From Lie Systems to the Geometry of Quantum Mechanics

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Abstract

After a quick presentation of our common work for more than thirty years, some results and recent progress about two common research lines, Lie-Scheffers systems and Geometry of Quantum Mechanics, will be described

Outline

Motivation: A long scientific collaboration

I met Beppe by the first time, 30 years ago, during the 1st Workshop on Diff. Geom. Methods in Classical Mechanics at Ghent University and our scientific collaboration is more than 20 years long

This was the starting point for a fruitful collaboration along the series of Workshops on Diff. Geom. Methods in Classical Mechanics:

Jaca,

Ferrara,

Trieste

Windsor

Stirling,

El Escorial (including a visit to Segocia),

⋮

Levico(2010)


SEP 12 '97

As our collaboration has covered many different subjects of Mathematical Physics, I only will show you first some coworked papers and then...

I will fix my attention on two specific problems in which Beppe's motivation has been crucial for me:

- A) Theory of Lie systems and its applications
- B) Geometric approach to Quantum mechanics

On the Geometry of Lie algebras and Poisson tensors, J. Phys. A: Math. Gen. **27**, 7425–7449 (1994) (JFC, Luis A. Ibort, Giuseppe Marmo and Askold Perelomov)

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Related operators and exact solutions of Schrödinger equations, Int. J. Mod. Phys. **13**, 4913–4929 (1998) (JFC, Giuseppe Marmo, Askold M. Perelomov and Manuel F. Rañada)

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COLLABORATION ON HJ WITH BARCELONA COLLEAGUES

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It also opened a new field of research and collaboration.

For instance, some Ph. D. Thesis on the subject:

- Aplicación de Métodos Algebraicos y Geométricos al estudio de la evolución dinámica (Javier A. Nasarre)
- Sistemas de Lie y sus aplicaciones en Física y Teoría de Control (Arturo Ramos)
- Sistemas de Lie y aplicaciones en Mecánica Cuántica (Javier de Lucas)

As well as many other people have being involved:

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Lie–Scheffers systems: a quick review

Lie–Scheffers systems = Non-autonomous systems of first-order differential equations admitting a ...

Superposition rule: a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}$, $\quad x = \Phi(u_1, \dots, u_m; k_1, \dots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)(t)}; k_1, \dots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \dots, m\}$(x. being a generic set of particular solutions of the system and where $k_1, \dots, k_n$ are real numbers.

They are a generalisation of linear superposition rules for homogeneous linear systems for which $m = n$ and $x = \Phi(x_{(1)}, \dots, x_{(n)}; k_1, \dots, k_n) = k_1\, x_{(1)} + \cdots + k_n\, x_{(n)}$ but

i) The number m may be different from the dimension n

ii) The function Φ is nonlinear in this more general case

They appear quite often in many different branches of Science ranging from pure mathematics to classical and quantum physics, control theory, economy, etc. Forgotten for a long time they had a revival due to the work of Winternitz and coworkers.

One particular example is Riccati equation, of a fundamental importance in physics (for instance factorisation of second order differential operators, Darboux transformations and in general Supersymmetry in Quantum Mechanics) and mathematics

These systems are related with equations in Lie groups and in general connections in fibre bundles

In the solution of such non-autonomous systems of first-order differential equations we can use techniques imported form group theory, for instance Wei–Norman method, and reduction techniques coming from the theory of connections

Recent generalisations have also been shown to be useful for dealing with other systems of differential equations (e.g. Emden–Fowler equations, Abel equations)

The existence of additional compatible geometric structures, like symplectic or Poisson structures may be useful in the search for solutions

Theorem: *Given a non-autonomous system of n first order differential equations*

$$\frac{dx^i}{dt} = X^i(x^1, \ldots, x^n, t), \quad i = 1 \ldots, n,$$

a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$, $\quad x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ being a set of particular solutions of the system and where $k_1, \ldots, k_n$, are n arbitrary constants, is that the system can be written as

$$\frac{dx^i}{dt} = Z^1(t)\xi_1^i(x) + \cdots + Z^r(t)\xi_r^i(x), \qquad i = 1 \ldots, n,$$

where $Z^1, \ldots, Z^r$, are r functions depending only on t and ξ_α^i, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by

$$X_\alpha \equiv \sum_{i=1}^{n} \xi_\alpha^i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \ , \qquad \alpha = 1, \ldots, r,$$

close on a real finite-dimensional Lie algebra, i.e. the X_α are l.i. and there are r^3 real numbers, $c_{\alpha\beta}{}^\gamma$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma \ .$$

The number r satisfies $r \leq mn$.

The solutions of the system are the integral curves of the t-dependent vector field

$$X(x,t) = \sum_{i=1}^{n} X^i(x,t)\frac{\partial}{\partial x^i}$$

and the condition of the theorem is that $X(x,t)$ be a linear combination

$$X(x,t) = \sum_{\alpha=1}^{r} Z^\alpha(t)\, X_\alpha(x).$$

The t-dependent vector field can be seen as a family of vector fields $\{X_t \mid t \in \mathbb{R}\}$, one for each value of t.

Definition. *The minimal Lie algebra of a given a t-dependent vector field X over M is the smallest real Lie algebra, V^X, containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$.*

Definition. *The vector field associated to a non-autonomous system X allows us to define a generalised distribution* $\mathcal{D}^X : x \in M \mapsto \mathcal{D}_x^X \subset TM$, *where* $\mathcal{D}_x = \{Y_x \mid Y \in V^X\} \subset T_xM$, *and X also gives rise to a generalised co-distribution* $\mathcal{V} : x \in M \mapsto \mathcal{V}_x \subset T^*M$, *where* $\mathcal{V}_x = \{\omega_x \mid \omega_x(Y_x) = 0, \forall Y_x \in \mathcal{D}_x^X\}$.

Remark that the Lie–Scheffers theorem can be reformulated as follows:

Theorem: *A system X admits a superposition rule if and only if the minimal Lie algebra V^X is finite-dimensional.*

Definition. *A function $f : U \subset U^X \to \mathbb{R}$ is a local first integral (or t-independent constant of the motion) for a given t-dependent vector field X over $\mathbb{R}^n$ if $Xf = 0$*

Then f is a first integral if and only if $df \in \mathcal{V}^X|_U$.

One can easily prove that:

Property. *Given a t-dependent vector field X on a n-dimensional manifold M and a point $x \in U^X$ where the rank of $\mathcal{D}^X$ is equal to k, the associated co-distribution $\mathcal{V}^X$ admits, in a neighbourhood of x, a local basis of the form, $df_1, \ldots, df_{n-k}$, where, $f_1, \ldots, f_{n-k}$, is a family of fist integrals of X. Additionally, the space $\mathcal{I}^X|_U$ of first-integrals of the system X over an open U of M, can be put in the form*

$$\mathcal{I}^X|_U = \{g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{n-k} \to \mathbb{R},\ g = F(f_1, \ldots, f_{n-k})\}.$$

There exist different procedures to derive superposition rules for Lie systems. We can use a method based on the *diagonal prolongation* notion.

Definition. *Given a t-dependent vector field X over M, its diagonal prolongation to M^{m+1} is the t-dependent vector field $\widetilde{X}$ over M^{m+1} such that*

- $\widetilde{X}$ *projects onto X by the map* $\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in M^{m+1} \mapsto x_{(0)} \in M$, *that is,* $\mathrm{pr}_* \widetilde{X} = X$.
- $\widetilde{X}$ *is invariant under permutation of the variables $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \ldots, m$.*

The procedure to determine superposition rules described is:

i) Take a basis $X_1, \ldots, X_r$ of the Vessiot–Guldberg Lie algebra V associated with the Lie system.

ii) Choose the minimum integer m such that the diagonal prolongations to M^m of the elements of the previous basis are linearly independent at a generic point.

ii) Obtain n common first-integrals for the diagonal prolongations, $\widetilde{X}_1, \ldots, \widetilde{X}_r$, to M^{m+1} (for instance, by means of *the method of characteristics*).

iii) Obtain the expression of the variables of one of the spaces M only in terms of the other variables of M^{m+1} and the above mentioned n first-integrals.

The so obtained expressions give rise to a superposition rule in terms of any generic family of m particular solutions and n constants corresponding to the possible values of the derived first-integrals.

Some particular examples

A) Inhomogeneous linear systems:

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \ , \qquad i = 1, \ldots, n.$$

It is related with the (n^2+n)-dimensional Lie algebra of the affine group..

In this case $r = n^2 + n$ and $m = n+1$ and the equality $r = m\, n$ also follows. The superposition function $\Phi : \mathbb{R}^{n(n+1)} \to \mathbb{R}^n$ is:

$$x = \Phi(u_1, \ldots, u_{n+1}); k_1, \ldots, k_n) = u_1 + k_1(u_2 - u_1) + \cdots + k_n(u_{n+1} - u_1).$$

B) The Riccati equation $(n = 1)$

$$\frac{dx(t)}{dt} = a_2(t)\, x^2(t) + a_1(t)\, x(t) + a_0(t) \ .$$

Now $m = r = 3$ and the superposition principle comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k \ ,$$

or in other words,

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) + k\, x_2(t)(x_1(t) - x_3(t))}{(x_3(t) - x_2(t)) + k\,(x_1(t) - x_3(t))} \ .$$

The value $k = \infty$ must be accepted, otherwise we do not obtain the solution x_2.

The associated Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$.

C) Lie–Scheffers systems on Lie groups

Consider a basis of are eitherleft–invariant (or right–invariant) vector fields X_α in G as corresponding to the Lie algebra $\mathfrak{g}$ of G or its opposite algebra.

If $\{a_1, \ldots, a_r\}$ is a basis for the tangent space T_eG and X^R_α denotes the right-invariant vector field in G such that $X^R_\alpha(e) = a_\alpha$, a Lie–Scheffers system is

$$\dot g(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) X^R_\alpha(g(t)) \ .$$

When applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides we obtain the equation on T_eG

$$(R_{g(t)^{-1}})_{*g(t)}(\dot g(t)) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ , \qquad (**)$$

This is usually written with a slight abuse of notation:

$$(\dot{g}\, g^{-1})(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ .$$

Such equation is right-invariant. Then,

If $\bar{g}(t)$ is a solution of (**) with initial condition $\bar{g}(0) = e$, the solution $g(t)$ with initial conditions $g(0) = g_0$ is given by $\bar{g}(t)g_0$.

Moreover, there is a superposition rule $\Phi : G \times G \to G$ involving one solution

$$\Phi(g, g_0) = g\, g_0.$$

This example is very useful because there are many other examples related with them as explained next.

D) Lie-Scheffers systems on homogeneous spaces for Lie groups

Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. The right–invariant vector fields X_α^R are τ-projectable and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M.

$$\tau_{*g} X_\alpha^R(g) = -X_\alpha(gH) \ ,$$

and we will have an associated Lie–Scheffers system on M:

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x) \ .$$

Therefore, a solution of this last system starting from x_0 will be:

$$x(t) = \Phi(g(t), x_0) \ ,$$

with $g(t)$ being a solution of (**).

The converse property is true: Given a Lie Scheffers system defined by complete vector fields with associated Lie algebra $\mathfrak{g}$, we can see these as fundamental vector fields relative to an action which can be found by integrating the vector fields.

The reduction method

Given an equation on a Lie group

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\, a_\alpha \in T_eG \ , \qquad (\bullet)$$

with $g(0) = e \in G$, it may happen that the only nonvanishing coefficients are those corresponding to a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the equation reduces to a simpler equation on a subgroup, involving less coordinates.

The fundamental result is that if we know a particular solution of the problem associated in a homogeneous space, the original solution reduces to one on the subgroup.

Let us choose a curve $g'(t)$ in the group G, and define the curve $\overline{g}(t)$ by $\overline{g}(t) = g'(t)g(t)$. The new curve in G, $\overline{g}(t)$, determines a new Lie system.

Indeed,

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}\,(g'(t))a_\alpha \ ,$$

which is an equation similar to the original one but with a different right hand side.

In this way we can define an action of the group of curves in the Lie group G on the set of Lie systems on the group. This can be used to reduce a given Lie system to a simpler one.

The aim is to choose the curve $g'(t)$ in such a way that the new equation be simpler. For instance, we can choose a subgroup H and look for a choice of $g'(t)$ such that the right hand side lies in T_eH, and hence $\overline{g}(t) \in H$ for all t.

If $\Psi : G \times M \to M$ is a transitive action of G on a homogeneous space M, which can be identified with the set G/H of left-cosets, by choosing a fixed point x_0, then the integral curves starting from the point x_0 associated to both Lie systems are related by

$$\overline{x}(t) = \Psi(\overline{g}(t), x_0) = \Psi(g'(t)g(t), x_0) = \Psi(g'(t), x(t)) \ .$$

Therefore, this gives an action of the group of curves in G on the set of associated Lie systems in homogeneous space s.

More explicitly, if we consider ta curve $g'(t)$ in the group, the Lie system transforms into a new one

$$\dot{\bar{x}} = \sum_{\alpha=1}^{r} \bar{b}_\alpha(t) X_\alpha(\bar{x}) \ ,$$

in which

$$\bar{b} = \mathrm{Ad}\,(g'(t)) b(t) + \dot{g}'\, g'^{-1} \ .$$

The important result is that the knowledge of a particular solution of the associated Lie system in G/H allows us to reduce the problem to one in the subgroup H.

Theorem: *Each solution of* $(\bullet)$ *on the group* G *can be written in the form* $g(t) = g_1(t)\, h(t)$, *where* $g_1(t)$ *is a curve on* G *projecting onto a solution* $\tilde{g}_1(t)$ *for the left action* λ *on the homogeneous space* G/H *and* $h(t)$ *is a solution of an equation but for the subgroup* H, *given explicitly by*

$$(\dot{h}\, h^{-1})(t) = -\mathrm{Ad}\,(g_1^{-1}(t)) \left(\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha + (\dot{g}_1\, g_1^{-1})(t) \right) \in T_e H \ .$$

Structure preserving Lie systems

There are particularly interesting cases in which the manifod M is endowed with additional structures. For instance, let (M,Ω) be a symplectic manifold and the vector fields arising in the expression of the t-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra.

These vector fields correspond to a symplectic action of the Lie group G on (M,Ω).

The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = -dh_\alpha$, do not close on the same Lie algebra when the Poisson bracket is considered, but we can only say that

$$d\left(\{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]}\right) = 0 \ ,$$

and then they span a Lie algebra extension of the original one.

The important fact is that we can define a t-dependent Hamiltonian

$$h_t = \sum_\alpha b_\alpha(t)\, h_\alpha,$$

with the functions h_α closing a Lie algebra, in such a wat hat $i(X_t)\Omega = -dh_t$.

As an example we can consider the differential equation of an n-dimensional Winternitz–Smorodinsky oscillator of the form

$$\left\{ \begin{array}{rcl} \dot{x}_i & = & p_i, \\ \dot{p}_i & = & -\omega^2(t)x_i + \dfrac{k}{x_i^3}, \end{array} \right. \qquad i = 1, \ldots, n.$$

which describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}^n$

$$X_t = \sum_{i=1}^n \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i + \frac{k}{x_i^3} \right) \frac{\partial}{\partial p_i} \right],$$

which can be written as $X_t = X_2 + \omega^2(t)X_1$ with X_1, X_2 and $X_3 = -[X_1, X_2]$ being given by

$$X_1 = -\sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, \quad X_2 = \sum_{i=1}^n \left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3} \frac{\partial}{\partial p_i} \right), \quad X_3 = \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i} \right).$$

Note that X_t is a Lie system, because X_1, X_2 and X_3 close on a $\mathfrak{sl}(2, \mathbb{R})$ algebra:

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = -X_2.$$

Moreover, the preceding vector fields are Hamiltonian vector fields with respect to the usual symplectic form $\omega_0 = \sum_{i=1}^{n} dx^i \wedge dp_i$ with Hamiltonian functions

$$h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad h_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \frac{k}{x_i^2}\right), \qquad h_3 = \sum_{i=1}^{n} x_i p_i,$$

which obey that

$$\{h_1, h_2\} = h_3, \qquad \{h_1, h_3\} = -h_1, \qquad \{h_2, h_3\} = h_2.$$

Consequently, every curve h_t that takes values in the Lie algebra $(W, \{\cdot,\cdot\})$ spanned by h_1, h_2 and h_3 gives rise to a Lie system which is Hamiltonian in $\mathrm{T}^*\mathbb{R}^n$ with respect to the symplectic structure ω_0 in such a way that the t-dependent vector field is given by

$$X_t = X_2 + \omega^2(t) X_1 = \widehat{\omega}_0^{-1}(dh_2 + \omega^2(t) dh_1),$$

i.e. the Hamiltonian is $h_t = h_2 + \omega^2(t) h_1$.

We can go a step further and consider Lie systems in (may be degenerate) Poisson manifolds.

Definition. *A Poisson manifold is a pair* (M, Λ) *where* Λ *is a bivector field in the differentiable manifold* M *in such a way that the Schouten bracket* $[\cdot, \cdot]_{\mathrm{S.B.}} = 0$. *The bivector field gives by contraction a map denoted* $\widehat{\Lambda}$ *such that*

$$\widehat{\Lambda}(\alpha)(\beta) = \Lambda(\alpha, \beta)$$

In particular, if $f_1, f_2 \in C^\infty(M)$, *we define the Poisson bracket* $\{f_1, f_2\}$ *by*

$$\{f_1, f_2\} = \Lambda(df_1, df_2),$$

and this Poisson bracket satisfies Jacobi identity because of the vanishing of the Schouten bracket condition

The Lie bracket over $C^\infty(M)$ holds the Leibnitz rule

$$\{fg, h\} = \{f, h\}g + \{g, h\}f, \qquad \forall f, g, h \in C^\infty(M).$$

Consequently, the above Lie bracket becomes a derivation in each entry: given a function $f \in C^\infty(M)$, there exists a vector field X_f over M such that $X_f g = \{g, f\}$ for each $g \in C^\infty(M)$, i.e. $X_f = \widehat{\Lambda}(-df)$. T he vector field X_f is called the Hamiltonian vector field associated with f. The Jacobi identity for the Poisson structure entails that

$$X_{\{f,g\}} = -[X_f, X_g], \qquad \forall f, g \in C^\infty(M).$$

In other words, the mapping $f \mapsto X_f$ is a Lie algebra anti-homomorphism between the Lie algebras $(C^\infty(M), \{\cdot,\cdot\})$ and $(\Gamma(\tau_M), [\cdot,\cdot])$.

Equivalently, $\widehat{\Lambda} \circ d : C^\infty(M) \to \mathfrak{X}_H(M, \Lambda)$ is a Lie algebra homomorphism.

Definition. *The elements of the kernel of the previous homomorphism are called Casimir functions. The set of such Casimir functions will be denoted* $\mathcal{C}$.

This can be summarising by saying that the following sequence is exact:

$$0 \longrightarrow \mathcal{C} \longrightarrow C^\infty(M) \xrightarrow{\widehat{\Lambda}\circ d} \mathfrak{X}_H(M,\Lambda) \longrightarrow 0$$

Definition. *A Lie–Hamiltonian structure is a triple* (M, Λ, h)*, where* (M, Λ) *is a Poisson manifold and* h *is a* t*-parametrised family of functions* $h_t : M \to \mathbb{R}$ *such that* $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ *is a finite-dimensional real Lie algebra.*

Definition. *A* t*-dependent system* X *on* M *is said to admit a Lie–Hamilton structure if there exists a Lie–Hamiltonian structure* (M, Λ, h) *such that* $X_t \in \widehat{\Lambda}(-dh_t)$*, for every* $t \in \mathbb{R}$*. The triple* (M, Λ, X) *is called a Lie–Hamilton triple.*

There is a generalization to the framework of Dirac manifolds. Recall that a Pontryagin bundle $\mathcal{P}N$ is a vector bundle $TN \oplus_N T^*N$ on N and that an almost-Dirac manifold is a pair (N, L), where L is a maximally isotropic subbundle of $\mathcal{P}N$ with respect to the pairing

$$\langle X_x + \alpha_x, \bar{X}_x + \bar{\alpha}_x \rangle_+ \equiv \frac{1}{2}(\bar{\alpha}_x(X_x) + \alpha_x(\bar{X}_x)),$$

where $X_x + \alpha_x, \bar{X}_x + \bar{\alpha}_x \in T_xN \oplus T_x^*N = \mathcal{P}_xN$, i.e., L is isotropic and has rank $n = \dim N$.

A Dirac manifold is an almost-Dirac manifold (N, L) whose subbundle L, its Dirac structure, is involutive relative to the Courant–Dorfman bracket, namely

$$[[X + \alpha, \bar{X} + \bar{\alpha}]]_C \equiv [X, \bar{X}] + \mathcal{L}_X \bar{\alpha} - \iota_{\bar{X}} d\alpha \,,$$

where $X + \alpha, \bar{X} + \bar{\alpha} \in \Gamma(TN \oplus_N T^*N)$.

A vector field X on N is said to be an L-Hamiltonian vector field (or simply a Hamiltonian vector field if L is fixed) if there exists an $f \in C^\infty(N)$ such that $X + df \in \Gamma(L)$. In this case, f is an L-Hamiltonian function for X and an admissible function of (N, L). Let us denote by $\mathrm{Ham}(N, L)$ and $\mathrm{Adm}(N, L)$ the spaces of Hamiltonian vector fields and admissible functions of (N, L), respectively.

The space $\mathrm{Adm}(N,L)$ becomes a Poisson algebra $(\mathrm{Adm}(N,L), \bullet, \{\cdot,\cdot\}_L)$ relative to the standard product of functions and the Lie bracket given by

$$\{f,\bar{f}\}_L = X\bar{f}\,,$$

where X is an L-Hamiltonian vector field for f.

Moreover, if X and $\bar{X}$ are L-Hamiltonian vector fields with Hamiltonian functions f and $\bar{f}$, then $\{f,\bar{f}\}_L$ is a Hamiltonian for $[X,\bar{X}]$:

$$[[X+df,\bar{X}+d\bar{f}]]_C = [X,\bar{X}] + \mathcal{L}_X d\bar{f} - \iota_{\bar{X}} d^2 f = [X,\bar{X}] + d\{f,\bar{f}\}_L.$$

One can proceed in a very a similar way to the case of Poisson manifolds

See e.g.

Dirac–Lie systems and Schwarzian equations, J. Diff. Eqns. **257**, 2303–2340 (2014) (JFC, Janusz Grabowski, Javier de Lucas and Cristina Sardón)

Geometric approach to Quantum Mechanics

The Schrödinger picture of Quantum mechanics admits a geometric interpretation similar to that of classical mechanics.

A separable complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ can be considered as a real linear space, to be then denoted $\mathcal{H}_{\mathbb{R}}$. The norm in $\mathcal{H}$ defines a norm in $\mathcal{H}_{\mathbb{R}}$, where $\|\psi\|_{\mathbb{R}} = \|\psi\|_{\mathbb{C}}$.

The linear real space $\mathcal{H}_{\mathbb{R}}$ is endowed with a natural symplectic structure as follows:

$$\omega(\psi_1, \psi_2) = 2\,\mathrm{Imag}\,\langle\psi_1, \psi_2\rangle.$$

The Hilbert $\mathcal{H}_{\mathbb{R}}$ can be considered as a real manifold modelled by a Banach space admitting a global chart.

The tangent space $T_\phi\mathcal{H}_{\mathbb{R}}$ at any point $\phi \in \mathcal{H}_{\mathbb{R}}$ can be identified with $\mathcal{H}_{\mathbb{R}}$ itself: the isomorphism associates $\psi \in \mathcal{H}_{\mathbb{R}}$ with the vector $\dot\psi \in T_\phi\mathcal{H}_{\mathbb{R}}$ given by:

$$\dot\psi f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0}, \qquad \forall f \in C^\infty(\mathcal{H}_{\mathbb{R}})\ .$$

The real manifold can be endowed with a symplectic 2-form ω:

$$\omega_\phi(\dot\psi, \dot\psi') = 2\,\mathrm{Imag}\,\langle \psi, \psi' \rangle \ .$$

One can see that the constant symplectic structure ω in $\mathcal{H}_\mathbb{R}$, considered as a Banach manifold, is exact, i.e., there exists a 1-form $\theta \in \bigwedge^1(\mathcal{H}_\mathbb{R})$ such that $\omega = -d\theta$. Such a 1-form $\theta \in \bigwedge^1(\mathcal{H})$ is, for instance, the one defined by

$$\theta(\psi_1)[\dot\psi_2] = -\mathrm{Imag}\,\langle \psi_1, \psi_2 \rangle.$$

This shows that the geometric framework for usual Schrödinger picture is that of symplectic mechanics, as in the classical case.

A continuous vector field in $\mathcal{H}_\mathbb{R}$ is a continuous map $X\colon \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$. For instance for each $\phi \in \mathcal{H}$, the constant vector field X_ϕ defined by

$$X_\phi(\psi) = \dot\phi.$$

It is the generator of the one-parameter subgroup of transformations of $\mathcal{H}_\mathbb{R}$ given by

$$\Phi(t, \psi) = \psi + t\,\phi\,.$$

As another particular example of vector field consider the vector field X_A defined by the $\mathbb{C}$-linear map $A : \mathcal{H} \to \mathcal{H}$, and in particular when A is skew-selfadjoint.

With the natural identification natural of $T\mathcal{H}_\mathbb{R} \approx \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}$, X_A is given by

$$X_A : \phi \mapsto (\phi, A\phi) \in \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}\,.$$

When $A = I$ the vector field X_I is the Liouville generator of dilations along the fibres, $\Delta = X_I$, usually denoted Δ given by $\Delta(\phi) = (\phi, \phi)$.

Given a selfadjoint operator A in $\mathcal{H}$ we can define a real function in $\mathcal{H}_\mathbb{R}$ by

$$a(\phi) = \langle \phi, A\phi \rangle\,,$$

i.e.,

$$a = \langle \Delta, X_A \rangle\,.$$

Then,

$$\begin{aligned} da_\phi(\psi) &= \frac{d}{dt} a(\phi + t\psi)_{t=0} = \frac{d}{dt}\left[\langle \phi + t\psi, A(\phi + t\psi)\rangle\right]_{|t=0} \\ &= 2\,\mathrm{Re}\,\langle \psi, A\phi \rangle = 2\,\mathrm{Imag}\,\langle -\mathrm{i}\,A\phi, \psi \rangle = \omega(-\mathrm{i}\,A\phi, \psi). \end{aligned}$$

If we recall that the Hamiltonian vector field defined by the function a is such that for each $\psi \in T_\phi\mathcal{H} = \mathcal{H}$,

$$da_\phi(\psi) = \omega(X_a(\phi), \psi)\,,$$

we see that

$$X_a(\phi) = -\mathrm{i}\, A\phi\,.$$

Therefore if A is the Hamiltonian H of a quantum system, the Schrödinger equation describing time-evolution plays the rôle of 'Hamilton equations' for the Hamiltonian dynamical system $(\mathcal{H}, \omega, h)$, where $h(\phi) = \langle \phi, H\phi \rangle$: the integral curves of X_h satisfy

$$\dot{\phi} = X_h(\phi) = -\mathrm{i}\, H\phi\,.$$

The real functions $a(\phi) = \langle \phi, A\phi \rangle$ and $b(\phi) = \langle \phi, B\phi \rangle$ corresponding to two selfadjoint operators A and B satisfy

$$\{a, b\}(\phi) = -\mathrm{i}\, \langle \phi, [A, B]\phi \rangle\,,$$

because

$$\{a, b\}(\phi) = [\omega(X_a, X_b)](\phi) = \omega_\phi(X_a(\phi), X_b(\phi)) = 2\,\mathrm{Imag}\, \langle A\phi, B\phi \rangle\,,$$

and taking into account that

$$2\,\mathrm{Imag}\, \langle A\phi, B\phi \rangle = -\mathrm{i}\, [\langle A\phi, B\phi \rangle - \langle B\phi, A\phi \rangle] = -\mathrm{i}\, [\langle \phi, AB\phi \rangle - \langle \phi, BA\phi \rangle]\,,$$

we find the above result.

In particular, on the integral curves of the vector field X_h defined by a Hamiltonian H,

$$\dot{a}(\phi) = \{a, h\}(\phi) = -\mathrm{i}\,\langle \phi, [A, H]\phi \rangle\,,$$

what is usually known as Ehrenfest theorem:

$$\frac{d}{dt}\langle \phi, A\phi \rangle = -\mathrm{i}\,\langle \phi, [A, H]\phi \rangle\,.$$

There is another relevant symmetric $(0, 2)$ tensor field which is given by the Real part of the inner product. It endows $\mathcal{H}_\mathbb{R}$ with a Riemann structure and we have also a complex structure J such that

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \qquad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with

$$g(Jv_1, Jv_2) = g(v_1, v_2), \qquad \omega(Jv_1, Jv_2) = \omega(v_1, v_2)\,.$$

The triplet (g, J, ω) defines a Kähler structure in $\mathcal{H}_\mathbb{R}$ and the symmetry group of the theory must be the unitary group $U(\mathcal{H})$ whose elements preserve the inner product, or in an alternative but equivalent way (in the finite-dimensional case), by the intersection of the orthogonal group $O(2n, \mathbb{R})$ and the symplectic group $Sp(2n, \mathbb{R})$.

The time evolution from time t_0 to time t, even in the non-autonomous case, is described in terms of the evolution operator $U(t, t_0)$ which, as it must be a symmetry of the theory, is for each fixed t_0 a curve $U(t, t_0)$ in the unitary group $U(\mathcal{H})$. Assume by simplicity that $\mathcal{H}$ is finite-dimensional, and then as

$$\frac{dU(t, t_0)}{dt} \in T_{U(t,t_0)}U(\mathcal{H}) \Longrightarrow \frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} \in T_I U(\mathcal{H}) \approx \mathfrak{u}(\mathcal{H}),$$

and therefore, there exists a curve $H(t)$ in $\mathrm{Herm}(n, \mathbb{C})$ such that

$$\frac{dU(t, t_0)}{dt} = -\mathrm{i}\, H(t)\, U(t, t_0).$$

In this equation $H(t)$ does not depend on t_0 because of the relation

$$U(t, t_0) = U(t, t_1)U(t_1, t_0)$$

This is a Lie system in the unitary group $U(\mathcal{H})$ with associated Lie algebra $\mathfrak{u}(\mathcal{H})$ in the most general case. Sometimes however we can deal with some of its subalgebras.

Every curve $H(t)$ in $\mathfrak{u}(\mathcal{H})$ can be written as a linear combination of at most n^2 elements, those of a basis of $\mathfrak{u}(\mathcal{H})$, and therefore these (finite-dimensional) quantum systems are Lie systems.

As the elements of the Vessiot-Guldberg Lie algebra are skew-Hermitians, all of them define simultaneously Hamiltonian vector fields and Killing vector fields, and the system is a Lie-Kähler system.

As an example consider a Hamiltonian operator $H(t)$ that can be written as a linear combination, with some t-dependent real coefficients $b_1(t), \ldots, b_r(t)$, of some Hermitian operators,

$$H(t) = \sum_{k=1}^{r} b_k(t) H_k \, ,$$

where the H_k form a basis of a real finite-dimensional Lie algebra V relative to the Lie bracket of observables, i.e. $[\![H_j, H_k]\!] = \sum_{l=1}^{r} c_{jkl} H_l$, with $c_{jkl} \in \mathbb{R}$ and $j, k, l = 1, \ldots, r$.

It determines a t-dependent Schrödinger equation

$$\frac{d\psi}{dt} = -\mathrm{i} H(t)\psi = -\mathrm{i} \sum_{k=1}^{r} b_k(t) H_k \psi.$$

The vector fields X_k such that $X_k(\psi) = -\mathrm{i} H_k \, \psi$ are such that the t-dependent vector vector field X corresponding to the equation is $X = \sum_{k=1}^{r} b_k(t) X_k$ and

$$[X_j, X_k] = -X_{[\![H_j, H_k]\!]} = -\sum_{l=1}^{r} c_{jkl} X_l, \qquad j, k = 1, \ldots, r.$$

If $\mathcal{H} = \mathbb{C}^2$, the time evolution is described by a curve $-\mathrm{i}H(t) := \dot{U}_t U_t^{-1}$ in the Lie algebra $\mathfrak{u}(2)$ of $U(2)$. Using the basis

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and denoting $\mathbf{S} = (\sigma_1, \sigma_2, \sigma_3)/2$ and $\mathbf{B} := (B_1, B_2, B_3)$, the Hamiltonian can be written as

$$H(t) := B_0(t) I_0 + \mathbf{B}(t) \cdot \mathbf{S}.$$

Using the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, the Schrödinger equation is

$$\begin{pmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \end{pmatrix} = \frac{1}{2} \left(\begin{matrix} 0 & 2B_0(t) + B_3(t) & -B_2(t) & B_1(t) \\ -2B_0(t) - B_3(t) & 0 & -B_1(t) & -B_2(t) \\ B_2(t) & B_1(t) & 0 & 2B_0(t) - B_3(t) \\ -B_1(t) & B_2(t) & B_3(t) - 2B_0(t) & 0 \end{matrix} \right] \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

while the vector fields are now

$$
\begin{aligned}
X_0 &= -\Gamma = p_1 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial p_2}, \\
X_1 &= \frac{1}{2}\left(p_2 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_2} - q_1 \frac{\partial}{\partial p_2}\right), \\
X_2 &= \frac{1}{2}\left(-q_2 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial p_1} + q_1 \frac{\partial}{\partial q_2} + p_1 \frac{\partial}{\partial p_2}\right), \\
X_3 &= \frac{1}{2}\left(p_1 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial p_2}\right)
\end{aligned}
$$

satisfying

$$
[X_0, \cdot] = 0, \qquad [X_1, X_2] = -X_3, \qquad [X_2, X_3] = -X_1, \qquad [X_3, X_1] = -X_2.
$$

The vector fields X_0, X_1, X_2, X_3 are Hamiltonian with Hamiltonian functions given by

$$
\begin{aligned}
h_0(\psi) &= \tfrac{1}{2}\langle \psi, \psi \rangle = \tfrac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2), \\
h_1(\psi) &= \tfrac{1}{2}\langle \psi, S_1\psi \rangle = \tfrac{1}{2}(q_1 q_2 + p_1 p_2), \\
h_2(\psi) &= \tfrac{1}{2}\langle \psi, S_2\psi \rangle = \tfrac{1}{2}(q_1 p_2 - p_1 q_2), \\
h_3(\psi) &= \tfrac{1}{2}\langle \psi, S_3\psi \rangle = \tfrac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2).
\end{aligned}
$$

h_1, h_2, h_3 are functionally independent, but $h_0^2 = 4(h_1^2 + h_2^2 + h_3^2)$.

When $\mathcal{H}$ is not finite-dimensional Lie system theory applies when the t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of Hamiltonians H_i closing on, under the commutator bracket, a real finite-dimensional Lie algebra.

Note however that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension.

On the other hand, as the fundamental concept for measurements is the expectation value of observables, two vector fields such that

$$\frac{\langle\psi_2, A\psi_2\rangle}{\langle\psi_2, \psi_2\rangle} = \frac{\langle\psi_1, A\psi_1\rangle}{\langle\psi_1, \psi_1\rangle}, \quad \forall A \in \mathrm{Her}(\mathcal{H})$$

should be considered as indistinguishable. This is only possible when ψ_2 is proportional to ψ_1, and therefore we must consider rays rather than vectors the elements describing the quantum states. The space of states is not $\mathbb{C}^n$ but the projective space $\mathbb{CP}^{n-1}$.

It is possible to define a Kähler structure on $\mathbb{CP}^{n-1}$ and therefore to study Lie-Kähler systems leading to superposition rules and to study time evolution in this projective space.

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THANKS FOR YOUR ATTENTION !!!

CONGRATULATIONS, BEPPE!!!

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