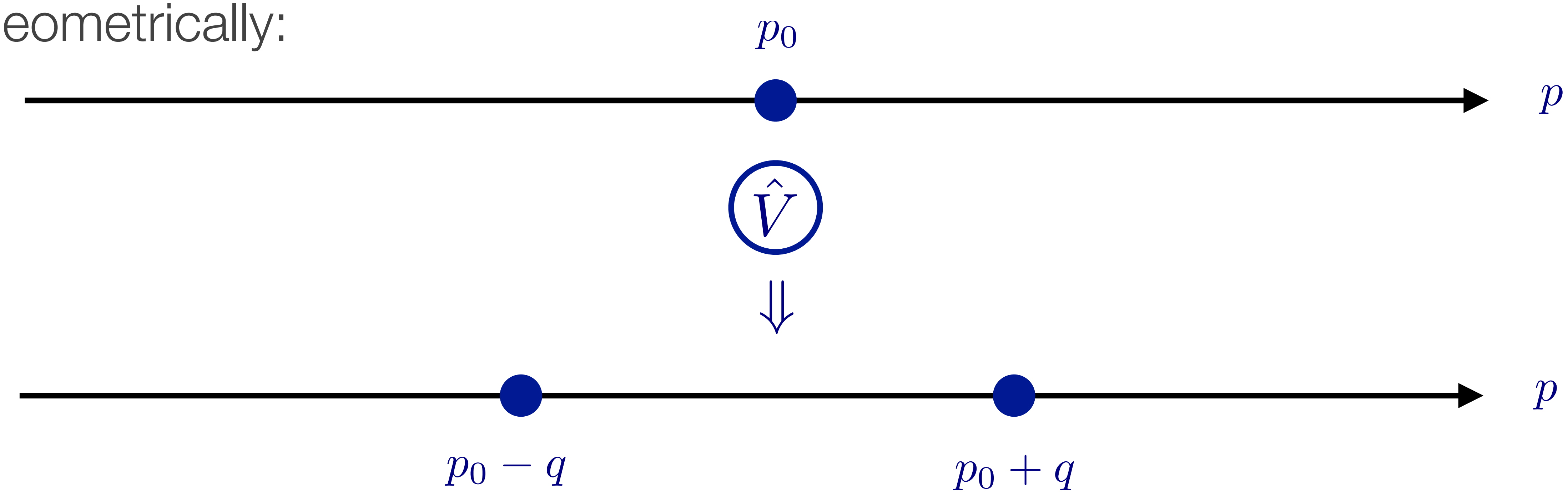

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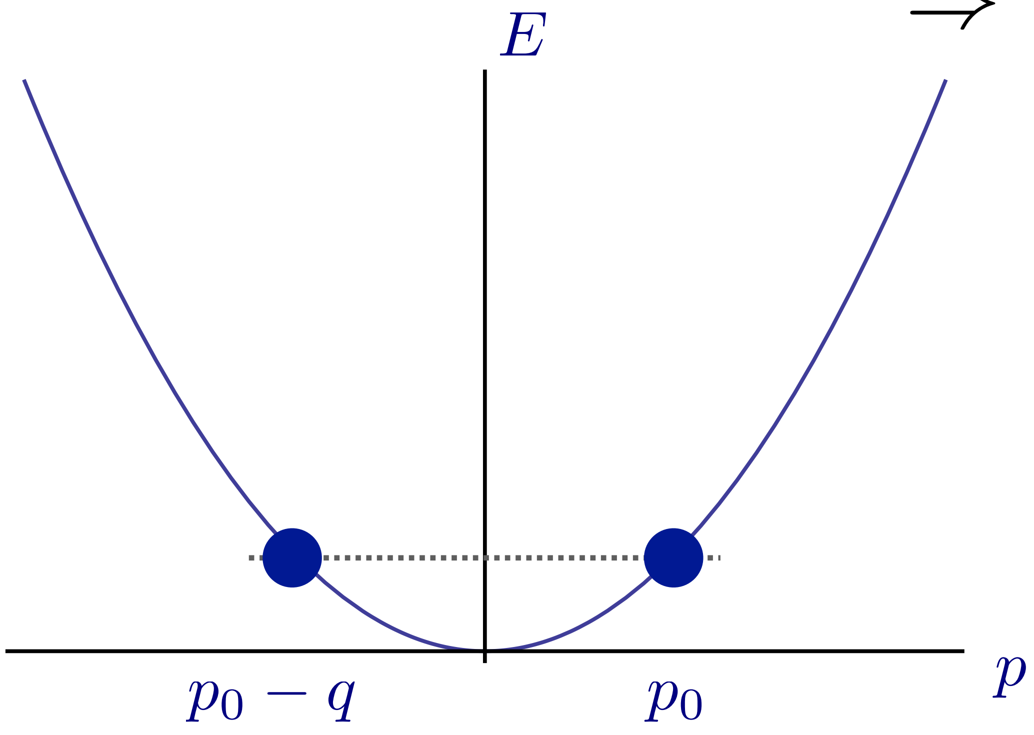

No first order energy shifts?

If $p_0 = \pm q/2$
⇒ transition between

degenerate states

Degenerate perturbation theory

$$p = q/2$$

$$H = \begin{pmatrix} \frac{p^2}{2m} & \lambda V_0 \\ \lambda V_0 & \frac{p^2}{2m} \end{pmatrix}$$

$$H\left\{|q/2\rangle \pm |-q/2\rangle\right\} = \left(E_{q/2} \pm \lambda V_0\right)\left\{|q/2\rangle \pm |-q/2\rangle\right\}$$

$$p \sim q/2$$

Degenerate perturbation theory

- Exact degeneracy: $p = q/2$

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- Consider mixing on almost-degenerate states $p \sim q/2$

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Eigenvalues

$$\frac{p^2 + (p-q)^2}{4m} \pm \sqrt{\frac{q^2(q-2p)^2}{16m^2} + \lambda^2 V_0^2}$$

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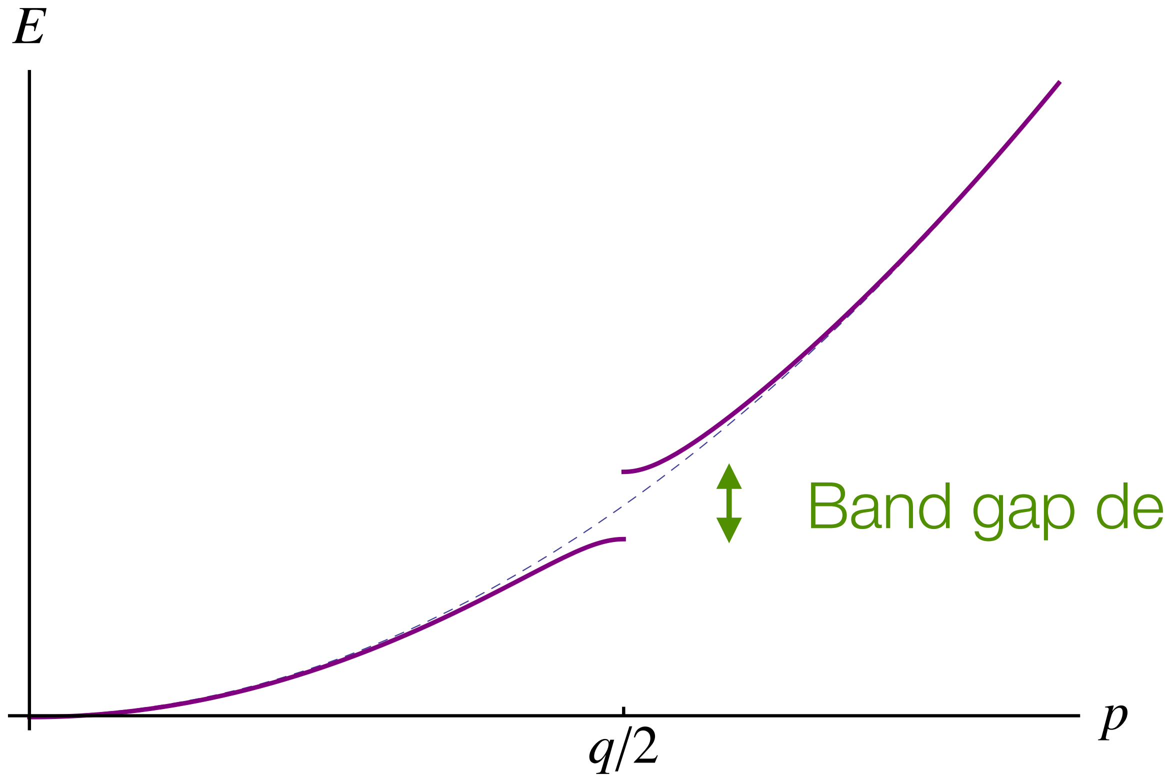

Band gap determined by strength of perturbation

$$\Delta E = 2\lambda V_0$$

External (momentum) field on the lattice

- Modify Lagrangian with external field containing a spatial Fourier transform [constant in time]

$$\mathcal{L}(y) \to \mathcal{L}_0(y) + \lambda 2\cos(\vec{q}.\vec{y})\overline{q}(y)\gamma_\mu q(y)$$

- Project onto "back-to-back" momentum state: $|\vec{q}/2\rangle + |-\vec{q}/2\rangle$

- E.g. pion form factor

$$\langle \pi(\vec{p}')|\overline{q}(0)\gamma_\mu q(0)|\pi(\vec{p})\rangle = (p+p')_\mu F_\pi(q^2)$$

- "Feynman-Hellmann"

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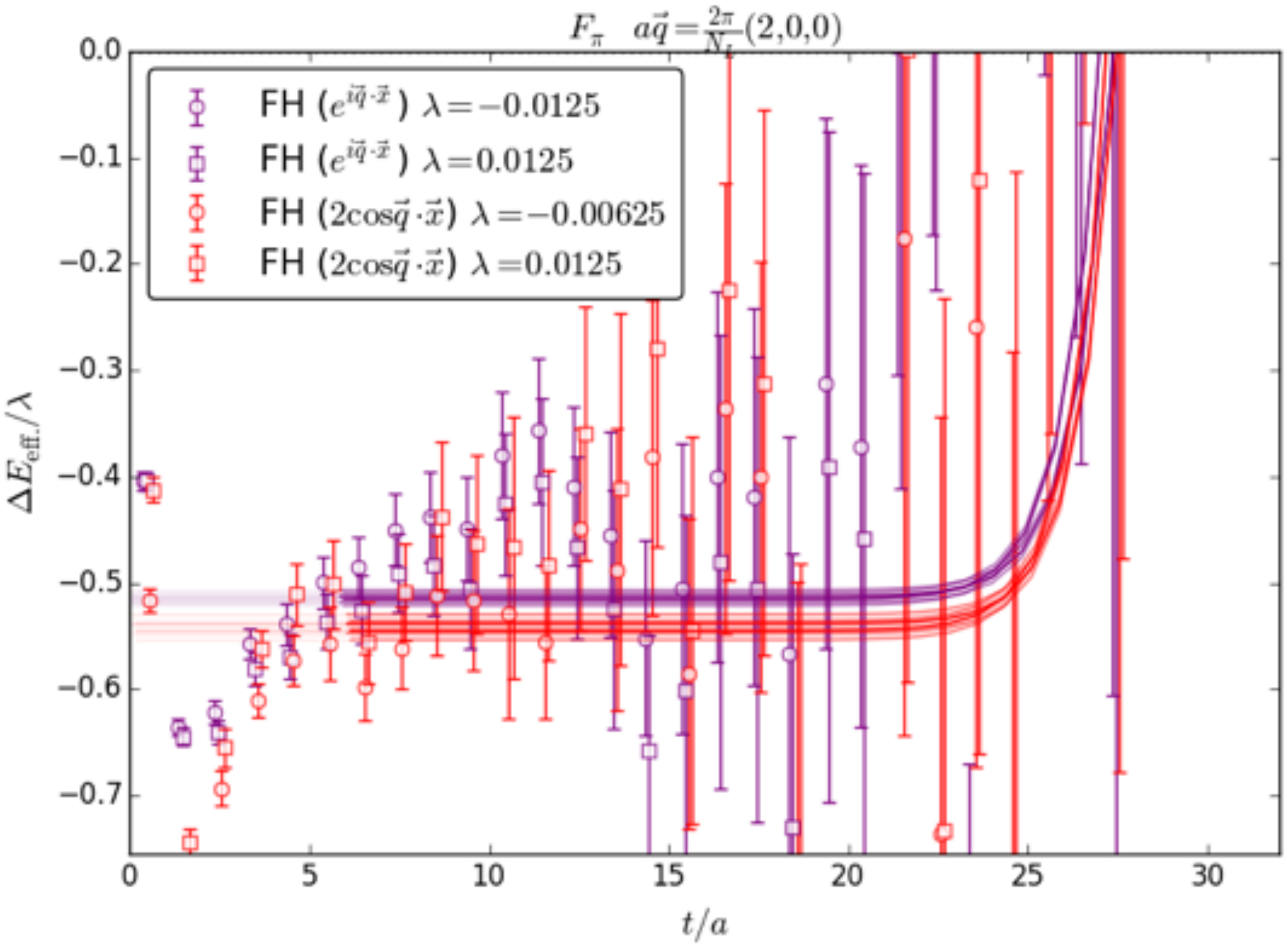
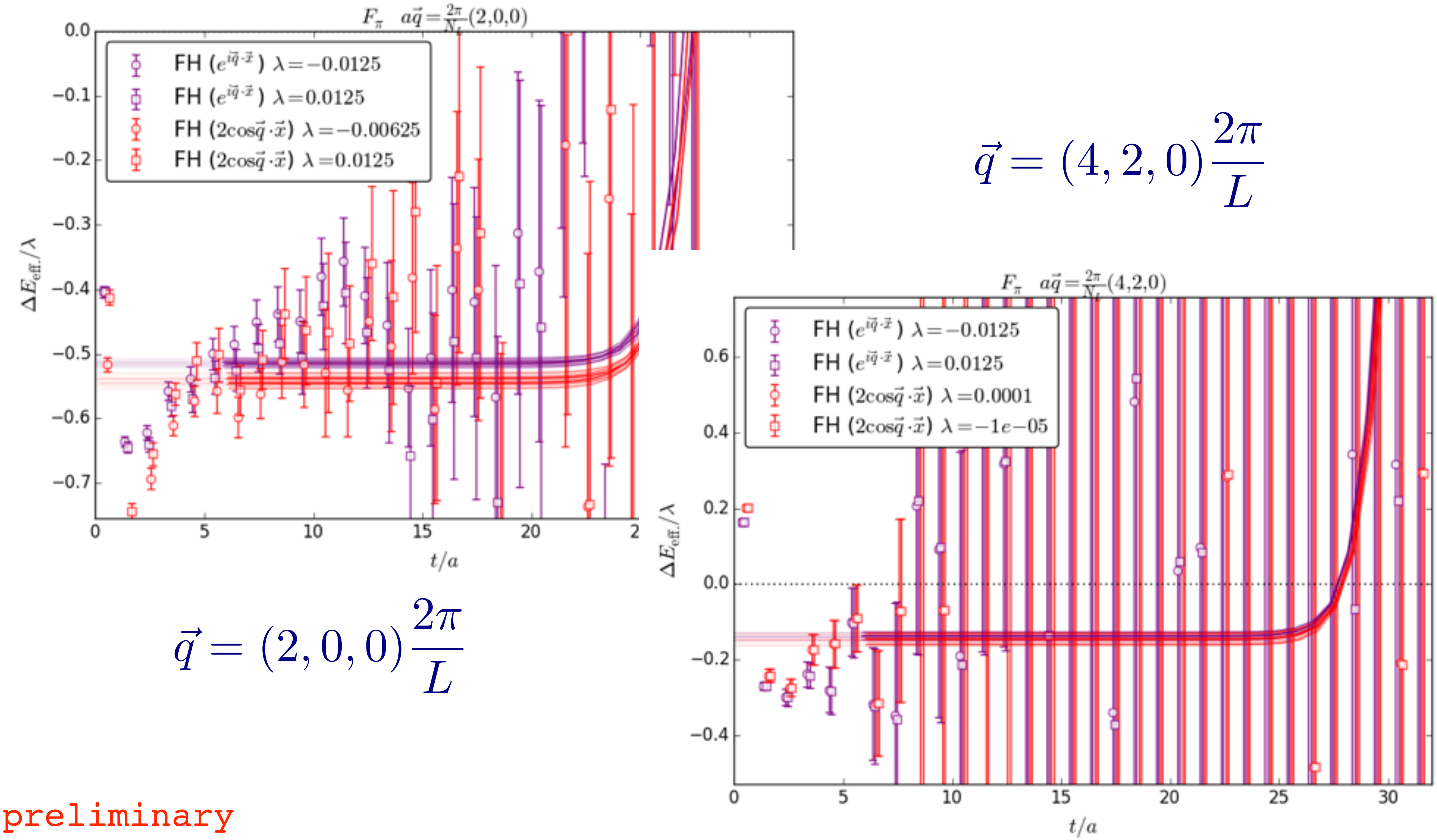


$$\vec{q}=(2,0,0)\frac{2\pi}{L}$$

`preliminary`

Pion energy shift

“Effective mass plot”



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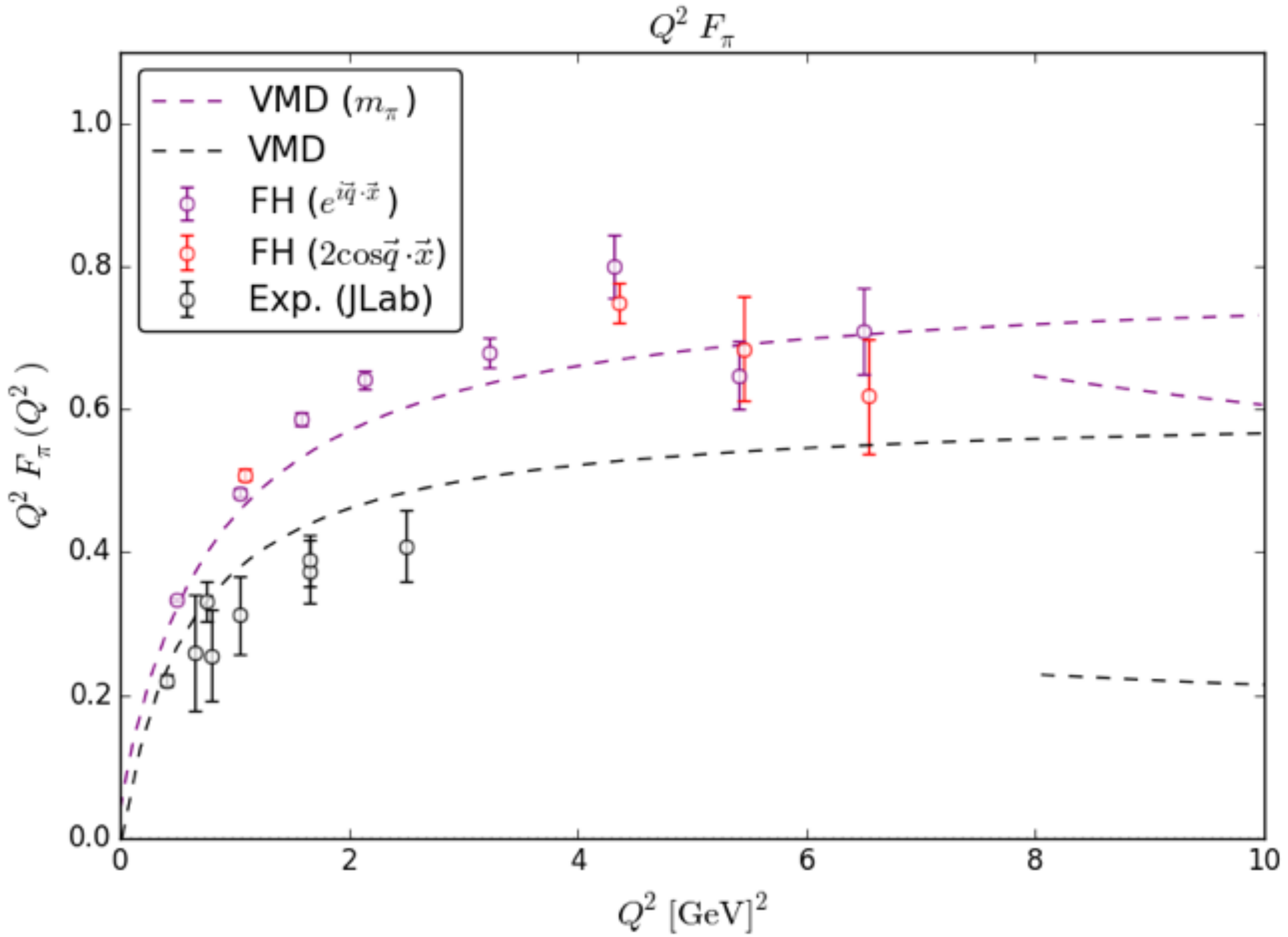
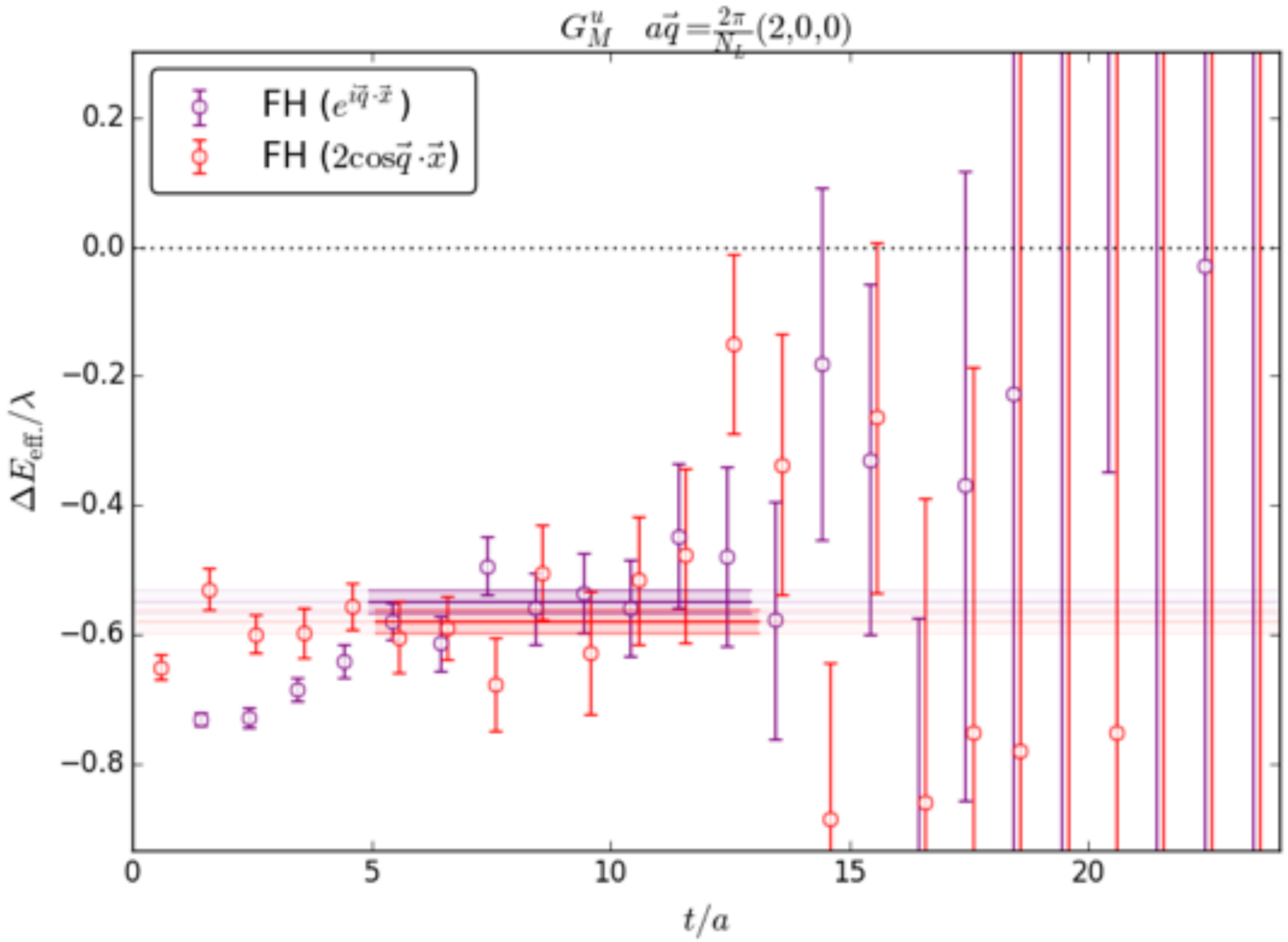


preliminary

Pion form factor

$m_\pi \sim 430\,\mathrm{MeV}$

Statistically encouraging signal

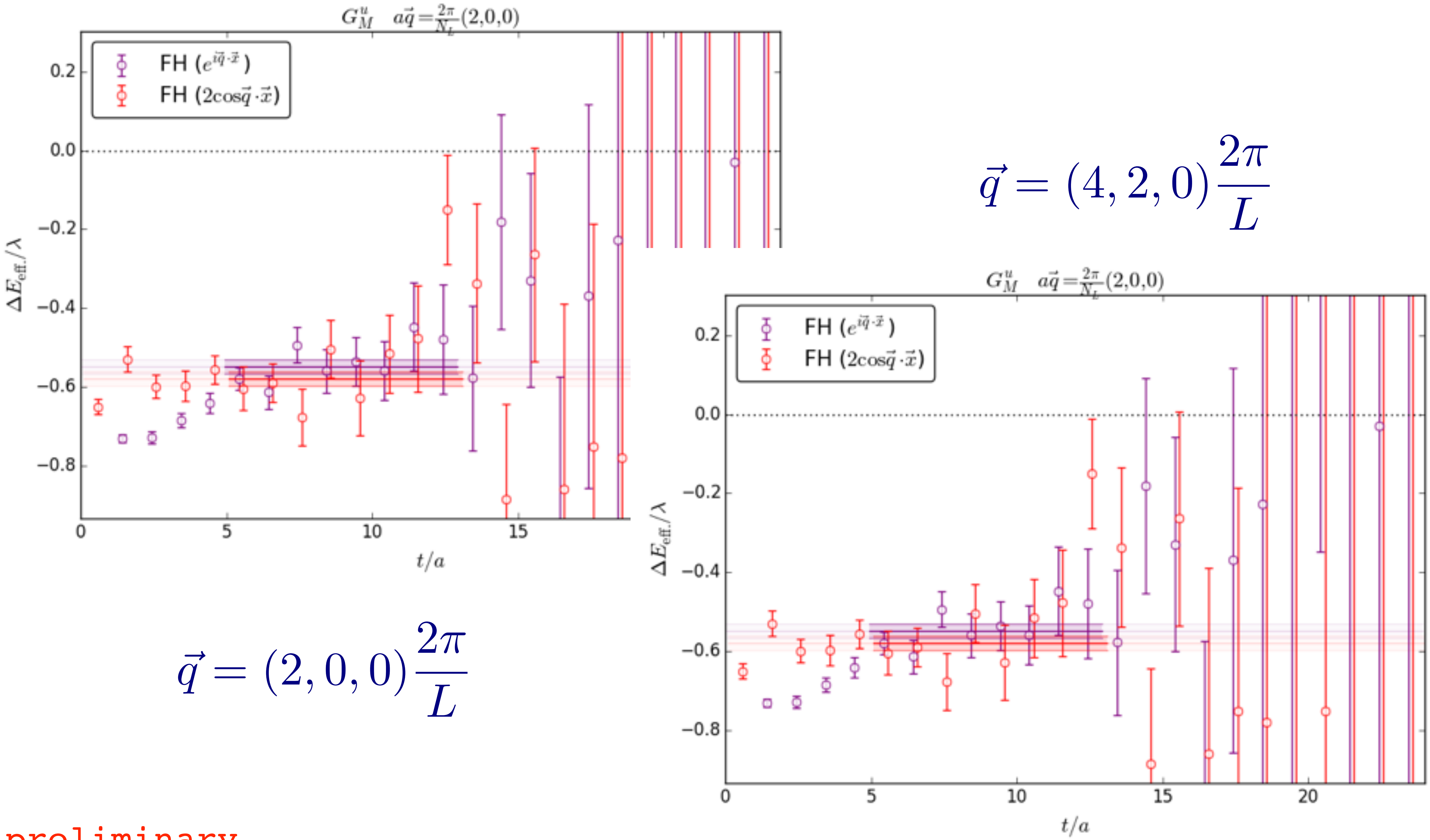


$$\vec{q} = (2,0,0)\frac{2\pi}{L}$$

preliminary

Nucleon *up* quark magnetic FF

Energy shift
“Effective mass plot”



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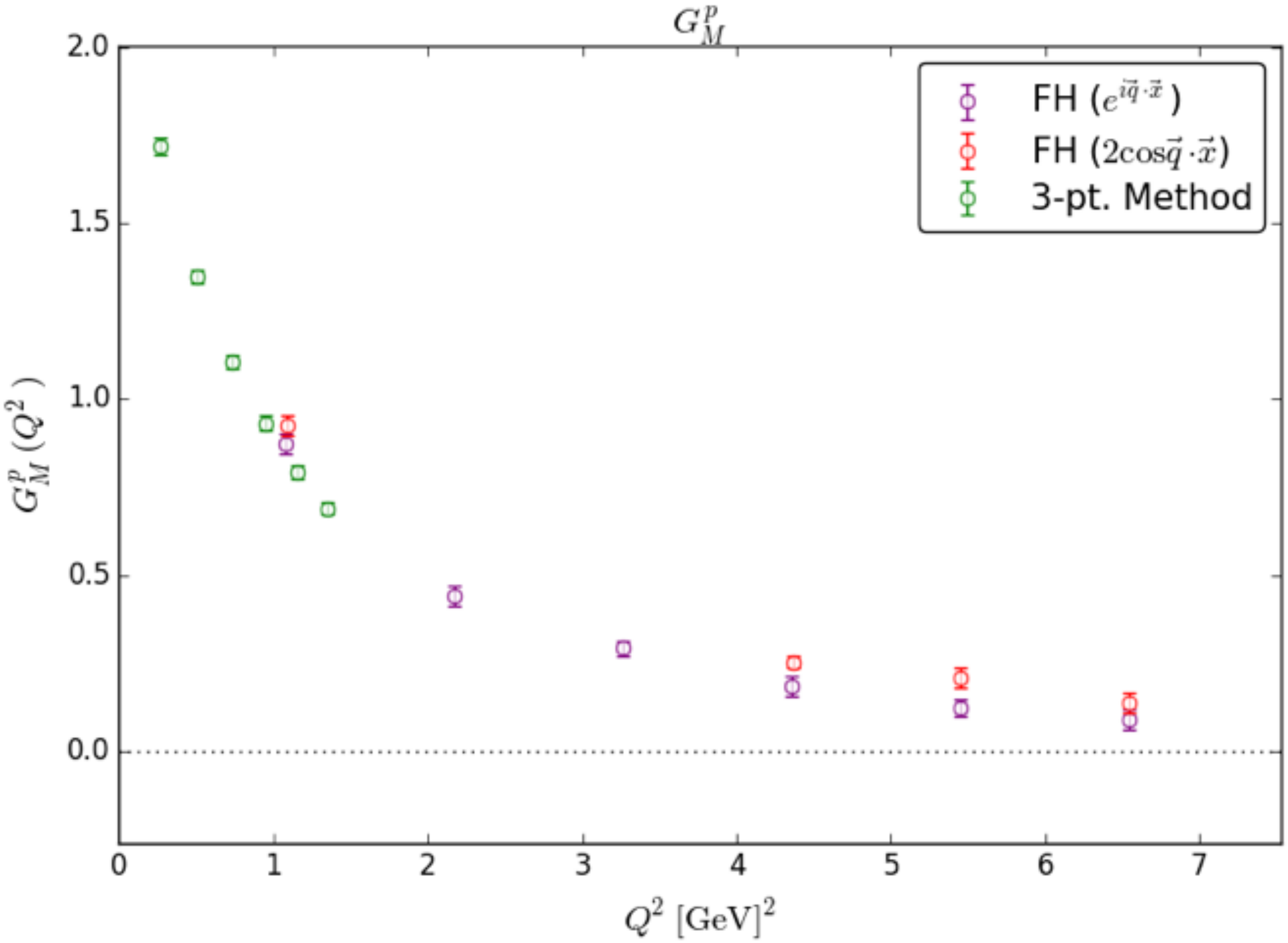
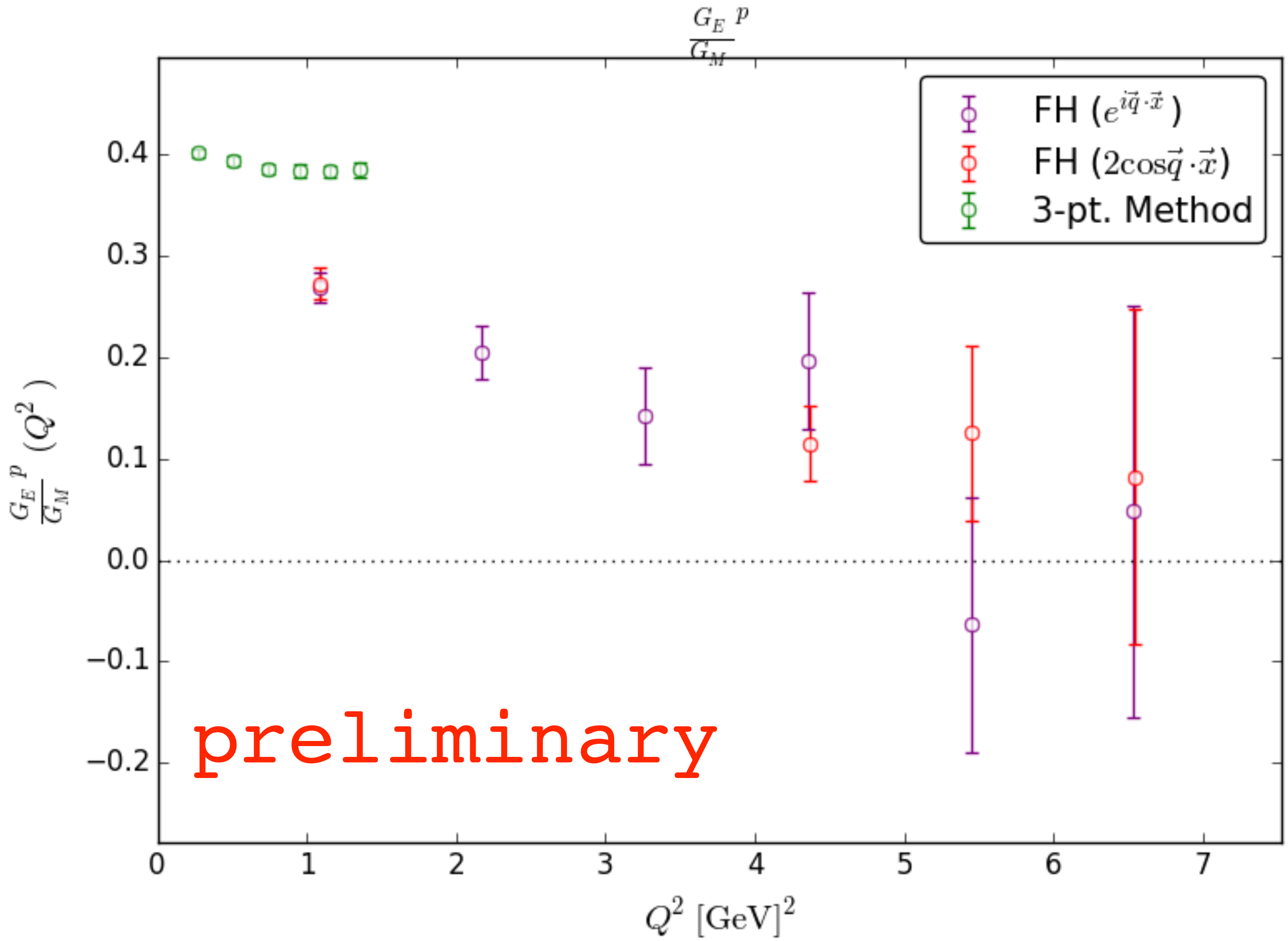


preliminary

Proton magnetic FF

$m_\pi \sim 430\,\mathrm{MeV}$

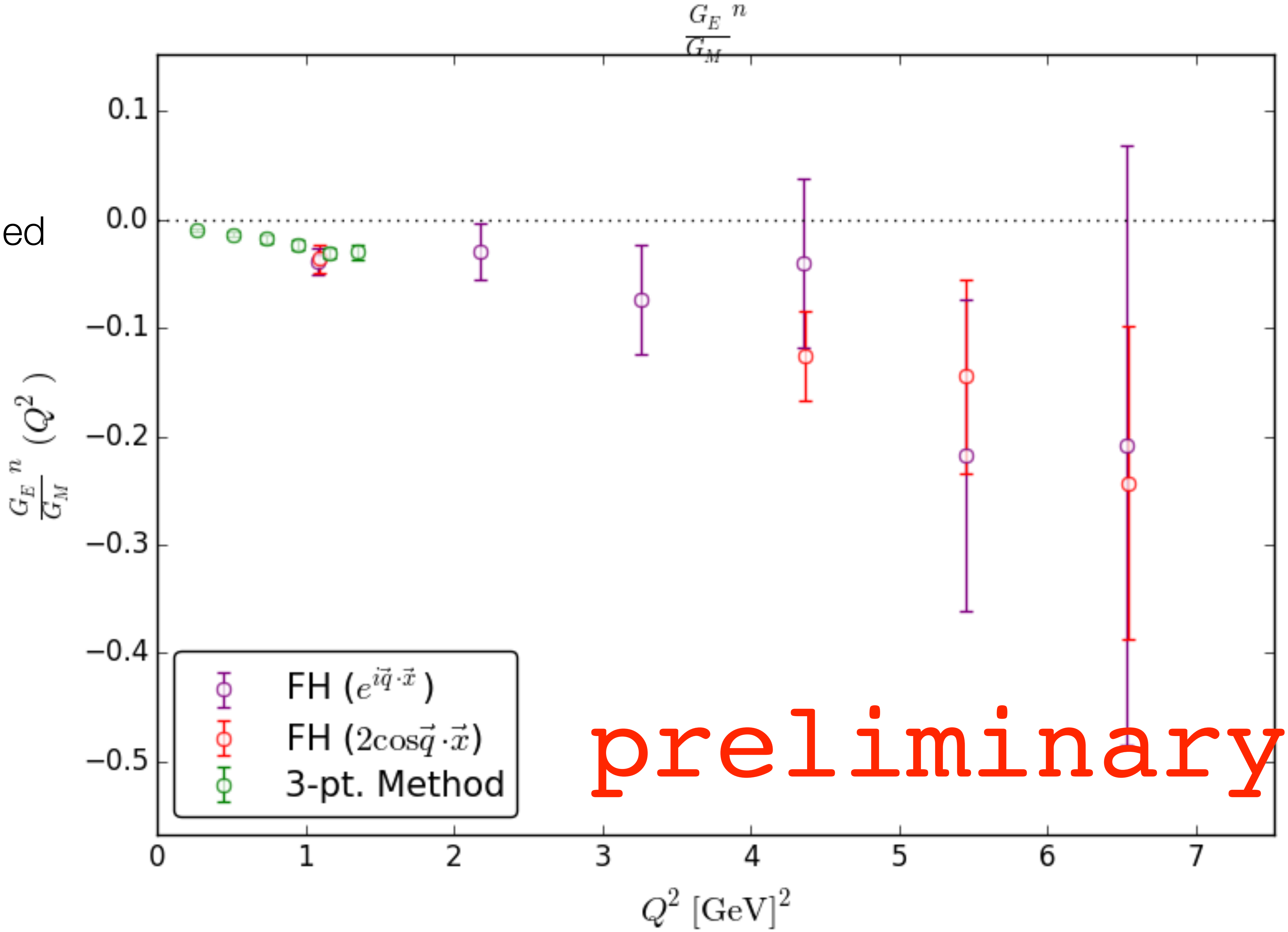
Statistical signature
still working on systematics



Proton GE/GM

$m_\pi \sim 430\,\mathrm{MeV}$

Not divided by μ_n



Neutron GE/GM

$m_\pi \sim 430\,\mathrm{MeV}$

Concluding remarks & excitement

- Tremendous advances in studies of vector form factors
- New progress in quark tensor form factors
- Feynman–Hellmann offering an alternative method for extracting hadronic matrix elements
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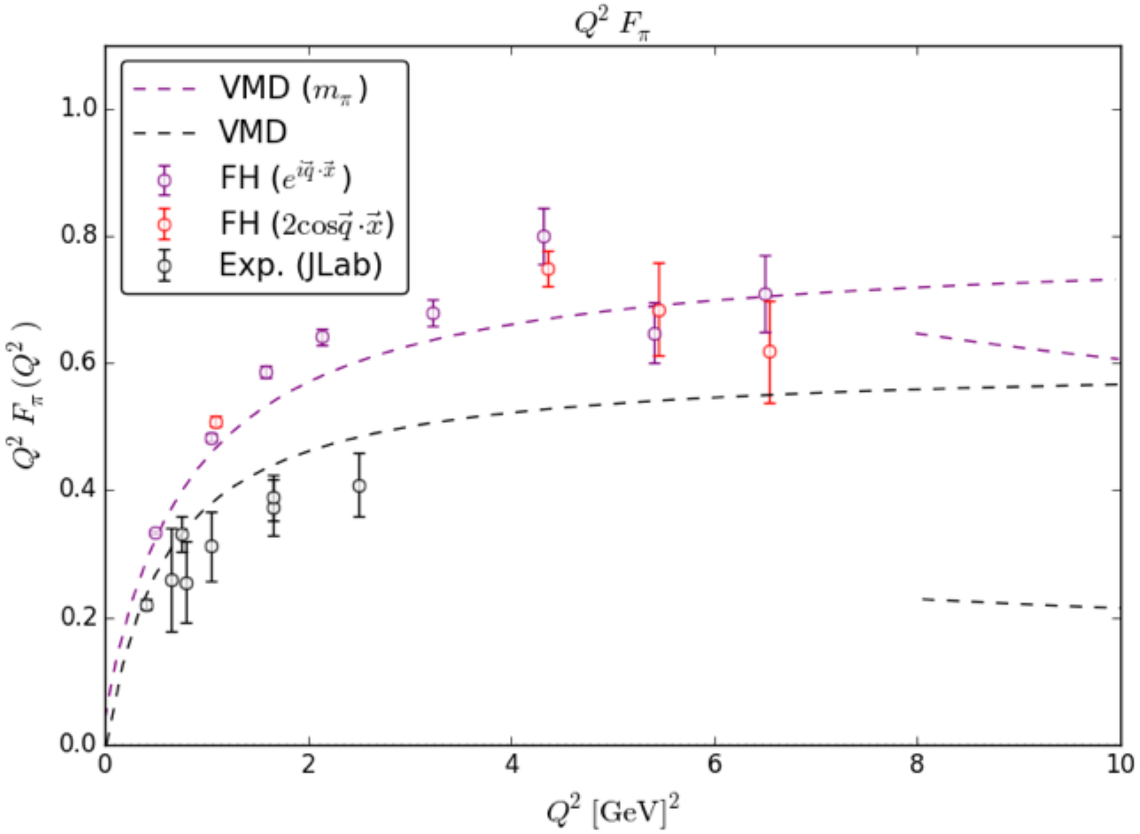
    - Still work to do: more statistics; improved operator basis at large momenta; chiral & continuum extrapolation



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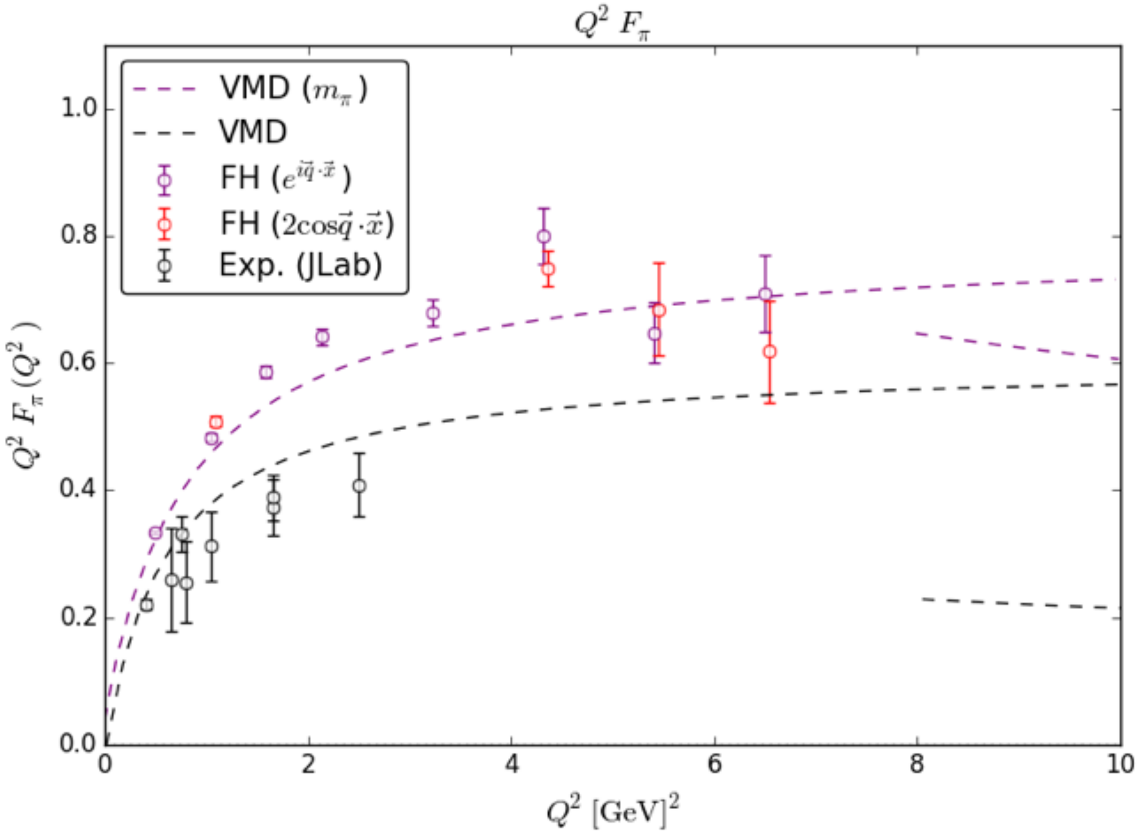
- Extending FH to external momenta offering access to unprecedented scales
    - Still work to do: more statistics; improved operator basis at large momenta; chiral & continuum extrapolation



Challenge? Can lattice determine existence of GE/GM crossover before experiment?

Back-up slides

[*or old slides I didn’t delete yet*]

Lattice Specs

- Nf =2+1 O(a)-improved Clover fermions (“SLiNC” action)
  - Tree-level Symanzik gluon action (plaq. + rect.)
- Results from a single lattice spacing (a~0.074fm), and volume (32^3 x 64)
- Most results are at the SU(3)-symmetric point (m_{pi}~470 MeV)
  - Total spin contribution (also m_{pi}~330 MeV)
- ~500 measurements per mass per field strength

Twist-2 Generalised Quark Distributions

[Diehl & Hägler,…]

$$P = \tfrac{1}{2}(p+p')$$
$$\Delta = p' - p$$
$$t = \Delta^2$$
$$\xi = -\frac{1}{2}\frac{\Delta^+}{P^+}$$

$$F(x,\xi,t) = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-}\, \langle p'|\, \bar{q}(-\tfrac{1}{2}z)\gamma^+ q(\tfrac{1}{2}z)|\, p\rangle \Big|_{z^+=0,\, \mathbf{z}=0}$$
$$= \frac{1}{2P^+}\left[H(x,\xi,t)\, \bar{u}\gamma^+ u + E(x,\xi,t)\, \bar{u}\frac{i\sigma^{+\alpha}\Delta_\alpha}{2m}u\right],$$

$$\tilde{F}(x,\xi,t) = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-}\, \langle p'|\, \bar{q}(-\tfrac{1}{2}z)\gamma^+\gamma_5\, q(\tfrac{1}{2}z)|\, p\rangle \Big|_{z^+=0,\, \mathbf{z}=0}$$
$$= \frac{1}{2P^+}\left[\tilde{H}(x,\xi,t)\, \bar{u}\gamma^+\gamma_5 u + \tilde{E}(x,\xi,t)\, \bar{u}\frac{\gamma_5\Delta^+}{2m}u\right],$$

$$F_T^j(x,\xi,t) = -i\int \frac{dz^-}{4\pi}\, e^{ixP^+z^-}\, \langle p'|\, \bar{q}(-\tfrac{1}{2}z)\, \sigma^{+j}\gamma_5\, q(\tfrac{1}{2}z)|\, p\rangle \Big|_{z^+=0,\, \mathbf{z}=0}$$
$$= -\frac{i}{2P^+}\Bigg[H_T(x,\xi,t)\, \bar{u}\sigma^{+j}\gamma_5\, u + \tilde{H}_T(x,\xi,t)\, \bar{u}\frac{\epsilon^{+j\alpha\beta}\Delta_\alpha P_\beta}{m^2}u$$
$$+ E_T(x,\xi,t)\, \bar{u}\frac{\epsilon^{+j\alpha\beta}\Delta_\alpha\gamma_\beta}{2m}u + \tilde{E}_T(x,\xi,t)\, \bar{u}\frac{\epsilon^{+j\alpha\beta}P_\alpha\gamma_\beta}{m}u\Bigg].$$

Disconnected Spin Contributions

- Include operator in HMC
- For Hermitian spin operator, the Fermion matrix is modified by

$$M \to M(\lambda) = M_0 + \lambda\, i\gamma_5\gamma_3$$

  - Does not satisfy γ_5 Hermiticity ⇒ sign problem

- Hence we simulate with γ_5 Hermitian operator

$$M \to M(\lambda) = M_0 + \lambda\, \gamma_5\gamma_3$$

- Correlation function picks up complex phase

$$C(\lambda, t) \xrightarrow{\text{large } t} A(\lambda)e^{-E(\lambda)t}e^{i\phi(\lambda)t}$$

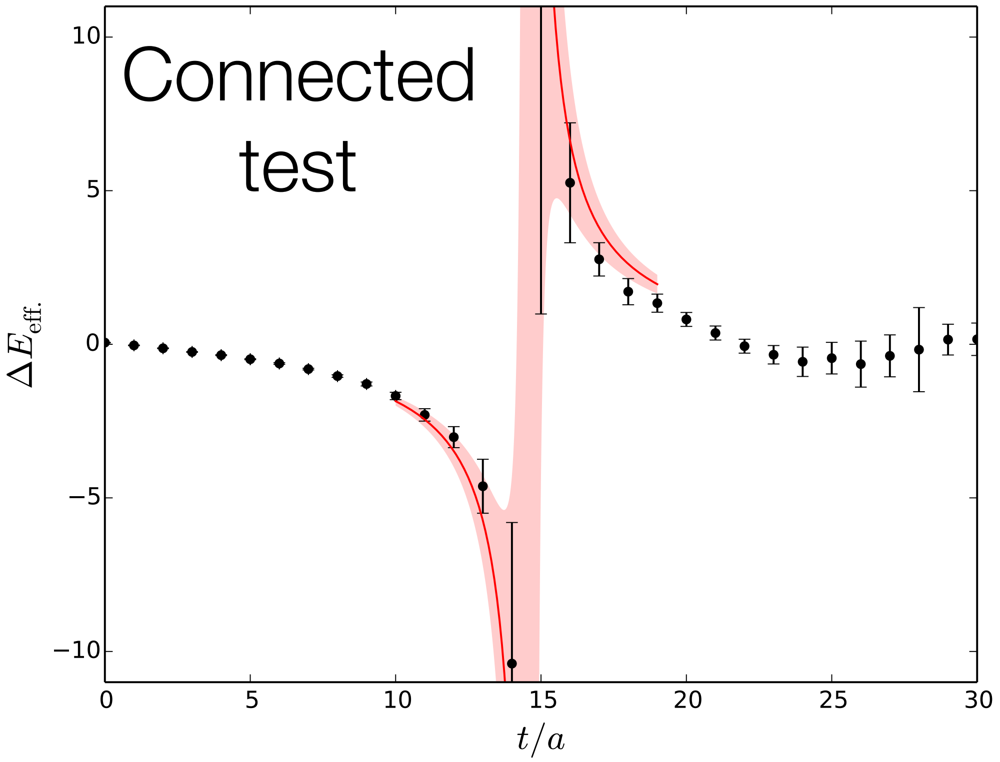
  - Extract matrix element from phase

$$\phi(\lambda) = \lambda\Delta q + \mathcal{O}(\lambda^3)$$

Disconnected Spin Contributions

- Isolate complex phase:
  - "Imaginary spin difference" / "Real spin average"

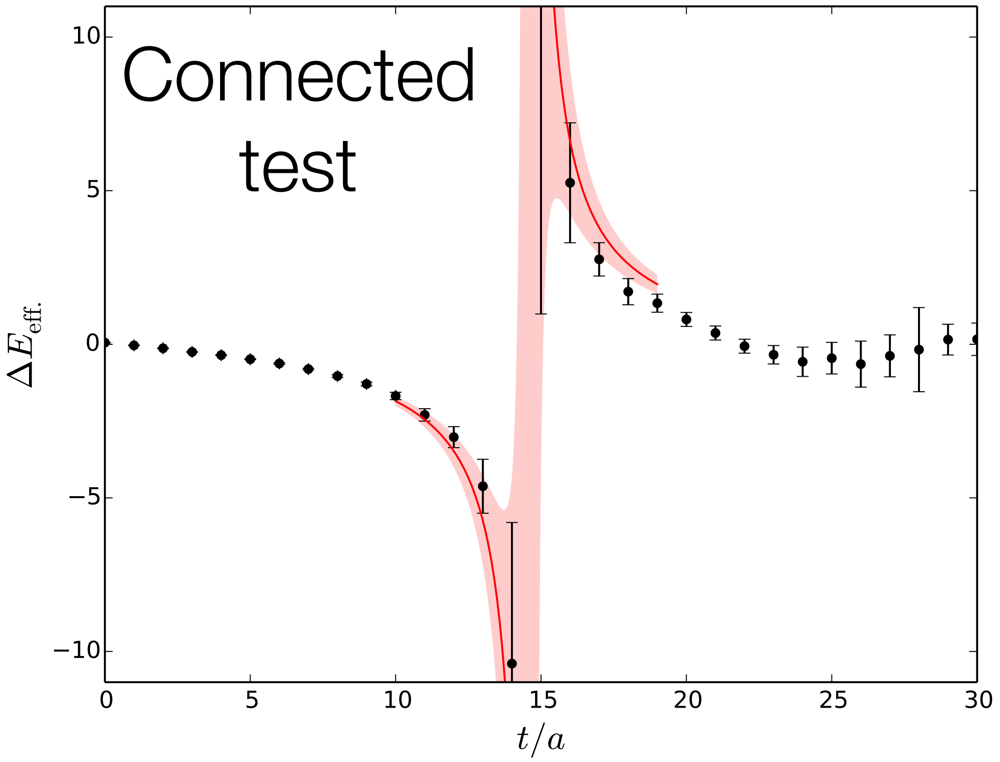
$$\frac{\mathcal{I}\Big[C_+(\lambda,t)\Big]-\mathcal{I}\Big[C_-(\lambda,t)\Big]}{\mathcal{R}\Big[C_+(\lambda,t)\Big]+\mathcal{R}\Big[C_-(\lambda,t)\Big]}\xrightarrow{\text{large } t}\tan(\phi t)$$

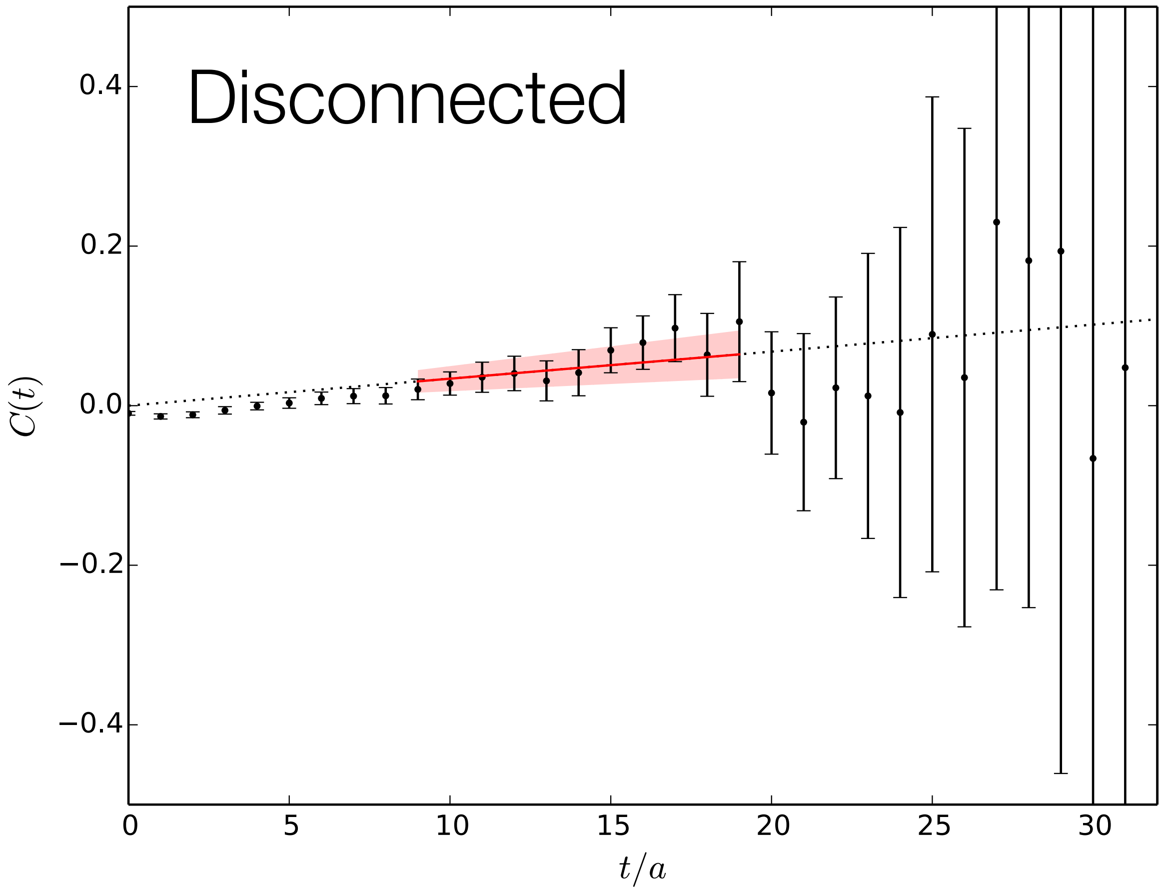


Disconnected Spin Contributions

- Isolate complex phase:
  - “Imaginary spin difference” / “Real spin average”

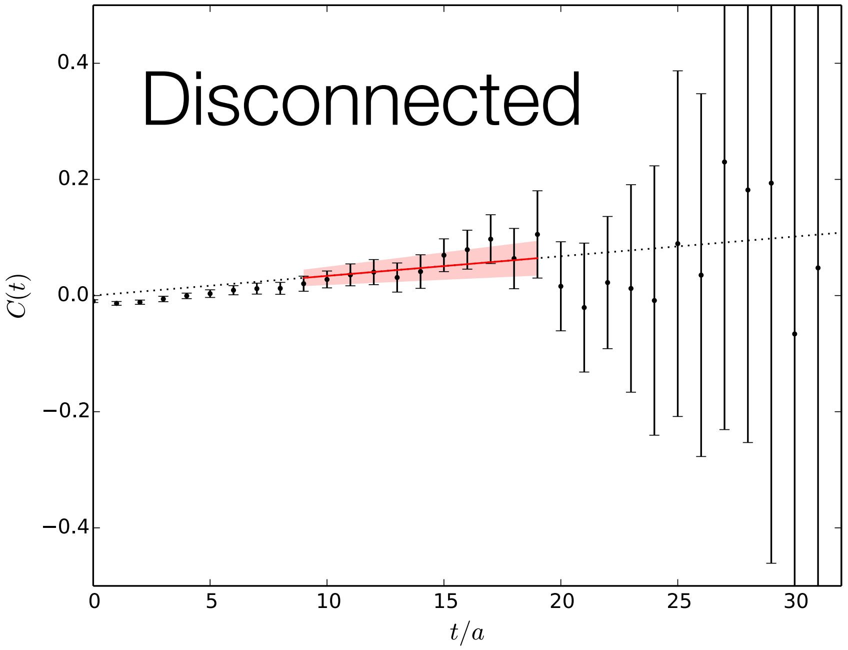
$$\frac{\mathcal{I}\Big[C_+(\lambda,t)\Big] - \mathcal{I}\Big[C_-(\lambda,t)\Big]}{\mathcal{R}\Big[C_+(\lambda,t)\Big] + \mathcal{R}\Big[C_-(\lambda,t)\Big]} \xrightarrow{\text{large } t} \tan(\phi t)$$





Disconnected Spin Contributions

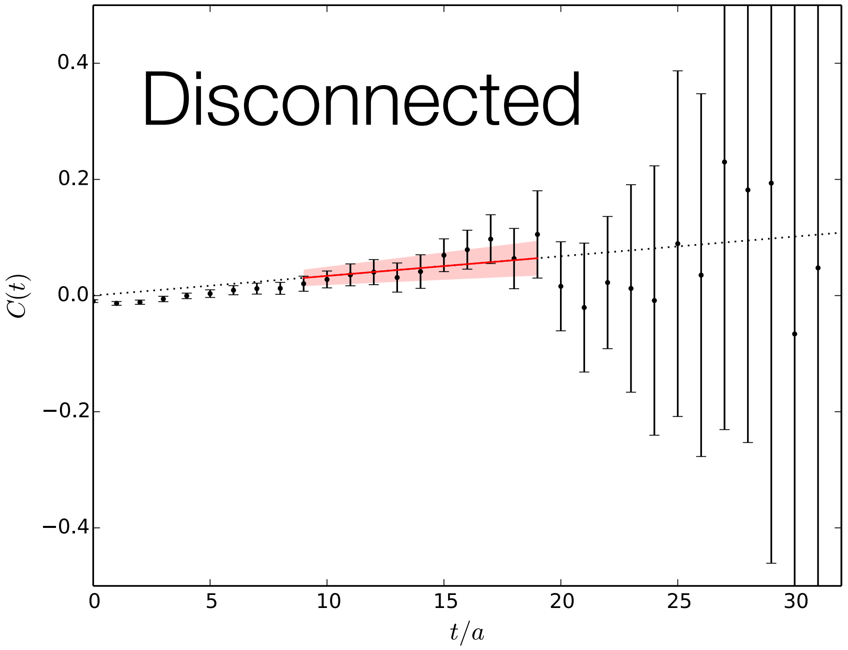
- Difficult to distinguish tangent behaviour from excited states



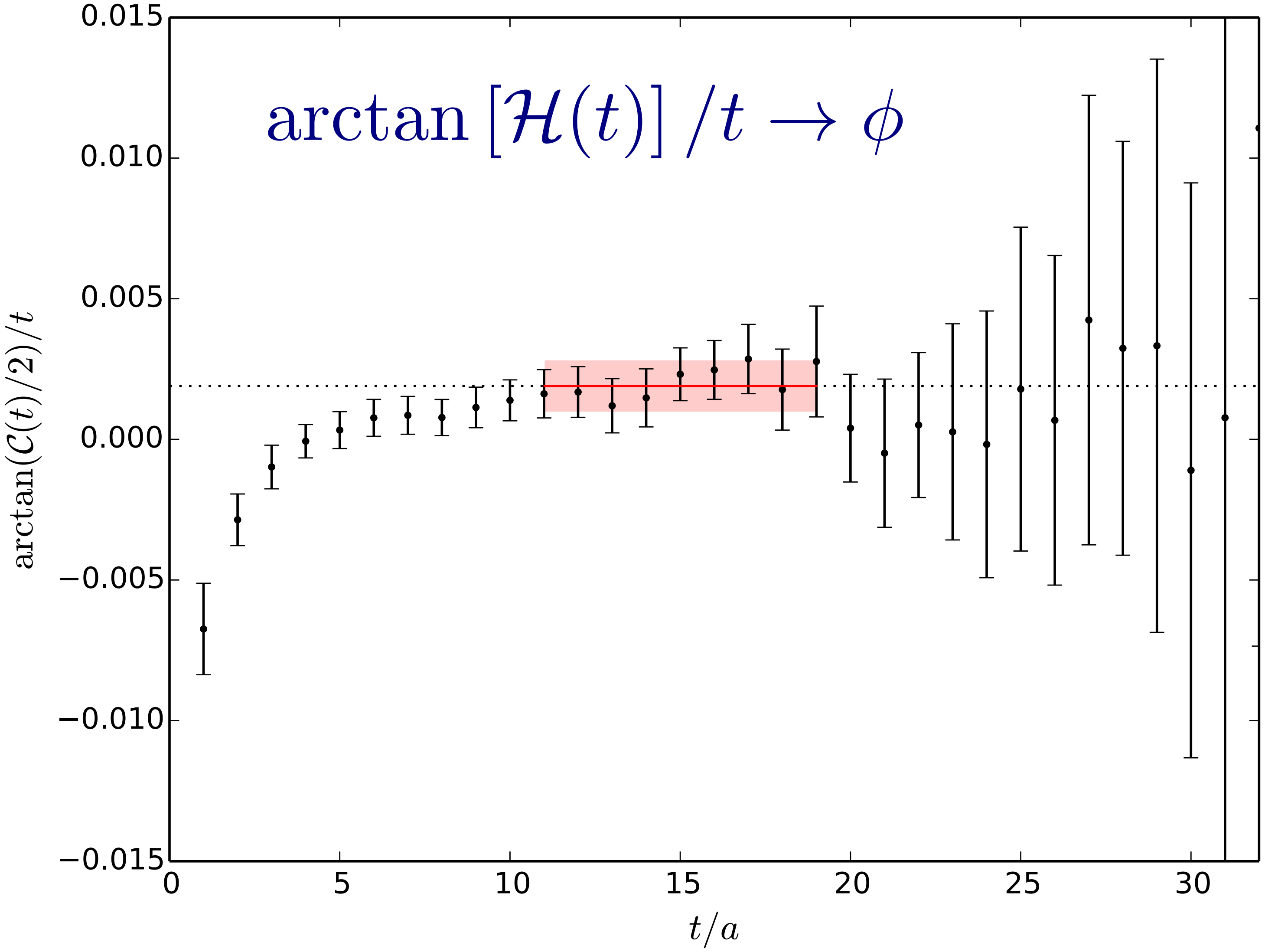
$$\mathcal{H}(t) = \frac{\mathfrak{Im}\left[C_+(t) - C_-(t)\right]}{\mathfrak{Re}\left[C_+(t) + C_-(t)\right]} \to \tan(\phi t)$$

Disconnected Spin Contributions

- Difficult to distinguish tangent behaviour from excited states

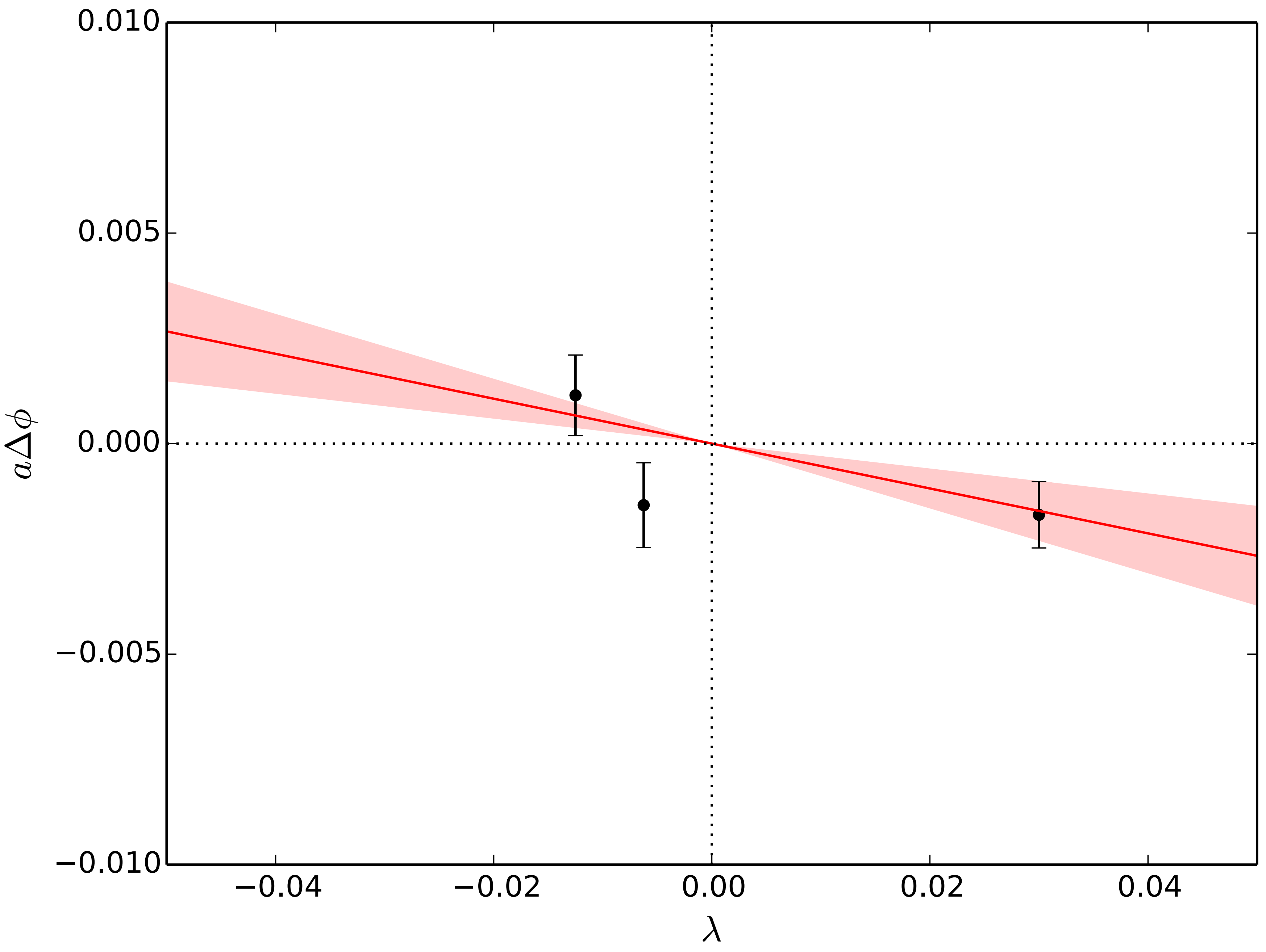


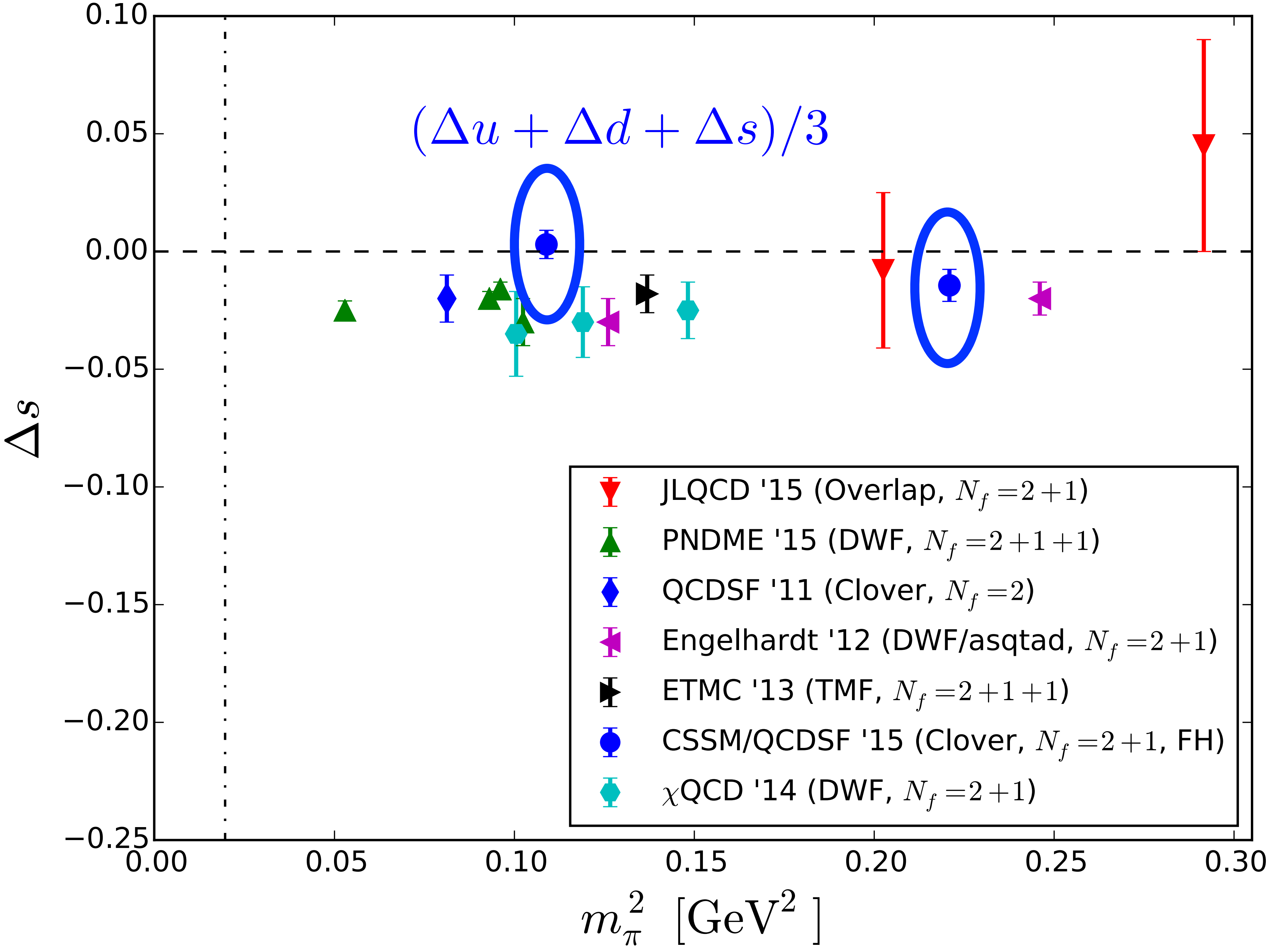
$$\mathcal{H}(t) = \frac{\mathfrak{Im}\left[C_+(t) - C_-(t)\right]}{\mathfrak{Re}\left[C_+(t) + C_-(t)\right]} \rightarrow \tan(\phi t)$$



Disconnected Spin Contributions

- SU(3) symmetric point; 3 field strengths





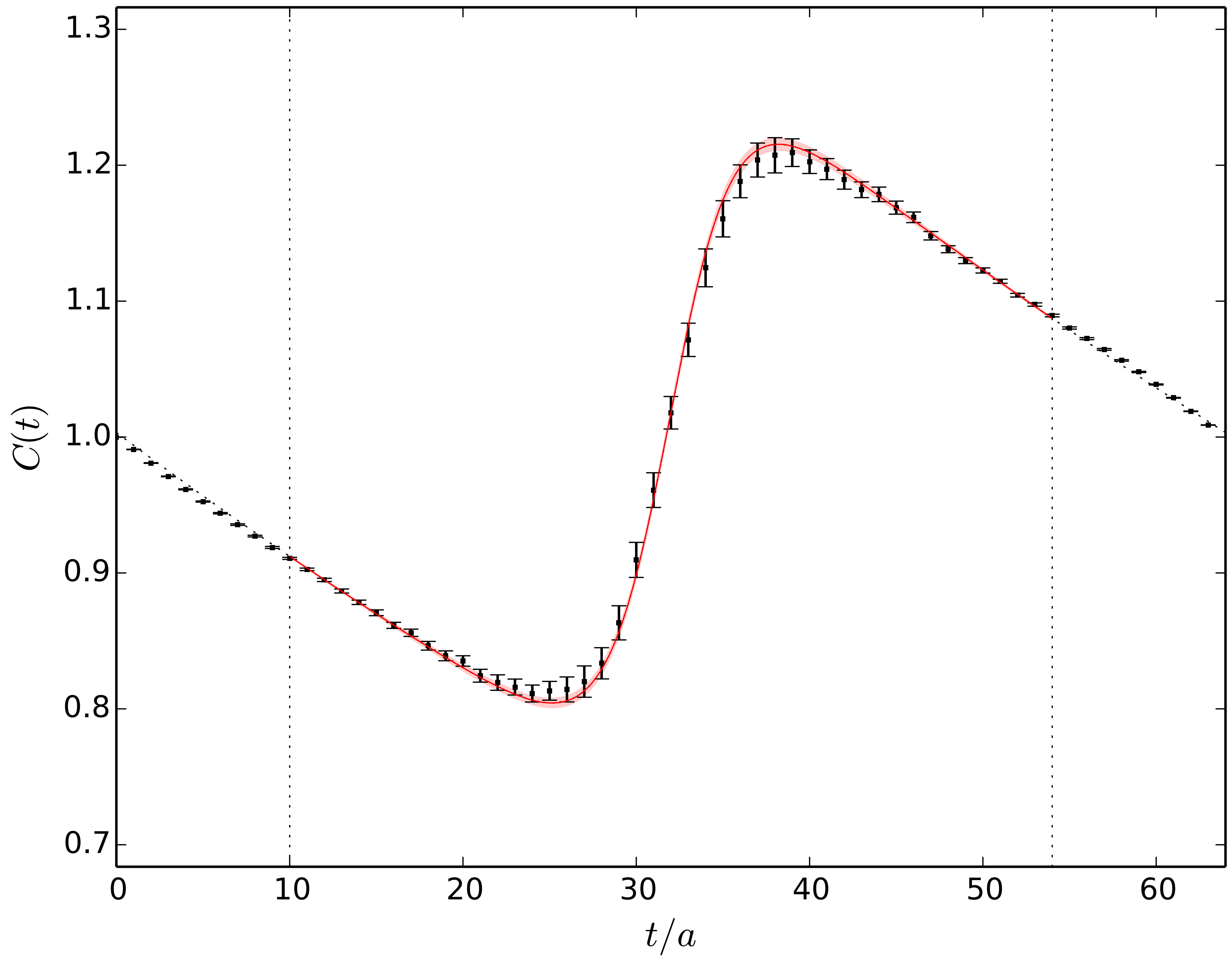
Strangeness spin

Global comparison

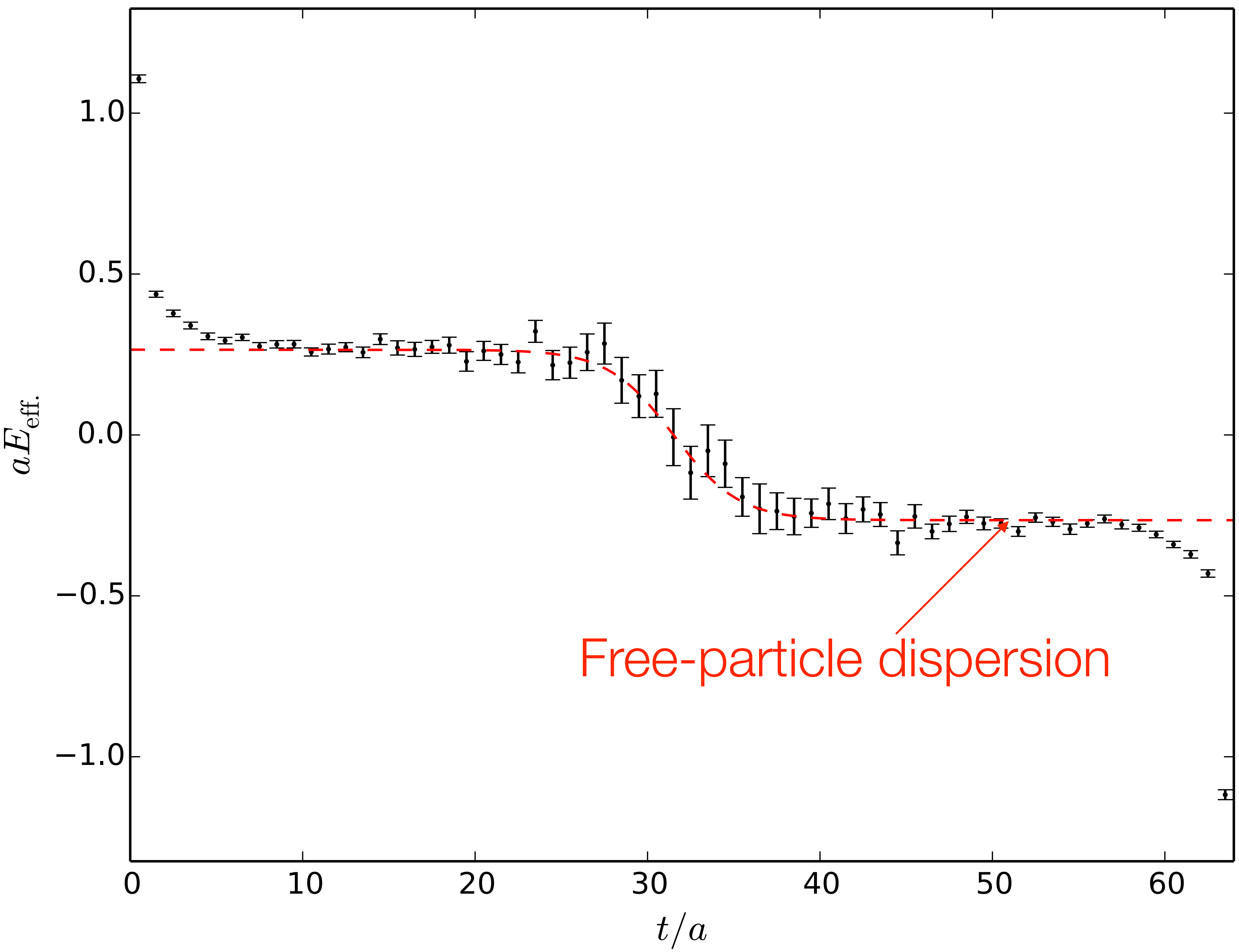
FH Non-forward: Pion

- Ratio of correlators

$$C(t) = \frac{G(\lambda,t)}{G(0,t)} \overset{t\to\infty}{\simeq} \frac{A(\lambda)}{A(0)} \frac{\left[e^{(E+\Delta E)t} + e^{(E-\Delta E)(T-t)}\right]}{e^{-Et} + e^{-E(T-t)}}$$



$$\vec{p} = (-1,0,0)$$
$$\vec{q} = (\ \ 1,1,0)$$
$$\vec{p}' = (\ \ 0,1,0)$$

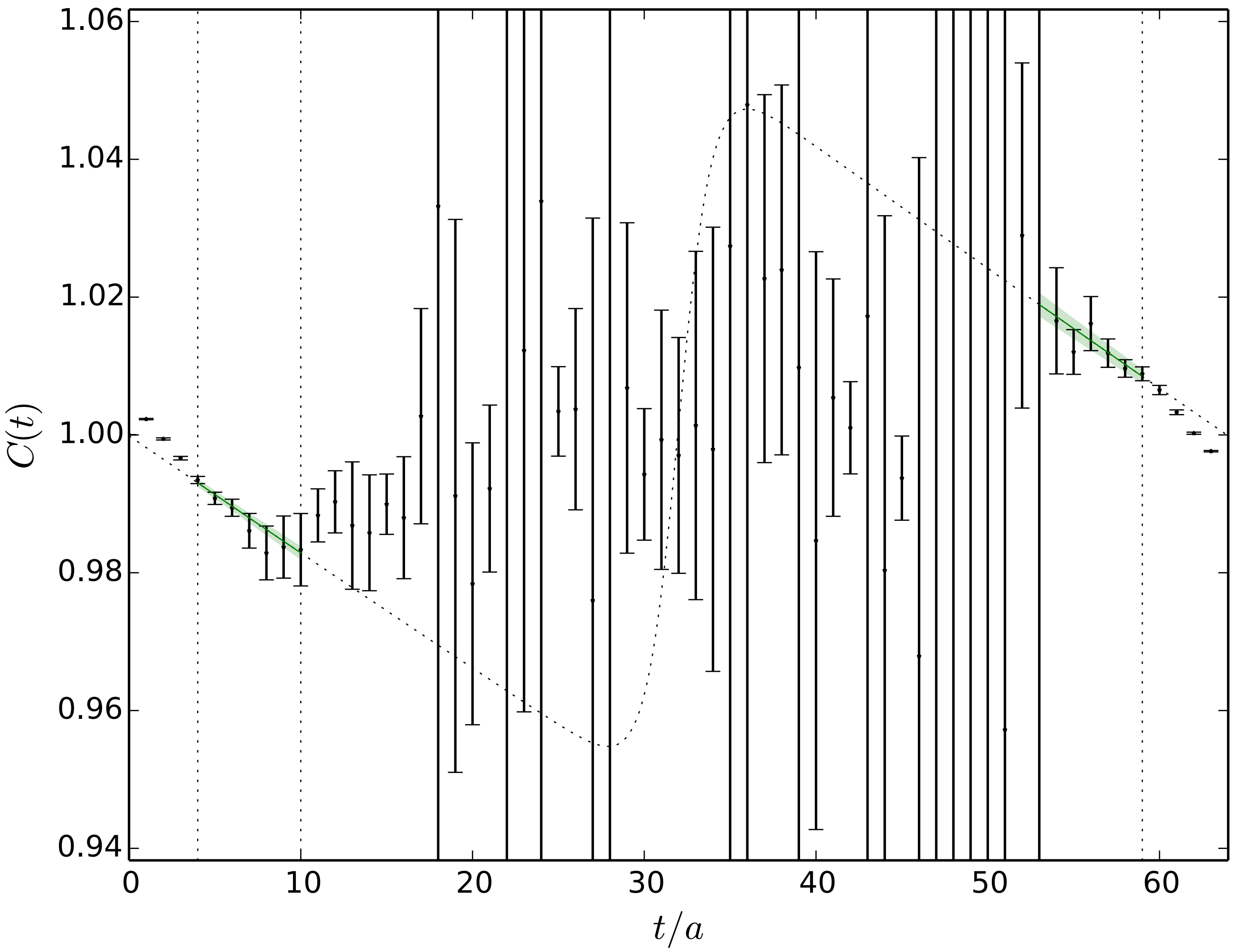


Effective "Mass" plot

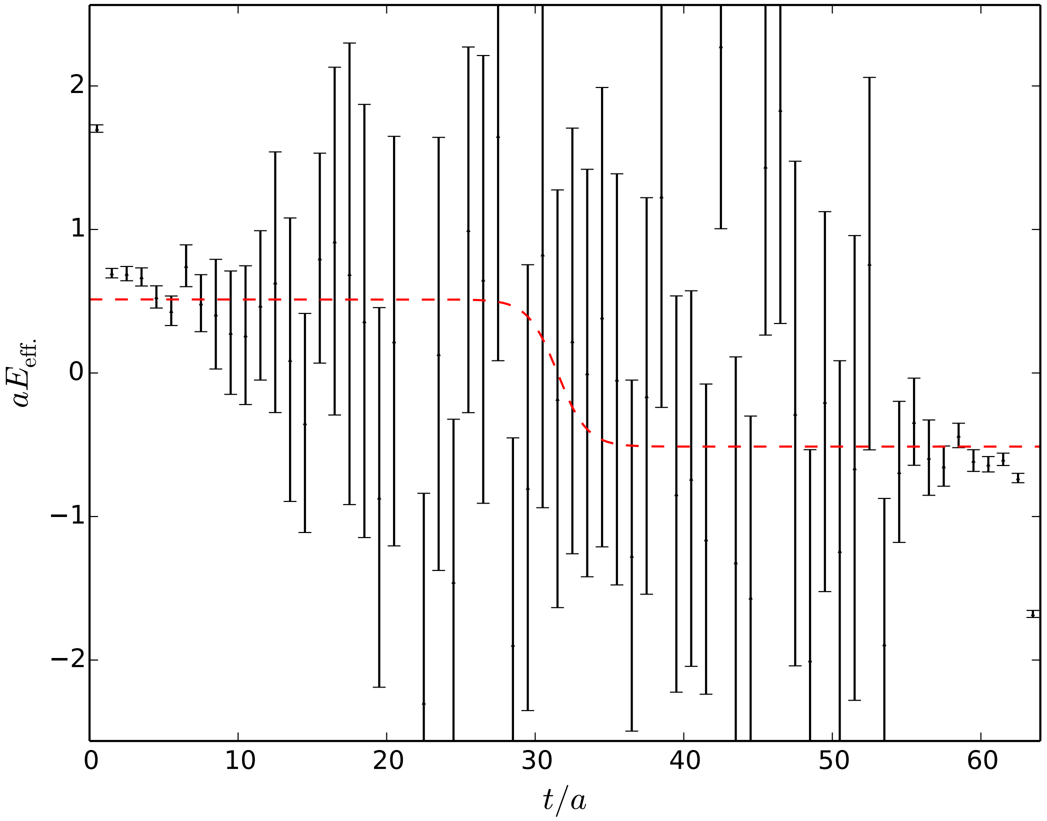
Only determine energy shift
where ground state saturated

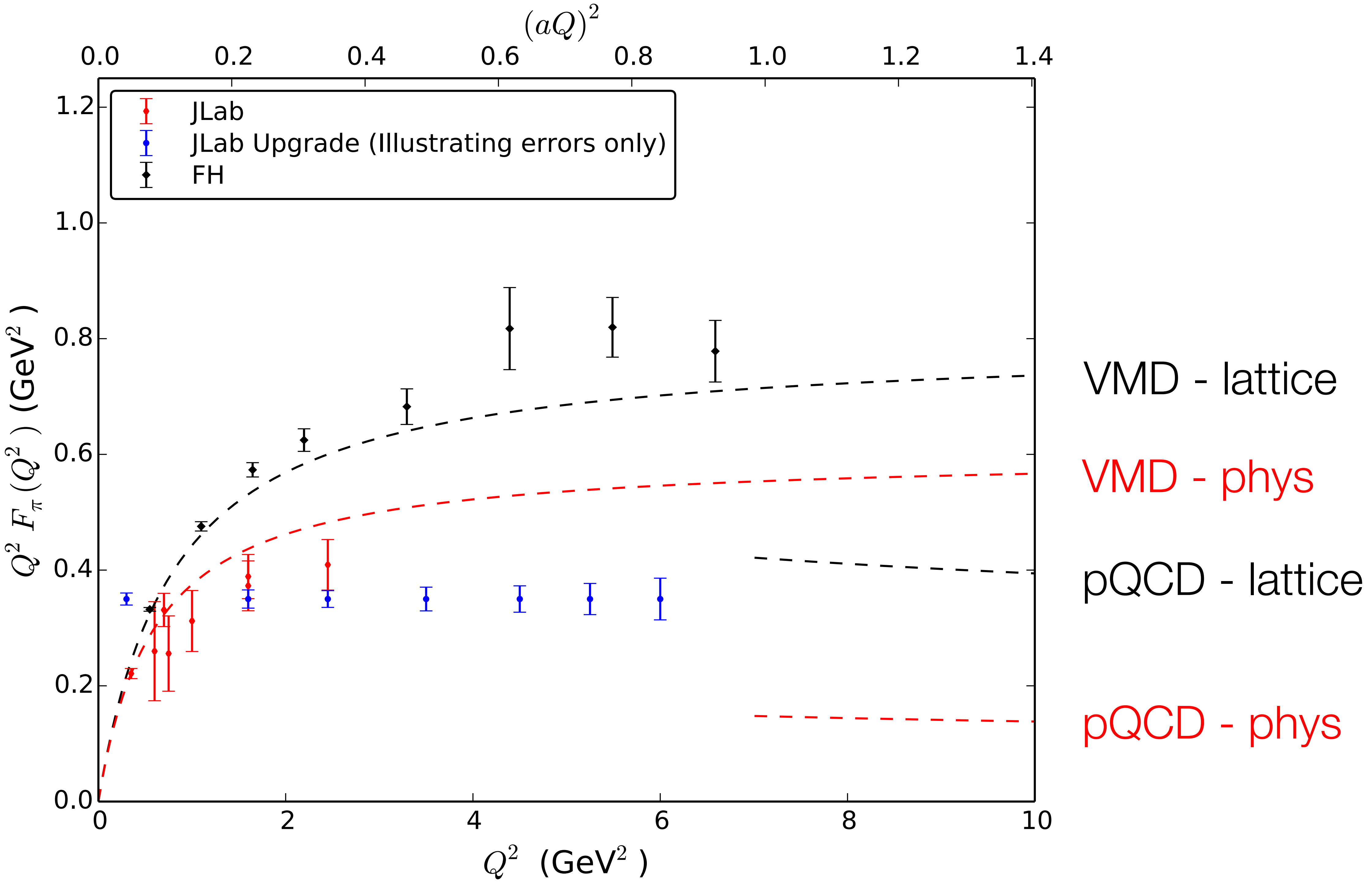
Pushing the limits

- “Extreme” momentum injection



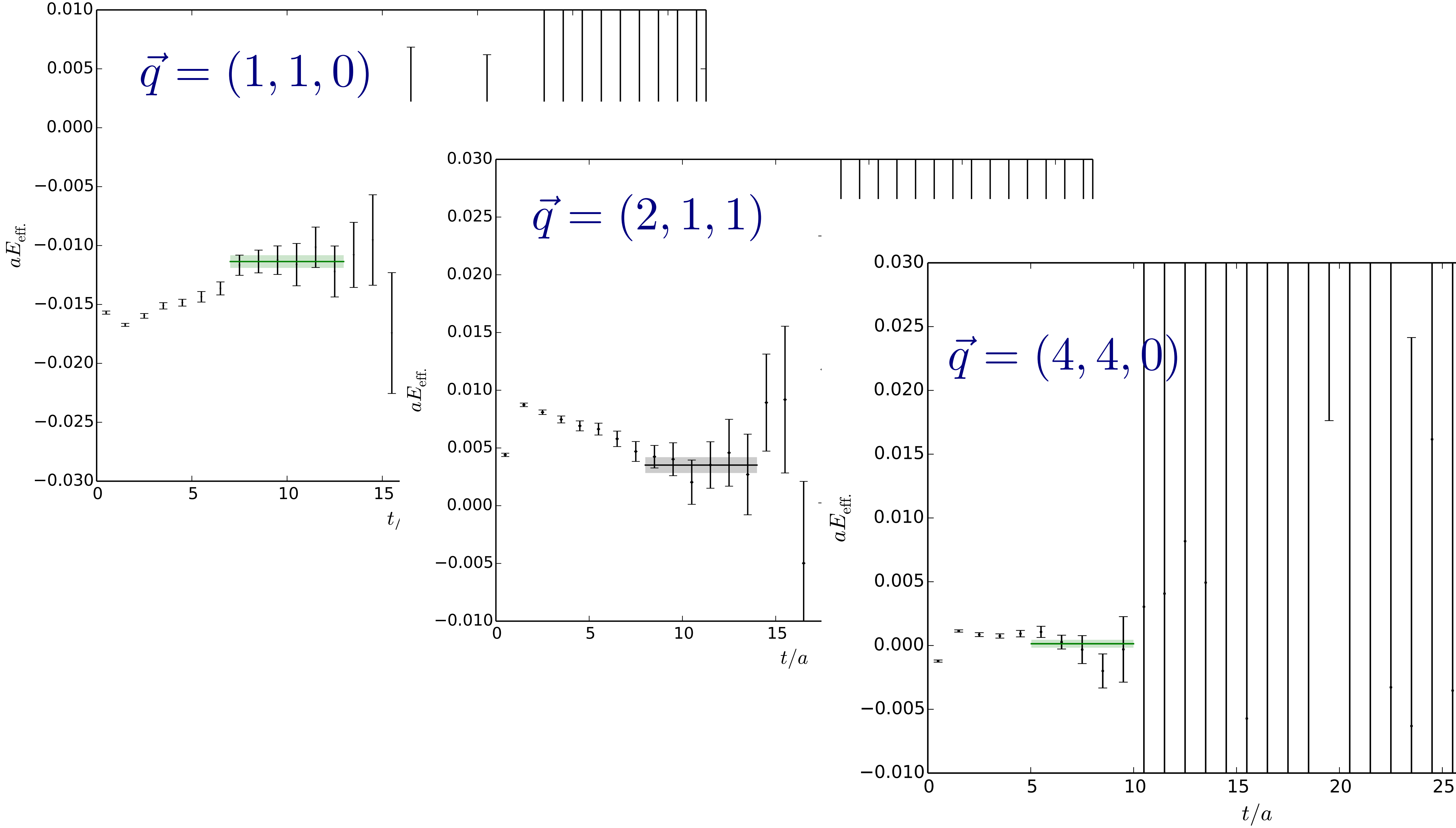
$$\vec{p} = (-2, -1, -1)$$
$$\vec{q} = (\ \ 4, \ \ 2, \ \ 2)$$
$$\vec{p}\,' = (\ \ 2, \ \ 1, \ \ 1)$$





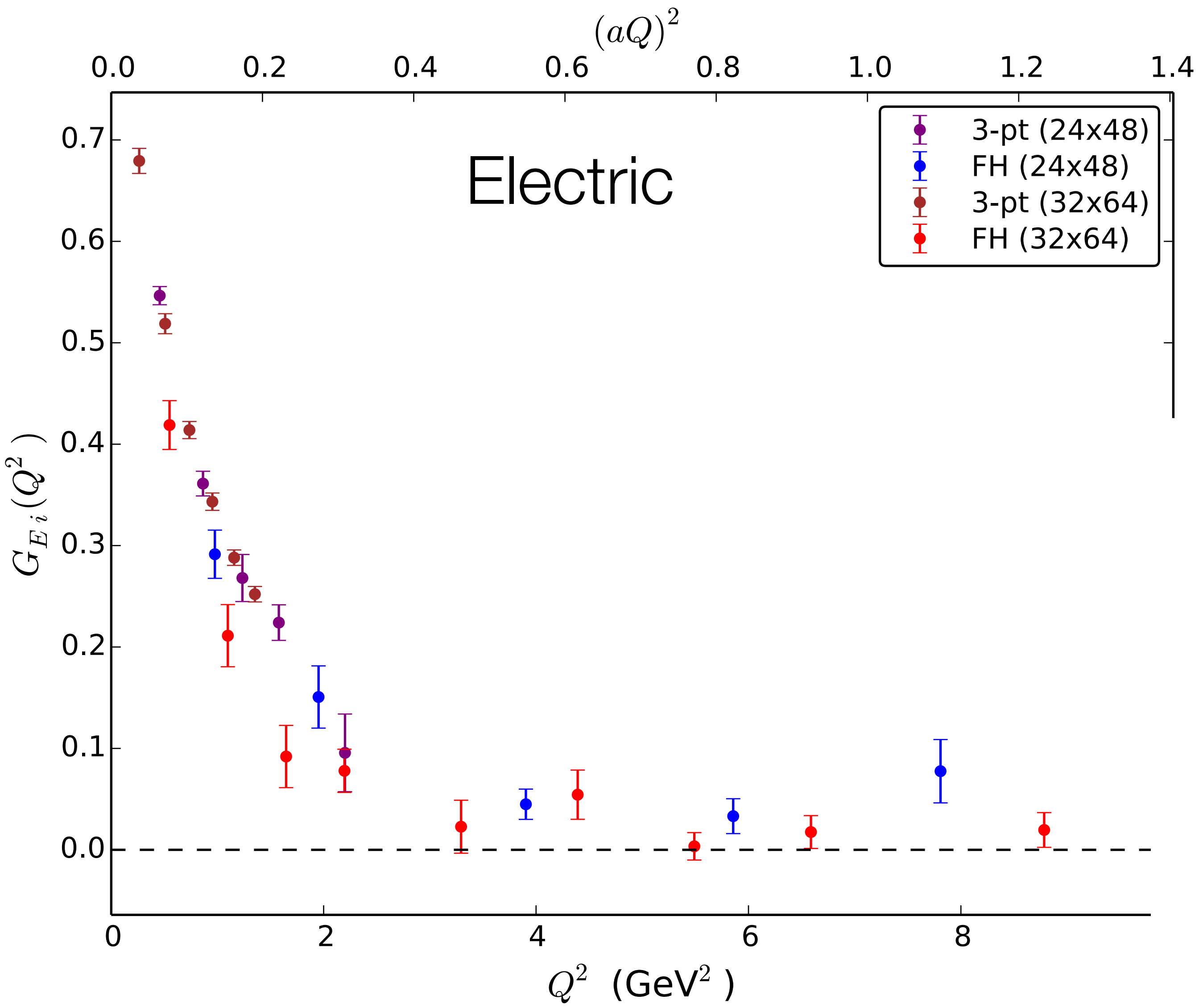
Pion Form Factor

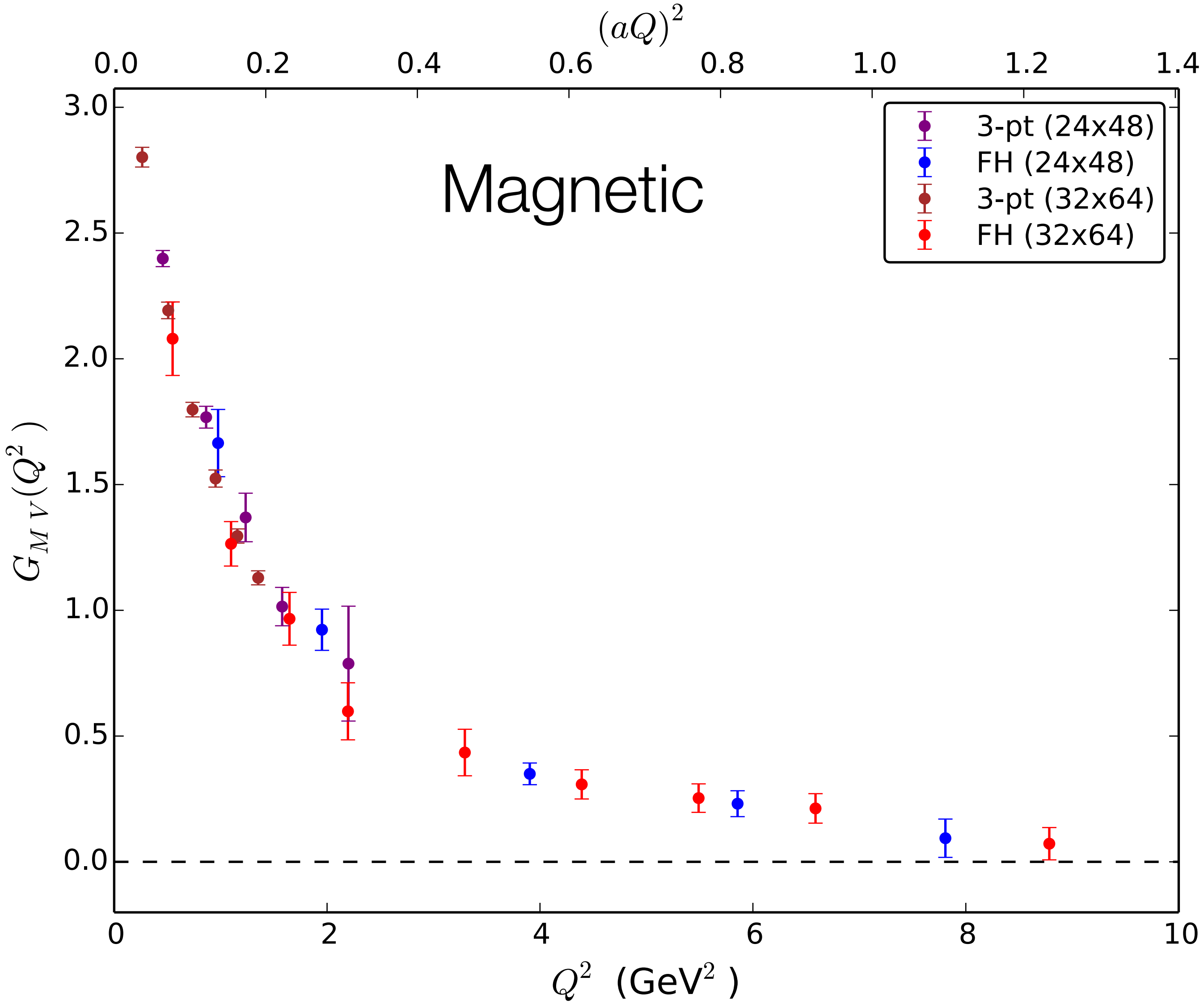
Can probe momentum range of JLab 12 GeV proposal



Nucleon Form Factors

“Up quark” temporal current energy shifts





Nucleon Form Factors

Isovector

Nucleon Form Factors

Dirac and Pauli form factors defined by

$$\langle\, \vec{p}'\ \vec{s}'\,|\,\bar{q}(0)\gamma_\mu q(0)\,|\,\vec{p}\ \vec{s}\,\rangle = \bar{u}(\vec{p}', \sigma')\left[\gamma_u F_1(Q^2) + \sigma_{\mu\nu}\frac{q_\nu}{2m}F_2(Q^2)\right] u(\vec{p}, \sigma)$$

Make idential modification to the action as for the pion case

$$\mathcal{L}(y) \to \mathcal{L}(y) + \lambda\, e^{i\vec{q}\cdot(\vec{y}-\vec{x})}\, \bar{q}(y)\, \gamma_\mu\, q(y)$$

Feynman-Hellmann relation gives

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{F_3(\Gamma, \mathcal{J}_{\mathcal{O}}; \vec{p}, \vec{p}, m)}{F_2(\Gamma; \vec{p}, m)}$$

Need to make choice of projection matrix Γ

Nucleon Form Factors

For temporal current, choose projection matrix

$$\Gamma_{\text{unpol.}} = \frac{1}{2}(1 + \gamma_4)$$

then the Feynman-Hellmann relation gives

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{m}{E}\left[\left(1 + \frac{(\vec{p} + \vec{p}\,')^2}{4m(E+m)}\right) F_1(Q^2) - \frac{Q^2}{4m^2} F_2(Q^2)\right]$$

For $\vec{p}\,' = -\vec{p}$ we have simply

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{m}{E} G_E(Q^2)$$

Nucleon Form Factors

For spatial current, choose $\Gamma_\pm$ as in axial charge calculation

$$\Re\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \pm\frac{1}{2mE}\left[\vec{s}\times\vec{q}\,F_1(Q^2)\right.$$
$$\left.\left[\left[1+\frac{(\vec{p}+\vec{p}\,')^2}{4m(E+m)}\right]\vec{s}\times\vec{q}-\frac{\vec{s}\cdot(\vec{p}+\vec{p}\,')\vec{p}\times\vec{p}\,'}{2m(E+m)}\right]F_2(Q^2)\right]$$

For $\vec{p}\,' = -\vec{p}$ we have simply

$$\Re\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \pm\frac{\vec{s}\times\vec{q}}{2mE}G_M(Q^2)$$