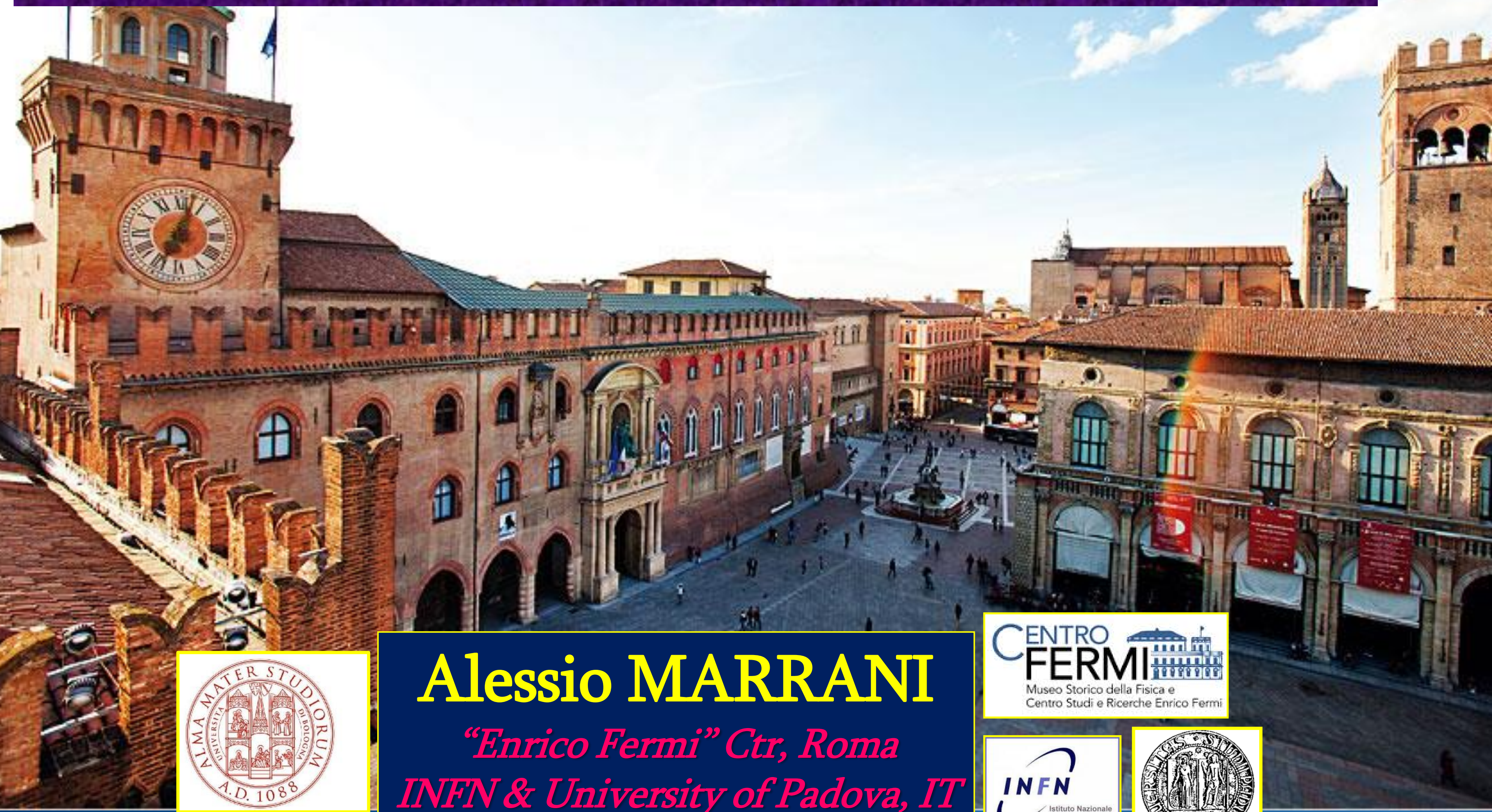
Bekenstein and Hawking meet Jordan and Freudenthal : Non-Linear Symmetries of Black Hole Entropy



Alessio MARRANI

"Enrico Fermi" Ctr, Roma

INFN & University of Padova, IT

Bologna, May 11, 2016

Bibliography

Borsten, Dahanayake, Duff, Rubens, Phys. Rev. **D80** (2009) 026003, arXiv:0903.5517 [hep-th]

Ferrara, AM, Yeranyan, Phys. Lett. **B701** (2011) 640, arXiv:1102.4857 [hep-th]

Ferrara, AM, Orazi, Trigiante, JHEP **1311** (2013) 056, arXiv:1305.2057 [hep-th]

AM, *Freudenthal Duality in Gravity : from Groups of Type E7 to Pre-Homogeneous Vector Spaces*, arXiv:1509.01031 [hep-th]

see also (other applications)

Borsten, Duff, Ferrara, AM, Class. Quant. Grav. **30** (2013) 235003, arXiv:1212.3254 [hep-th]

Galli, Meessen, Ortin, JHEP **1305** (2013) 011, arXiv:1211.7296 [hep-th]

AM, Qiu, Shih, Tagliaferro, Zumino, JHEP **1303** (2013) 132, arXiv:1208.0013 [hep-th]

Fernandez-Melgarejo, Torrente-Lujan, JHEP **1405** (2014) 081, arXiv:1310.4182 [hep-th]

Summary

Maxwell-Einstein-Scalar Gravity Theories

Symmetric Scalar Manifolds :
Application to (**Super)Gravity** and **Extremal Black Holes**

Attractor Mechanism

U-Duality Orbits, Stability of **Attractors**,
Flat Directions and ***"Moduli Spaces"***

The matrix **M** and **Freudenthal Duality**

Groups "of type E_7"

Rigid **Special Kaehler Geometry** of **U-Orbits**
and **Pre-Homogeneous Vector Spaces**

Hints for the Future…

Maxwell-Einstein-Scalar Theories

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\,\partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

$$H := \left(F^\Lambda, G_\Lambda\right)^T;$$

$$^*G_{\Lambda|\mu\nu} := 2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.$$

D=4 Maxwell-Einstein-scalar system (with no potential)

[may be the bosonic sector of D=4 (ungauged) sugra]

Abelian 2-form field strengths

static, spherically symmetric, asympt. flat, **extremal BH**

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right]$$

$\tau := -1/r$

$$Q \ : \ = \int_{S^2_\infty} H = \left(p^\Lambda, q_\Lambda\right)^T;$$

$$p^\Lambda \ : \ = \frac{1}{4\pi}\int_{S^2_\infty} F^\Lambda, \ \ q_\Lambda = \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda.$$

dyonic vector of e.m. fluxes
(**BH charges**)

$$S_{D=1} = \int [(U')^2 + g_{ij}\varphi'^i\varphi'^j + e^{2U}V_{BH}(\varphi(\tau),\mathcal{Q})]d\tau \qquad ' \equiv \frac{d}{d\tau}$$

reduction D=4 →D=1 :effective 1-dimensional (radial) Lagrangian

$$V_{BH}(\varphi,\mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$$

BH effective potential

Ferrara,Gibbons,Kallosh

eoms

$$\begin{cases} \frac{d^2U}{d\tau^2} = e^{2U}V_{BH}; \\ \frac{d^2\varphi^i}{d\tau^2} = g^{ij}e^{2U}\frac{\partial V_{BH}}{\partial\varphi^j}. \end{cases}$$

in N=2 ungauged sugra, hyper mults. decouple, and we thus disregard them : scalar fields belong to vector mults.

Attractor Mechanism : $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\to-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

conformally flat geometry $AdS_2 \times S^2$ near the horizon

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}}dt^2 - \frac{M^2_{\text{B-R}}}{r^2}(dr^2 + r^2 d\Omega)$$

near the horizon, the scalar fields are **stabilized** purely in terms of charges

$$S = \frac{A_H}{4} = \pi\, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$$

Bekenstein-Hawking entropy-area formula for extremal dyonic BH

Symmetric Scalar Manifolds

Let's specialize the discussion to theories with scalar manifolds which are **symmetric cosets G/H**

[**N>2** : general, **N=2** : particular, **N=1** : special cases]

H = isotropy group = *linearly* realized; scalar fields sit in an **H**-repr.

G = (global) electric-magnetic duality group
[in string theory : **U-duality**]

G is an *on-shell* symmetry of the Lagrangian

The 2-form field strengths (F,G) vector and the BH e.m. charges sit in a **G**-repr. **R** which is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \mathbb{C}_{MN} = -\langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

symplectic product

$$\begin{aligned} G &\subset Sp(2n, \mathbb{R}); \\ \mathbf{R} &= \mathbf{2n} \end{aligned}$$

Gaillard-Zumino embedding
(generally maximal, but not symmetric)

Dynkin, Gaillard-Zumino

Symmetricity : algebraic definition :

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k},$ Cartan decomposition of a Lie algebra g

h = compact Lie subalgebra

k = complementary of h in g

$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ from the definition of subalgebra

$[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$ by the adjoint action, h acts on k as a repr. whose real dim. is dim(G/H) (it holds in any coset G/H)

$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$ symmetricity condition; in gen. it simply holds $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{g}$

Symmetricity : differential definition **:** :

$D_m R_{ijkl} = 0$ the Riemann tensor is covariantly constant

All symmetric scalar manifolds in supergravity are:

- strictly positive definite metric;
- ***Einstein spaces***, with (constant) *negative* scalar curvature : $R_{ij} = \lambda g_{ij}$

❖ **symmetric** scalar manifolds of **N=2, D=4 sugra**

all special Kaehler of projective type	$\frac{G_V}{H_V}$	r	$dim_{\mathbb{C}} \equiv n_V$
quadratic sequence $n \in \mathbb{N}$	$\frac{SU(1,n)}{U(1)\otimes SU(n)}$	1	n
$\mathbb{R} \oplus \mathbf{\Gamma}_n,\ n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\otimes SO(n)}$	$2\ (n=1)$ $3\ (n \geqslant 2)$	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\otimes U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3)\otimes U(3))} = \frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\overline{C}_{\overline{jlp}}g^{m\bar{p}}$$

$J_3^{\mathbb{A}}, \quad \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is the *Jordan algebra* of degree 3 of Hermitian 3x3 matrices over the 4 *division algebras* of real (**R**), complex (**C**), quaternions (**H**), octonions (**O**)

$\Gamma_{m,n}$ is the Jordan algebra of degree 2 with a quadratic form with Lorentzian signature (m,n)

Jordan algebras were completely classified by Jordan, Von Neumann and Wigner in an attempt to generalize *Quantum Mechanics* beyond **C**

Gunaydin,Sierra,Townsend

They are related to the Magic Square of Freudenthal, Rosen and Tits

All Magic Squares of order 3 recently **classified and interpreted in sugra** in Cacciatori, Cerchiai, AM, **arXiv:1208.6153 [math-ph]**

J	$\mathrm{Aut}(J)$	$\mathrm{Str}_0(J)$	$\mathrm{Conf}(J)$	$\mathrm{QConf}(J)$
$\mathbb{R}$	1	1	$Sl(2,\mathbb{R})$	$G_{2(2)}$
$\mathbb{R} \oplus \Gamma_{n-1,1}$	$SO(n-1)$	$SO(n-1,1)$	$Sl(2) \times SO(n,2)$	$SO(n+2,4)$
$J_3^{\mathbb{R}}$	$SO(3)$	$Sl(3,\mathbb{R})$	$Sp(6)$	$F_{4(4)}$
$J_3^{\mathbb{C}}$	$SU(3)$	$Sl(3,\mathbb{C})$	$SU(3,3)$	$E_{6(+2)}$
$J_3^{\mathbb{H}}$	$USp(6)$	$SU^*(6)$	$SO^*(12)$	$E_{7(-5)}$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$	$E_{8(-24)}$

symmetric scalar manifolds **G/H** (including symm. SKGs of N=2, D=4 sugra) :

The **G**-representation space **R** of the BH em charges gets **stratified**, under the action of **G**, in **U-orbits** (*non-symmetric* cosets **G/$\mathcal{H}$**).

Ferrara, Gunaydin

$\mathcal{H}$ is the **stabilizer** (isotropy) group of the **U-orbit** = symmetry of the charge configs., it relates equivalent BH charge configs

each **U-orbit** supports a class of crit. pts. of V_{BH} , corresponding to specific ***SUSY-preserving properties*** of the near-horizon geometry

*[We will here be considering the so-called **"large" U-orbits**, corresponding to extremal BHs with classical non-vanishing entropy]*

When $\mathcal{H}$ is **non-compact**, there is a residual compact symmetry linearly acting on scalars, such that the scalars belonging to the
***"moduli space"* $\mathcal{H}$/mcs($\mathcal{H}$)** (symmetric **submanifold** of **G/H**)
are **not** stabilized in terms of BH charges at the event horizon of the extremal BH

Ferrara, AM

The Attractor Mechanism is **inactive** on these **unstabilized** scalar fields, which are ***flat directions*** of V_{BH} at its critical points.

symmetric scalar manifolds **G/H** (cont'd) :

The **absence** of flat directions at **N=2 ½-BPS attractors** can thus be explained by the fact that the stabilizer of the ½-BPS orbit is **compact** : **$\mathcal{H}$=H/U(1)**, where H is the stabilizer of the scalar manifold **G/H** itself

The **massless Hessian modes**, ubiquitous at non-BPS crit pts of V_{BH}, are actually **all *flat directions*** of V_{BH} itself at the considered class of crit. pts.

In other words, ***at each class of its crit pts***, **V_{BH}**, and thus the (semi)classical **Bekenstein-Hawking BH entropy**, does **not** depend on a certain subset of the scalars

Such a set of scalars is thus **not stabilized** at the BH event horizon. *Nevertheless…*

BH Entropy is Independent on All Unstabilized Scalars

Thus, the ***flat directions*** of V_{BH} at its critical points span various "***moduli spaces"***, related to the solutions of the ***Attractor Eqs*.**

❖ *"large"* **U-Orbits** of symmetric **N=2, D=4 sugras**

	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_0}$	non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{non-BPS,Z\neq 0} = \frac{G}{\tilde{H}}$	non-BPS, $Z = 0$ orbits $\mathcal{O}_{non-BPS,Z=0} = \frac{G}{\hat{H}}$
Quadratic Sequence $(n = n_V \in \mathbb{N})$	$\frac{SU(1,n)}{SU(n)}$	$-$	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(n)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(1,1)\otimes SO(1,n-1)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Bellucci, Ferrara, Gunaydin, AM

in N=2 : $\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{[AB]} = \epsilon_{\alpha\beta}\epsilon^{AB} Z$

❖ non-BPS Z<>0 **attractor** ***"moduli spaces"*** of symmetric **N=2, D=4 sugras**

Ferrara,AM

$\hat{h}$ =mcs $\hat{H}$

	$\frac{\hat{H}}{\hat{h}}$	r	$dim_{\mathbb{R}}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$	$1 (n=1)$ $2 (n \geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	6
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,C)}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

They are nothing but the ***real special*** scalar manifolds
of symmetric **N=2, D=5 sugras**

let's reconsider the starting **Maxwell-Einstein-scalar Lagrangian density**

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

…and introduce the following real 2n x 2n matrix :

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + R I^{-1} R & -R I^{-1} \\ -I^{-1} R & I^{-1} \end{pmatrix}$$

$$\mathcal{M} = \mathcal{M}(R, I) = \mathcal{M}(\mathrm{Re}(\mathcal{N}), \mathrm{Im}(\mathcal{N})).$$

$$\mathcal{M}^T = \mathcal{M} \qquad \mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$$

$$\mathcal{M} = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1} = -\mathbf{L}^{-T}\mathbf{L}^{-1},$$

L = element of the **Sp(2n,R)**-bundle over the scalar manifold
(= *coset representative* for homogeneous spaces **G/H**)

…by virtue of this matrix, one can introduce a (scalar-dependent) **anti-involution in *any* Maxwell-Einstein-scalar gravity theory** with **symplectic structure** :

$$\mathcal{S}(\varphi) \quad : \quad = \mathbb{C}\mathcal{M}(\varphi)$$

$$\mathcal{S}^2(\varphi) \;=\; \mathbb{C}\mathcal{M}(\varphi)\,\mathbb{C}\mathcal{M}(\varphi) = \mathbb{C}^2 = -\mathbb{I},$$

Ferrara,AM,Yeranyan; Borsten,Duff, Ferrara,AM

…in turn, this allows to define an **anti-involution** on the dyonic charge vector Q, which has been called (**scalar-dependent**) **Freudenthal duality (F-duality)**

$$\mathfrak{F}(\mathcal{Q}) := -\mathcal{S}(\varphi)(\mathcal{Q}).$$

$$\mathfrak{F}^2 \;=\; -Id.$$

By recalling

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\,\mathcal{Q},$$

Freudenthal duality can be related to the **effective BH potential** :

$$\mathfrak{F} : \mathcal{Q} \to \mathfrak{F}(\mathcal{Q}) := \mathbb{C}\frac{\partial V_{BH}}{\partial \mathcal{Q}}.$$

All this enjoys a remarkable physical interpretation when evaluated **at the horizon** :

Attractor Mechanism $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\to-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

Bekenstein-Hawking entropy $S = \frac{A_H}{4} = \pi\, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$

…by evaluating the matrix M at the horizon $\lim_{\tau\to-\infty}\mathcal{M}(\varphi(\tau)) = \mathcal{M}_H(\mathcal{Q})$

one can define the **horizon Freudenthal duality** as:

$$\lim_{\tau\to-\infty}\mathfrak{F}(\mathcal{Q}) =: \mathfrak{F}_H(\mathcal{Q}) = -\mathbb{C}\mathcal{M}_H\mathcal{Q} = \frac{1}{\pi}\mathbb{C}\frac{\partial S_{BH}}{\partial Q} =: \tilde{\mathcal{Q}},$$

$$\mathfrak{F}^2_H(\mathcal{Q}) = \mathfrak{F}_H(\tilde{\mathcal{Q}}) = -\mathcal{Q}$$

non-linear (scalar-independent) anti-involutive map on Q (hom of degree one)

Bek.-Haw. entropy is **invariant** under its **non-linear symplectic gradient** (defined by **F-duali**ty) :

$$S(\mathcal{Q}) = S(\mathfrak{F}_H(\mathcal{Q})) = S\left(\frac{1}{\pi}\mathbb{C}\frac{\partial S}{\partial \mathcal{Q}}\right) = S(\tilde{\mathcal{Q}})$$

This can be extended to include *at least* **all quantum corrections** with **homogeneity 2** or **0** in the BH charges Q

Ferrara, AM, Yeranyan
(and late Raymond Stora)

Lie groups "of type E_7" : (G,**R**)

Brown (**1967**);
Garibaldi; Krutelevich;
Borsten,Duff *et al.*
Ferrara,Kallosh,AM;
AM,Orazi,Riccioni

❖ the (ir)repr. **R** is **symplectic** :

$$\exists! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \quad \langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = -\langle Q_2, Q_1 \rangle ;$$

symplectic product

❖ the (ir)repr. admits a unique completely symmetric **invariant rank-4** tensor

$$\exists! \; K_{MNPQ} = K_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s$$ (K-tensor)

G-invariant quartic polynomial

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon \, |I_4| \, , \quad \rightarrow \quad S_{BH} = \pi \sqrt{|I_4|}$$

❖ defining a triple map in **R** as

$$T : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \quad \langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q$$

it holds $\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle \, K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q$

this third property makes a **group of type E_7** amenable to a description in terms of **Freudenthal triple systems**

All electric-magnetic **(U-)duality** groups of D=4 sugras with **symmetric** scalar manifolds and *at least* **8** supersymmetries are **"of type E_7"**

$N = 2$

G	R
$U(1,n)$	$(\mathbf{1+n})$
$SL(2,\mathbb{R}) \times SO(2,n)$	$(\mathbf{2,2+n})$
$SL(2,\mathbb{R})$	$\mathbf{4}$
$Sp(6,\mathbb{R})$	$\mathbf{14'}$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

N	G	R
3	$U(3,n)$	$(\mathbf{3+n})$
4	$SL(2,\mathbb{R}) \times SO(6,n)$	$(\mathbf{2,6+n})$
5	$SU(1,5)$	$\mathbf{20}$
8	$E_{7(7)}$	$\mathbf{56}$

(E_7, 912 – embedding tensor) satisfies the first two axioms, *but **not** the third one!*

"degenerate" groups "of type E_7"

$$I_4(p,q) = (I_2(p,q))^2$$

$$S_{BH} = \pi\sqrt{|I_4(p,q)|} = \pi\,|I_2(p,q)|\,.$$

In sugras with electric-magnetic duality group **"of type E_7"**, the **G**-invariant **K-tensor** determining the extremal BH Bekenstein-Hawking entropy

$$S_{BH} = \pi\sqrt{|I_4|}$$

$$I_4 := K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q =: \epsilon\,|I_4|\,,$$

can generally be expressed as adjoint-trace of the product of **G**-generators (dim **R** = 2n, and dim **Adj** = d)

$$K_{MNPQ} = -\frac{n\,(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n(2n+1)}\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right]$$

The **horizon Freudenthal duality** can be expressed in terms of the **K-tensor**

$$\mathfrak{F}_H(\mathcal{Q})_M = \tilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

Borsten,Dahanayake,Duff,Rubens

the **invariance** of the BH entropy under **horizon Freudenthal duality** reads as

$$I_4\,(\mathcal{Q}) = I_4(\mathbb{C}\tilde{\mathcal{Q}}) = I_4\left(\mathbb{C}\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}}\right)$$

Are there other relevant symplectic matrices at the horizon ? **YES!**

$$\left(M_{-}^{H}\left(\mathcal{Q}\right)\right)^{T}\mathbb{C}M_{-}^{H}\left(\mathcal{Q}\right)=\epsilon\mathbb{C}$$

$$\epsilon:=I_{4}\left(\mathcal{Q}\right)/\left|I_{4}(\mathcal{Q})\right|$$

$$\left(M_{-}^{H}\left(\mathcal{Q}\right)\right)^{T}=M_{-}^{H}\left(\mathcal{Q}\right)$$

$$\mathcal{Q}^{T}M_{-}^{H}\left(\mathcal{Q}\right)\mathcal{Q}=-2\sqrt{\left|I_{4}(\mathcal{Q})\right|}$$

$$M_{-|MN}^{H}(\mathcal{Q})=\frac{1}{\sqrt{\left|I_{4}(\mathcal{Q})\right|}}\widetilde{\mathcal{Q}}_{M}\widetilde{\mathcal{Q}}_{N}-\epsilon\frac{6}{\sqrt{\left|I_{4}(\mathcal{Q})\right|}}K_{MN}$$

$$K_{MN}:=K_{MNPQ}\mathcal{Q}^{P}\mathcal{Q}^{Q}$$

$$M_{-|MN}^{H}=-\partial_{M}\partial_{N}\sqrt{\left|I_{4}(\mathcal{Q})\right|}=-\frac{1}{\pi}\partial_{M}\partial_{N}S_{BH}$$

(opposite of the) **Hessian of the BH entropy**

$$\mathfrak{F}_{H}\left(M_{-}^{H}\left(\mathcal{Q}\right)\right)=\epsilon M_{-}^{H}$$

$$\mathfrak{F}_{H}\left(\mathcal{M}_{H}\left(\mathcal{Q}\right)\right)=\epsilon\mathcal{M}_{H}\left(\mathcal{Q}\right)$$

This matrix is the (opposite of) **pseudo-Euclidean metric** of a non-compact, pseudo-Riemannian, **rigid special Kaehler manifold** related to the **U-orbit** of BH e.m. charges, which is an example of **pre-homogeneous vector space** (**PVS**)

Sato, Kimura

$$M^H_{-|MN}(\mathcal{Q}) = \frac{1}{\sqrt{|I_4(\mathcal{Q})|}}\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - \epsilon\frac{6}{\sqrt{|I_4(\mathcal{Q})|}}K_{MN}$$

This matrix is the (opposite of) **pseudo-Euclidean metric** of a non-compact, pseudo-Riemannian, **rigid special Kaehler manifold** related to the **U-orbit** of BH e.m. charges, which is an example of **pre-homogeneous vector space** (**PVS**)

Sato, Kimura

1st example : ***"large"*** **BPS U-orbit** in **maximal supergravity**

$$N = 8, D = 4 : scalar\ manifold\ \mathrm{M}_{N=8} = \frac{E_{7(7)}}{SU(8)},\ dim_{\mathbb{R}} = 70,\ rank = 7$$

$$I_4 > 0 : \frac{1}{8}-BPS\ E_{7(7)}-orbit\ in\ \mathbf{56}\ repr.space : \mathcal{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}$$

$$(quaternionic)\ moduli\ space\ \mathcal{M}_{I_4>0} = \frac{E_{6(2)}}{SU(6)\times SU(2)}\left(\subset \frac{E_{7(7)}}{SU(8)}\right),\ dim_{\mathbb{R}} = 40,\ rank = 4$$

$$M^H_- = -\partial^2\sqrt{I_4} : metric\ of\ \mathcal{O}_{I_4>0}\times\mathbb{R}^+ = \frac{E_{7(7)}}{E_{6(2)}}\times\mathbb{R}^+;\ \ (n_+, n_-) = (30, 26)$$

$$M^{H}_{-|MN}(\mathcal{Q}) = \frac{1}{\sqrt{|I_4(\mathcal{Q})|}}\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - \epsilon\frac{6}{\sqrt{|I_4(\mathcal{Q})|}}K_{MN}$$

2nd example : ***"large"*** **non-BPS U-orbit** in **maximal supergravity**

$$I_4 < 0 : non - BPS\ E_{7(7)}{-}orbit\ in\ \mathbf{56}\ repr.space : \mathcal{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}}$$

$$(real)\ m.s.\ \mathcal{M}_{I_4<0} = \frac{E_{6(6)}}{USp(8)} = \mathrm{M}_{N=8,D=5},\ dim_{\mathbb{R}} = 42,\ rank = 6$$

$$M^{H}_{-} = -\partial^2\sqrt{-I_4} : metric\ of\ \mathcal{O}_{I_4<0} \times \mathbb{R}^+ = \frac{E_{7(7)}}{E_{6(6)}} \times \mathbb{R}^+;\ \ (n_+, n_-) = (28, 28)$$

zero character (holding for all $\mathbf{I_4<0}$ U-orbits)

$\frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+$ and $\frac{E_{7(7)}}{E_{6(6)}} \times \mathbb{R}^+$ are non-compact, real forms of $\frac{E_7}{E_6} \times GL(1)$

Regular **Pre-Homogeneous Vector Space** (**PVS**) of type (29) in the classification by Sato and Kimura ('77) :

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$.

(i) $H \sim E_6$, (ii) $\deg f = 4$, (iii) $f(X) = T(x^\sharp, y^\sharp) - \xi N(x) - \eta N(y) - \frac{1}{4}(T(x,y) - \xi\eta)^2$ (see (1.16), or Proposition 52 in § 5).

A **PVS** is a finite-dimensional vector space ***V*** together with a subgroup ***G*** of **GL(*V*)** such that ***G*** has an **open**, **dense orbit** in ***V*** [Sato,Kimura; Knapp]

PVS are subdivided into two types, according to whether there exists a *homogeneous* polynomial **f** on ***V*** which is **invariant** under the semisimple part of ***G***.

In this case : **V** = **56** (fundamental irrep. of **G=E_7**), **f** = **quartic** invariant polynomial **I_4**
H= isotropy (stabilizer) group = **E_6**

Manifestly E_6 -invariant expression of the quartic invariant **I_4** of the **56** of **E_7** :
much before (**'77** = almost contemporary to sugra) the expression introduced by Ferrara,Gunaydin (**'97**) **!**

$$I_4\left(p^0, p^i, q_0, q_i\right) = -\left(p^0 q_0 + p^i q_i\right)^2 + 4\left[q_0 I_3(p) - p^0 I_3(q) + \left\{\frac{\partial I_3(p)}{\partial p}, \frac{\partial I_3(q)}{\partial q}\right\}\right]$$

Simple groups. **"of type E_7"** of **sugra** almost saturate list of **irr. PVS** with **invariant** deg **4**

G	V	n	Isotropy algebra	Degree	
$SL(2,\mathbb{C})$	$S^3\mathbb{C}^2$	1	0	4	N=2, T^3 model
$SL(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{sl}(3,\mathbb{C})$	4	N=2 magic on **R**
$SL(7,\mathbb{C})$	$\Lambda^3\mathbb{C}^7$	1	$\mathfrak{g}_2^{\mathbb{C}}$	7	
$SL(8,\mathbb{C})$	$\Lambda^3\mathbb{C}^8$	1	$\mathfrak{sl}(3,\mathbb{C})$	16	
$SL(3,\mathbb{C})$	$S^2\mathbb{C}^3$	2	0	6	
$SL(5,\mathbb{C})$	$\Lambda^2\mathbb{C}^3$	3,4	$\mathfrak{sl}(2,\mathbb{C}), 0$	5,10	
$SL(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^3$	2	$\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	6	
$SL(3,\mathbb{C}) \times SL(3,\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^3$	2	$\mathfrak{gl}(1,\mathbb{C}) \times \mathfrak{gl}(1,\mathbb{C})$	6	
$Sp(6,\mathbb{C})$	$\Lambda_0^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})$	4	N=2 magic on **C**
$Spin(7,\mathbb{C})$	$\mathbb{C}^8$	1,2,3	$\mathfrak{g}_2^{\mathbb{C}}, \mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{so}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$	2,2,2	
$Spin(9,\mathbb{C})$	$\mathbb{C}^{16}$	1	$\mathfrak{spin}(7,\mathbb{C})$	2	
$Spin(10,\mathbb{C})$	$\mathbb{C}^{16}$	2,3	$\mathfrak{g}_2^{\mathbb{C}} \times \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$	2,4	3-ctr. Inv. of **N=0** MESGT
$Spin(11,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(5,\mathbb{C})$	4	**?**
$Spin(12,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(6,\mathbb{C})$	4	N=2 magic on **H**, N=6
$Spin(14,\mathbb{C})$	$\mathbb{C}^{64}$	1	$\mathfrak{g}_2^{\mathbb{C}} \times \mathfrak{g}_2^{\mathbb{C}}$	8	
$G_2^{\mathbb{C}}$	$\mathbb{C}^7$	1,2	$\mathfrak{sl}(3,\mathbb{C}), \mathfrak{gl}(2,\mathbb{C})$	2,2	
$E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	1,2	$\mathfrak{f}_4^{\mathbb{C}}, \mathfrak{so}(8,\mathbb{C})$	3,6	
$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	1	$\mathfrak{e}_6^{\mathbb{C}}$	4	N=2 magic on **O**, N=8

In sugra, **n** can be associated to the # of centers of the **multi-centered** BH

Here we only consider *irreducible* **PVS**, with **G** simple and **complex** Lie group

→ Classification of all groups “**of type E_7**“ ? *Not yet complete….*

Some advances in rather recent papers,
e.g. in Garibaldi, Guralnick, arXiv:1309.6611v1 [math.GR]

G	V	$\dim V$	char k	G	V	$\dim V$	char k
B_n	λ_1	$2n+1$	$\neq 2$	A_1	$\lambda_1 + p^i \lambda_1$ $(i \geq 1)$	4	$= p \neq 0$
D_n	λ_1	$2n$	all	A_2	$\lambda_1 + \lambda_2$	7	3
A_1	$2\lambda_1$	3	$\neq 2$	A_3	λ_2	6	all
A_5	λ_3	20	2	B_4	λ_4	16	all
B_3	λ_3	8	all	B_5	λ_5	32	2
C_3	λ_3	8	2	C_3	λ_2	13	3
D_6	half-spin	32	2	G_2	λ_1	7	$\neq 2$
E_7	λ_7	56	2	F_4	λ_4	25	3

p=2 : T^3 model

?

known simple Lie groups “**of type E_7“** occurring in **D=4 (super)gravity theories**

The case **Spin(11), 32** is related to the classification of **susy** sols. in **M-theory** :
can it be realized in D=4 as global symmetry
of a Maxwell-Einstein (non-susy) system ?
[homogeneous non-symmetric scalar manifold, thus evading Gibbons, Breitenlohner and Maison ‘s classification (1988)]

Figueroa-O’ Farrill, Santi,
arXiv:1511.03460 [math.DG]

…some Hints for the Future…

- **F-Duality** applied to homogeneous **<u>non-symmetric</u>** special Kaehler manifolds
  [deWit, Van Proeyen; Alekseevsky, Cortes, Mohaupt]

- **Jordan Duality**, groups **"of type E_6", PVS** , and **D=5 (Super)Gravity**

- extension to **"small"** U-orbits : ***how to define "small" F-duality ?***
  [for ***intrinsically quantum black holes***…]

- extension to **Gauged** (Super)**Gravity**:
  embedding tensor, **omega-deformations** and **Freudenthal duality :**
  work in progress in defining the F-Duality for Abelian gaugings of N=2, D=4 sugra, also in presence of hypers. Possible extensions to Abelian gaugings N>2, D=4 sugras…
  Use of (Freudenthal triple systems over) <u>complexified</u> Jordan Algebras in some remarkable cases…
  Extensions to non-Abelian gaugings?
  D <>4 ?

- extension to **Multi-Centered (extremal) BH solutions**:
  some progress in Yeranyan, arXiv :1205.5618
  Ferrara,AM,Shcherbakov,Yeranyan, arXiv:1211.3262

- into the ***<u>quantum regime</u>*** of gravity [**U-duality over *discrete* fields**]:
  Freudenthal Duality for <u>integer</u>, quantized charges ? Borsten, Duff *et al.* arXiv:0903.5517

Thank You!