Generalized Twirling Semigroups

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Outline of the talk:

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Quantum dynamical maps and semigroups

A *quantum dynamical map* (or 'quantum channel') is a linear map

$$\mathfrak{Q}\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H}) \tag{1}$$

characterized by the following defining properties:

1. $\mathfrak{Q}$ is positive, i.e., $\widehat{\rho} \geq 0 \;\Rightarrow\; \mathfrak{Q}\,\widehat{\rho} \geq 0$;
2. $\mathfrak{Q}$ is trace-preserving, i.e., $\mathrm{tr}(\mathfrak{Q}\,\widehat{\rho}) = \mathrm{tr}(\widehat{\rho})$;
3. $\mathfrak{Q}$ is *completely* positive, i.e., for every n, the map

$$\mathfrak{Q} \otimes \mathrm{Id}_n\colon \mathcal{B}_1(\mathcal{H}) \otimes \mathbb{C}^{n\times n} \to \mathcal{B}_1(\mathcal{H}) \otimes \mathbb{C}^{n\times n} \tag{2}$$

   is positive.

The quantum dynamical maps form a semigroup $\mathrm{DM}(\mathcal{H})$, and a one-parameter semigroup of (super-)operators

$$\mathbb{R}^+ \ni t \mapsto \mathfrak{Q}_t \in \mathrm{DM}(\mathcal{H}), \quad \mathfrak{Q}_t\,\mathfrak{Q}_s = \mathfrak{Q}_{t+s}, \tag{3}$$

is called a *quantum dynamical semigroup* (QDS). A QDS describes the (reduced) dynamics of open quantum systems (Markovian approximation; suitable regimes, e.g., weak coupling limit).

Like any semigroup of operators, $\{\mathfrak{Q}_t\}_{t\in\mathbb{R}+}$ is completely characterized by its *infinitesimal generator* $\mathfrak{L}$:

$$\mathfrak{L}\,\widehat{\rho} := \lim_{t\downarrow 0} t^{-1}\Big(\mathfrak{Q}_t\,\widehat{\rho} - \widehat{\rho}\Big). \quad (\widehat{\rho} \in \mathrm{Dom}(\mathfrak{L})) \tag{4}$$

At least in the finite-dimensional case, one can show (*Gorini-Kossakowski-Sudarshan* and *Lindblad*, 1976) that the general form of such a generator is

$$\begin{aligned}\mathfrak{L}\,\widehat{\rho} &= -\mathrm{i}\Big[\widehat{H},\widehat{\rho}\Big] + \mathfrak{F}\,\widehat{\rho} - \frac{1}{2}\Big(\Big(\mathfrak{F}\,\widehat{I}\Big)\widehat{\rho} + \widehat{\rho}\Big(\mathfrak{F}\,\widehat{I}\Big)\Big) \\ &= -\mathrm{i}\Big[\widehat{H},\widehat{\rho}\Big] + \sum_{k=1}^{\mathtt{N}^2-1}\gamma_k\Big(\widehat{L}_k\,\widehat{\rho}\,\widehat{L}_k^* - \frac{1}{2}\Big(\widehat{L}_k^*\widehat{L}_k\,\widehat{\rho} + \widehat{\rho}\,\widehat{L}_k^*\widehat{L}_k\Big)\Big),\end{aligned} \tag{5}$$

where $\mathfrak{F}$ is a completely positive map, $\gamma_1,\ldots,\gamma_{\mathtt{N}^2-1}$ are non-negative real numbers, $\widehat{H}$ is a selfadjoint operator (Hamiltonian) and $\widehat{L}_1,\ldots,\widehat{L}_{\mathtt{N}^2-1}$ are operators that form an orthonormal basis, w.r.t. the Hilbert-Schmidt product, in the orthogonal complement of the space generated by the identity. More generally, $\widehat{H}$ and $\widehat{L}_1,\widehat{L}_2,\ldots$ will be bounded (norm-continuous QDSs) — or even unbounded — operators.

Twirling semigroups and generalizations

Given a locally compact group G, we consider the semigroup $\mathsf{PM}(G)$ of all **probability measures** on G, with respect to convolution. Recall that for $\mu, \nu \in \mathsf{PM}(G)$ the **convolution** of μ with ν is the probability measure $\mu \star \nu$ determined by

$$\int_G \mathrm{d}\mu \star \nu(g)\, f(g) = \int_G \mathrm{d}\mu(g) \int_G \mathrm{d}\nu(h)\, f(gh), \qquad f \in \mathsf{C}_\mathsf{c}(G;\mathbb{R}). \tag{6}$$

By a (continuous) *convolution semigroup of measures* on G we mean a subset $\{\mu_t\}_{t\in\mathbb{R}^+}$ of $\mathsf{PM}(G)$ such that the map $\mathbb{R}^+ \ni t \mapsto \mu_t \in \mathsf{PM}(G)$ is a homomorphism of semigroups,

$$\mu_t \star \mu_s = \mu_{t+s}, \qquad t, s \geq 0, \tag{7}$$

and

$$\lim_{t\downarrow 0} \mu_t = \delta, \qquad \delta \equiv \delta_e \text{ (Dirac measure at the identity } e \in G). \tag{8}$$

A further ingredient is a suitable representation of G.

Let U be a *projective representation* of G in a Hilbert space $\mathcal{H}$. We can define the *isometric representation*

$$U \vee U(g)\,\widehat{\rho} := U(g)\,\widehat{\rho}\,U(g)^*, \quad g \in G, \quad \widehat{\rho} \in \mathcal{B}_1(\mathcal{H}). \tag{9}$$

If $\widehat{\rho}$ is a *state*, this is the canonical *symmetry action* of G on $\widehat{\rho}$.
Given a convolution semigroup $\{\mu_t\}_{t\in\mathbb{R}+}$ on G, we set

$$\mathfrak{S}_t\,\widehat{\rho} := \int_G \mathrm{d}\mu_t(g)\,\Big(U \vee U(g)\,\widehat{\rho}\Big), \quad \widehat{\rho} \in \mathcal{B}_1(\mathcal{H}),\ t \geq 0\,. \tag{10}$$

The family of maps $\{\mathfrak{S}_t\}_{t\in\mathbb{R}+}$ is a *quantum dynamical semigroup*, a so-called **twirling semigroup**.

More generally, let $\mathfrak{V}$ be a (weakly continuous) representation of G in a Banach space $\mathcal{J}$. Setting

$$\mu_t[\mathfrak{V}]\,\Phi := \int_G (\mathfrak{V}(g)\,\Phi)\ \mathrm{d}\mu_t(g), \quad \Phi \in \mathcal{J}, \tag{11}$$

we obtain a semigroup of operators $\{\mu_t[\mathfrak{V}]\}_{t\in\mathbb{R}+}$, a **randomly generated semigroup (RGS)**. *A further generalization* can be obtained replacing the convolution semigroup $\{\mu_t\}_{t\in\mathbb{R}+}$ with a *suitable family of signed measures*.

The generators of twirling semigroups

How are the *generators of twirling semigroups* characterized within the *Gorini-Kossakowski-Lindblad-Sudarshan* classification?

Suppose, for the sake of simplicity, that $\mathcal{H}$ is finite-dimensional. A quantum dynamical map $\mathfrak{U}\colon \mathcal{B}(\mathcal{H}) \equiv \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to be a *random unitary map* if it is a convex combination of unitary transformations:

$$\mathfrak{U}\,\widehat{\rho} = \sum_{k=1}^{\mathcal{N}} p_k V_k\,\widehat{\rho}\,V_k^* \ . \tag{12}$$

Observe that the random unitary maps acting in $\mathcal{B}(\mathcal{H})$ form a semigroup $\mathsf{DM}_{\mathsf{ru}}(\mathcal{H})$ contained in the semigroup of quantum dynamical maps $\mathsf{DM}(\mathcal{H})$. Every twirling (super)operator

$$\mathcal{B}(\mathcal{H}) \ni \widehat{\rho} \mapsto \int_G \mathsf{d}\mu(g)\,\Big(U(g)\,\widehat{\rho}\,U(g)^*\Big) \tag{13}$$

is a random unitary map. Thus, every twirling semigroup is a random unitary semigroup *(a semigroup of ops. consisting of random unitary maps)*.

Theorem 1 *Let* $\{\mathfrak{Q}_t\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t\in\mathbb{R}^+}$ *be a quantum dynamical semigroup. Then, the following facts are equivalent.*

1. $\{\mathfrak{Q}_t\}_{t\in\mathbb{R}^+}$ *is a twirling semigroup.*

2. $\{\mathfrak{Q}_t\}_{t\in\mathbb{R}^+}$ *is a random unitary semigroup.*

3. *The infinitesimal generator* $\mathfrak{L}$ *of the quantum dynamical semigroup* $\{\mathfrak{Q}_t\}_{t\in\mathbb{R}^+}$ *is of the form*

$$\mathfrak{L}\,\widehat{\rho} = -\mathrm{i}\Big[\widehat{H},\widehat{\rho}\Big] + \sum_{k=1}^{\mathtt{N}^2-1}\gamma_k\bigg(\widehat{L}_k\widehat{\rho}\,\widehat{L}_k - \frac{1}{2}\Big(\widehat{L}_k^2\widehat{\rho} + \widehat{\rho}\,\widehat{L}_k^2\Big)\bigg) + \gamma_0\Big(\mathfrak{U} - \mathrm{Id}\Big)\widehat{\rho}, \qquad (14)$$

   where $\widehat{H}$ *is a trace-less selfadjoint operator,* $\widehat{L}_1,\ldots,\widehat{L}_{\mathtt{N}^2-1}$ *are trace-less selfadjoint operators such that*

$$\Big\langle \widehat{L}_j,\widehat{L}_k\Big\rangle_{\mathsf{HS}} = \delta_{jk}, \quad j,k = 1,\ldots,\mathtt{N}^2-1, \qquad (15)$$

   $\mathfrak{U}$ *is a random unitary map acting in* $\mathcal{B}(\mathcal{H})$ *and* $\gamma_0,\ldots,\gamma_{\mathtt{N}^2-1}$ *are non-negative numbers (compare with the G-K-L-S form of a generator).*

Note: clearly the *specific* form of $\mathfrak{L}$ depends on the details of the pair $(U,\{\mu_t\}_{t\in\mathbb{R}^+})$ giving rise to $\{\mathfrak{Q}_t\}_{t\in\mathbb{R}^+}$ (Lévy-Kintchine formula).

Classical-quantum semigroups

In classical (statistical) mechanics, *states* are probability measures on phase space, and the expectation value of an observable $f(q,p)$ in the state μ is given by

$$\langle f\rangle_\mu = \int_{\mathbb{R}\times\mathbb{R}} f(q,p)\,\mathrm{d}\mu(q,p)\;, \qquad f \in \mathrm{C}_0(\mathbb{R}\times\mathbb{R})\;. \tag{16}$$

It is often convenient to replace a probability measure μ with its (say, symplectic) Fourier transform

$$\chi(q,p) \equiv \widetilde{\mu}(q,p) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{\mathrm{i}(qp'-pq')}\,\mathrm{d}\mu(q',p')\;, \tag{17}$$

which is an ordinary function. By *Bochner's theorem*, the convex set formed by such functions *coincides* with the convex set of normalized, continuous *positive definite functions*; i.e., for every finite set

$$z_1 \equiv (q_1,p_1),\ldots,z_n \equiv (q_n,p_n)$$

we have

$$\sum_{j,k} \chi(z_j - z_k)\,c_j c_k^* \geq 0\;, \qquad \chi(0) = 1\;. \tag{18}$$

A similar picture is possible in the quantum setting as well.

Indeed, in the Weyl-Wigner-Groenewold-Moyal phase space formulation of quantum mechanics we have:

$$\widehat{\rho}_\psi = |\psi\rangle\langle\psi| \quad \mapsto \quad \varrho_\psi(q,p) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}px}\,\psi\Big(q-\frac{x}{2}\Big)^{*}\psi\Big(q+\frac{x}{2}\Big)\;\mathrm{d}x\,, \tag{19}$$

$$\langle\widehat{A}\rangle_{\widehat{\rho}} = \mathrm{tr}(\widehat{A}\,\widehat{\rho}) = \int_{\mathbb{R}\times\mathbb{R}} \mathcal{A}(q,p)\,\varrho(q,p)\,\mathrm{d}q\,\mathrm{d}p\;. \tag{20}$$

Is there any *intrinsic* characterization of *Wigner quasi-probability distributions*, as in the classical case? By a 'quantum version' of Bochner's theorem (**Kastler** 1965, **Loupias** and **Miracle-Sole** 1966), the Wigner functions $\varrho(q,p)$ are precisely those continuous functions on phase space satisfying

$$\sum_{j,k} \widetilde{\varrho}(z_j - z_k)\,\mathrm{e}^{\mathrm{i}\omega(z_k,z_j)/2}\;c_j c_k^{*} \geq 0\;, \quad \widetilde{\varrho}(0) = 1\;, \quad (\hbar = 1) \tag{21}$$

for every finite set $z_1 \equiv (q_1,p_1),\ldots,z_n \equiv (q_n,p_n)$ and numbers $c_1,\ldots,c_n$. Here, $\mathcal{Q} \equiv \widetilde{\varrho} = \mathcal{F}_{\mathrm{sp}}\,\varrho$ is a normalized *quantum positive definite function* and ω is the standard *symplectic form*.

Recall that, endowed with convolution, the convex set $\mathrm{PM}(G)$ of all probability measures on $G = \mathbb{R} \times \mathbb{R}$ is a *semigroup*, with *identity* δ_e. The convolution of probability measures corresponds — via the FT — to the *point-wise product of characteristic functions*.
Hence, the point-wise multiplication $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$ of two continuous (classical) positive definite functions on G gives rise to a continuous (classical) positive definite function.
What about the point-wise product of a (continuous) **classical** *positive definite function by a (continuous)* **quantum** *positive definite function?*
It turns out that *the point-wise product $\chi\mathcal{Q}$ of a positive definite function χ by a* quantum *positive definite function $\mathcal{Q}$ is still a* quantum *positive definite function.* In particular, it belongs to the convex set of quantum characteristic functions if χ and $\mathcal{Q}$ are normalized.

By linear superpositions, one can *extend* in a natural way the convex cone of *quantum* positive definite functions on $\mathbb{R}\times\mathbb{R}$ to a complex vector space $\mathscr{T}$ which, endowed with a suitable norm, becomes a (separable) Banach space. It turns out that $\mathscr{T}$ can be regarded as a *dense* linear span in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$. A semigroup of operators in the space $\mathscr{T}$ is defined as follows.

Consider a *multiplication semigroup of positive definite functions*

$$\{\chi_t \colon \mathbb{R}\times\mathbb{R}\to\mathbb{C}\}_{t\in\mathbb{R}^+}, \quad \chi_t\,\chi_s = \chi_{t+s},\ t,s\geq 0, \quad \chi_0\equiv 1 \tag{22}$$

(continuous w.r.t. the the topology of uniform convergence on compact sets). Such semigroups can be classified. As χ_t is a bounded continuous function, we can define a *bounded operator* $\widehat{\mathfrak{C}}_t$ in $\mathsf{L}^2(\mathbb{R}\times\mathbb{R})$:

$$\Big(\widehat{\mathfrak{C}}_t\,f\Big)(q,p) := \chi_t(q,p)\,f(q,p), \quad f\in\mathsf{L}^2(\mathbb{R}\times\mathbb{R}), \ \ t\geq 0. \tag{23}$$

The set $\{\widehat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of operators:

1. $\widehat{\mathfrak{C}}_t\widehat{\mathfrak{C}}_s = \widehat{\mathfrak{C}}_{t+s}$, $t,s\geq 0$;
2. $\widehat{\mathfrak{C}}_0 = \mathrm{Id}$;
3. $\lim_{t\downarrow 0}\|\widehat{\mathfrak{C}}_t f - f\| = 0$, $\forall f\in\mathsf{L}^2(\mathbb{R}\times\mathbb{R})$.

Since the product $\chi\mathcal{Q}$ of a positive definite function χ by a *quantum* positive definite function $\mathcal{Q}$ is still a function of the latter type, setting

$$\Big(\mathfrak{C}_t\,\mathcal{Q}\Big)(q,p) := \chi_t(q,p)\,\mathcal{Q}(q,p), \quad \mathcal{Q}\in\mathscr{T}\subset\mathsf{L}^2(\mathbb{R}\times\mathbb{R}), \tag{24}$$

we obtain a further semigroup of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ acting in the Banach space $\mathscr{T}$, a *classical-quantum semigroup*.

What is the relation with the twirling semigroups?

Tomographic semigroups

Consider the map $\mathcal{T}_{\mathsf{m}} \colon G \to \mathcal{U}(\mathsf{L}^2(G))$ defined by

$$\Big(\mathcal{T}_{\mathsf{m}}(g) f\Big)(h) := \Delta(g)^{\frac{1}{2}}\, \bar{\mathsf{m}}(g,h)\, f(g^{-1}hg), \quad f \in \mathsf{L}^2(G), \tag{25}$$

where Δ is the **modular function** and, given a **multiplier** $\mathsf{m} \colon G \times G \to \mathbb{T}$, the function $\bar{\mathsf{m}} \colon G \times G \to \mathbb{T}$ is defined as follows:

$$\bar{\mathsf{m}}(g,h) := \mathsf{m}(g, g^{-1}h)^* \, \mathsf{m}(g^{-1}h, g), \quad \forall g, h \in G. \tag{26}$$

$\mathcal{T}_{\mathsf{m}}$ is a (continuous) **unitary representation**. Given a convolution semigroup $\{\mu_t\}_{t \in \mathbb{R}^+}$ on G, we define the RGS associated with $(\mathcal{T}_{\mathsf{m}}, \{\mu_t\}_{t \in \mathbb{R}^+})$:

$$\mathfrak{T}_t^{\mathsf{m}} f = \int_G \mathcal{T}_{\mathsf{m}}(g) f \, \mathsf{d}\mu_t(g), \quad \forall f \in \mathsf{L}^2(G). \tag{27}$$

What is the physical meaning? If there is a **square integrable** projective representation $U \colon G \to \mathcal{U}(\mathcal{H})$ — with multiplier m — it turns out that the semigroup of operators $\{\mathfrak{T}_t^{\mathsf{m}}\}_{t \in \mathbb{R}^+}$ is *the twirling semigroup associated with the pair* $(U, \{\mu_t\}_{t \in \mathbb{R}^+})$, *but expressed in terms of the generalized Wigner functions* (quantum **tomograms**) associated with U. Thus, it is natural to call $\{\mathfrak{T}_t^{\mathsf{m}}\}_{t \in \mathbb{R}^+}$ the **tomographic semigroup** associated with m.

More precisely, these tomograms are functions living in the space $\mathsf{L}^2(G)$. In the case where the group G is **unimodular**, they are defined by

$$\widetilde{\varrho}(g) = \mathrm{tr}(U(g)^*\,\widehat{\rho}), \quad \widehat{\rho} \in \mathcal{B}_1(\mathcal{H}), \text{ in particular, a quantum } \textbf{state}. \tag{28}$$

The map $\widehat{\rho} \mapsto \widetilde{\varrho}$ extends to an **isometry** (generalized Wigner map)

$$\mathcal{V} \colon \mathcal{B}_2(\mathcal{H}) \to \mathsf{L}^2(G). \quad \text{(in general, } \mathrm{Ran}(\mathcal{V}) \subset \mathsf{L}^2(G)\text{)} \tag{29}$$

The tomograms form a **subspace** $\mathscr{T}(\mathcal{V})$ of $\mathsf{L}^2(G)$, which is **stable** under the action of the representation $\mathcal{T}_{\mathsf{m}}$ and of the semigroup of operators $\{\mathfrak{T}_t^{\mathsf{m}}\}_{t\in\mathbb{R}^+}$. *The restrictions to* $\mathscr{T}(\mathcal{V})$ *of* $\mathcal{T}_{\mathsf{m}}$ *and of* $\{\mathfrak{T}_t^{\mathsf{m}}\}_{t\in\mathbb{R}^+}$ *are the images through the* **dequantization map**

$$\mathcal{B}_1(\mathcal{H}) \ni \widehat{\rho} \;\mapsto\; \widetilde{\varrho} \in \mathscr{T}(\mathcal{V}) \subset \mathsf{L}^2(G) \tag{30}$$

of the representation $U \vee U$ — $U \vee U(g)\,\widehat{\rho} := U(g)\,\widehat{\rho}\,U(g)^*$ — *and of the twirling semigroup associated with the pair* $(U, \{\mu_t\}_{t\in\mathbb{R}^+})$*, respectively.*

Example: If G is the group of translations on phase space — the additive group $\mathbb{R}\times\mathbb{R}$ — and U is the projective representation (Weyl system)

$$U(q,p) := \exp(\mathrm{i}(p\widehat{q} - q\widehat{p}))\,, \quad q,p \in \mathbb{R}, \tag{31}$$

we get precisely a classical-quantum semigroup.

Generalized twirling semigroups

The role of *complete positivity* — a notion introduced by **Stinespring** (1955) — in the theory of open quantum systems has been recognized by **Kraus** (1971), **Accardi** (1976) and **Lindblad** (1976). Its justification on the physical ground is still controversial; see, e.g., the work of **Shaji** and **Sudarshan** (2005). *Let us then relax the complete positivity*, and let us consider a generalization of the twirling semigroups.

Let $\{\mathfrak{D}_t \colon \mathcal{B}(\mathcal{H}) \equiv \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{t\in\mathbb{R}^+}$ be a family of linear maps. $\{\mathfrak{D}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of *unital, trace-preserving positive maps* if and only if

$$\mathfrak{D}_t\,\widehat{\rho} = \int_G U(g)\,\widehat{\rho}\,U(g)^*\,\mathrm{d}\varsigma_t(g), \tag{32}$$

where $(U, \{\varsigma_t\}_{t\in\mathbb{R}^+})$ is a pair formed by a continuous unitary representation $U \colon G \to \mathrm{U}(\mathcal{H})$ of a locally compact group G and by a family $\{\varsigma_t\}_{t\in\mathbb{R}^+}$ of finite, signed Borel measures on G satisfying some technical conditions conditions `(M1)`$-$`(M3)` below.
(These can regarded as the defining conditions of a *generalized twirling semigroup*.)

(M1) $\varsigma_0 = \delta$ (Dirac measure at the identity of G) and $\varsigma_t(G) = 1$, for all $t \in \mathbb{R}^+_*$;

(M2) for some (hence, for every) orthonormal basis $\Psi \equiv \{\psi_k\}_{k=1}^{\mathtt{N}}$ in $\mathcal{H}$,

$$\lim_{t\downarrow 0} \int_G v_{jklm}(\Psi; g)\, \mathrm{d}\varsigma_t(g) = \delta_{jk}\delta_{lm} \tag{33}$$

and

$$\int_G v_{jklm}(\Psi; g)\, \mathrm{d}(\varsigma_s \star \varsigma_t)(g) = \int_G v_{jklm}(\Psi; g)\, \mathrm{d}\varsigma_{s+t}(g), \quad \forall s, t \in \mathbb{R}^+_*, \tag{34}$$

where

$$v_{jklm}(\Psi; g) := \langle \psi_j, U(g)\psi_k \rangle \langle U(g)\psi_l, \psi_m \rangle;$$

(M3) for every orthonormal basis $\Psi \equiv \{\psi_k\}_{k=1}^{\mathtt{N}}$ in $\mathcal{H}$,

$$\lim_{t\downarrow 0} t^{-1} \int_G v_{jkkj}(\Psi; g)\, \mathrm{d}\varsigma_t(g) \geq 0, \quad j \neq k. \tag{35}$$

(M1) + (M2) $\Rightarrow \{\mathfrak{D}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of unital trace-preserving maps

(M1) + (M2) + (M3) $\Leftrightarrow \{\mathfrak{D}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of unital, trace-preserving positive maps

What about the associated generators? Let us consider a *qubit system*.

Theorem 2 *For* $\dim(\mathcal{H}) = 2$, *the general form of the generator of a generalized twirling semigroup is given by*

$$\mathfrak{L}\,\widehat{\rho} = -\mathrm{i}\sum_{j=1}^{3} h_j\left[\widehat{S}_j,\widehat{\rho}\right] + \sum_{j,k=1}^{3} \varkappa_{jk}\left(\widehat{S}_j\,\widehat{\rho}\,\widehat{S}_k - \frac{1}{2}\Big(\widehat{S}_j\,\widehat{S}_k\,\widehat{\rho} + \widehat{\rho}\,\widehat{S}_j\,\widehat{S}_k\Big)\right), \qquad (36)$$

where $h_1, h_2, h_3 \in \mathbb{R}$, *the* 3×3 *matrix* $\mathscr{K} := (\varkappa_{jk})$ *is such that* $\varkappa_{jk} = \varkappa_{kj} \in \mathbb{R}$ *and — setting*

$\kappa_1 \equiv \varkappa_{22}+\varkappa_{33}$, $\kappa_2 \equiv \varkappa_{11}+\varkappa_{33}$, $\kappa_3 \equiv \varkappa_{11}+\varkappa_{22}$, $a \equiv -\varkappa_{23}$, $b \equiv -\varkappa_{13}$, $c \equiv -\varkappa_{12}$

— the associated symmetric real matrix

$$\mathscr{P} := \begin{pmatrix} \kappa_1 & c & b \\ c & \kappa_2 & a \\ b & a & \kappa_3 \end{pmatrix} \qquad (37)$$

is positive semidefinite.

The semigroup of linear maps associated with $\mathfrak{L}$ *is, in particular, completely positive if and only if the symmetric real matrix* $\mathscr{K}$ *is positive semidefinite.*

Conclusions and perspectives:

- An interesting class of quantum dynamical semigroups stems from the marriage between a group representation and a convolution semigroup of measures on that group: the *twirling semigroups*. This class arises as a suitable choice of the (type of) representation from a larger class of semigroups of operators: the *randomly generated semigroups*.

- Classical examples of physical systems described by twirling semigroups are: a finite-dim. system coupled to an infinite free boson bath whose time correlation functions are Gaussian; a finite-dim. system in the limit of singular coupling to a reservoir at infinite temperature (**Frigerio**, **Gorini**, **Kossakowski**, 1976).

- The class of twirling semigroups *coincides* with the class of *random unitary semigroups*, i.e., those quantum dynamical semigroups consisting of random unitary maps. **Gregoratti and Werner** (2004) have given a characterization of this class of maps as the only quantum channels enjoying the property of being *perfectly corrigible* by using, as the only side-resource, classical information form the environment.

- A natural problem is to express a twirling semigroup in terms of ‘phase space functions’. Undertaking a group-theoretical approach, this route leads to the definition of the so-called *tomographic semigroups*.

- In the case where the relevant group is the group of translations on (standard) phase space, one finds the *classical-quantum semigroups*.

- Interestingly, exploiting the notions of (classical) positive definite and of *quantum* positive definite function, and their properties, one can find the classical-quantum semigroups following a different route.

- In finite dimensions, the twirling semigroups are *unital*. (Not necessarily completely positive) unital dynamical semigroups correspond to the class of *generalized twirling semigroups*, associated with a suitable class of families of *signed* measures. The *standard* twirling semigroups are obtained by taking, in particular, convolution semigroups of probability measures.

- Whereas convolution semigroups of probability measures (on Lie groups) have been extensively studied, these more general families of signed measures will deserve further investigation.

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Thank you for your attention

and

long life to Beppe!