On the steps of Shannon's sampling theorem: towards a theory of quantum sampling

Geometria è Fisica: A geometrical vision of physics
Celebrating G. Marmo 70th birthday

Alberto Ibort
ICMAT/UCIIIM

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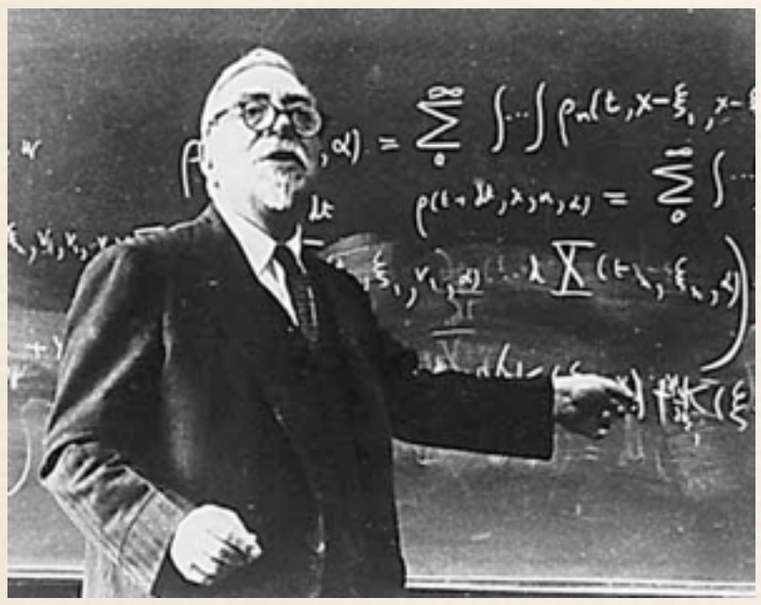
Work in progress with A. García, M.A. Hernández-Medina

1. Shannon's sampling theorem

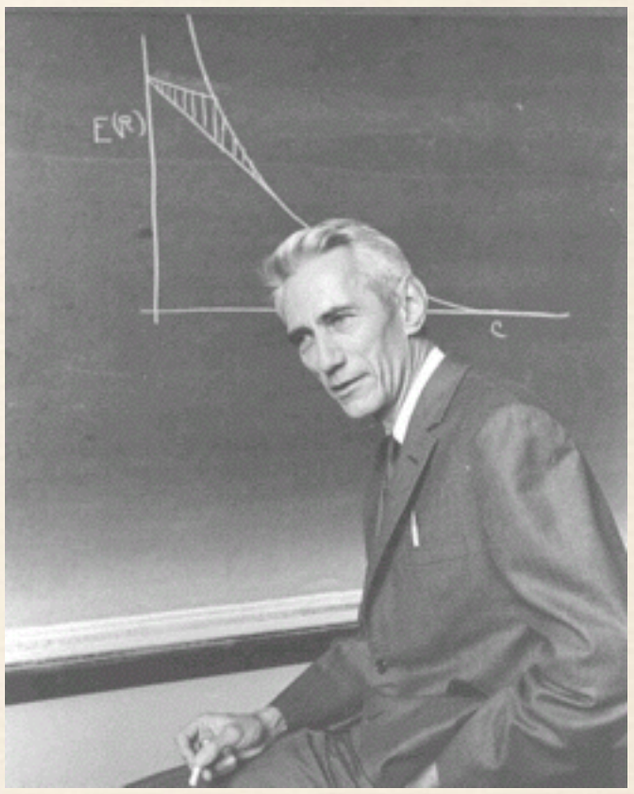
Theorem: If a signal $f(t)$ contains no frequencies higher than w cycles per second, then $f(t)$ is completely determined by its values $f(n/2w)$ at a discrete set of points with spacing $1/2w$, and can be reconstructed from these values by the formula:

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n}{2w}) \frac{\sin \pi(2wt - n)}{\pi(2wt - n)}.$$

Claude E. Shannon. *"Communications in the presence of noise"*. Proc. IRE, **137**, 10-21 (1949).



Norbert Wiener (1894-1964)



Claude E. Shannon (1916-2001)

3. Geometric sampling theory

Ω set $\quad\quad$ $\mathbb{H}$ complex separable Hilbert space

$K\colon \Omega \to \mathbb{H}$ kernel map

$\mathcal{F}\colon \mathbb{H} \to \mathcal{F}(\Omega) \quad\quad \mathcal{F}(\psi)(w) = \langle K(w), \psi \rangle_{\mathbb{H}}\,, \quad \psi \in \mathbb{H}$

Example $\quad \Omega = \mathbb{R} \quad\quad \mathbb{H} = L^2([-\pi, \pi])$

$$K\colon \mathbb{R} \to L^2([-\pi,\pi]) \qquad K(w) = \frac{1}{\sqrt{2\pi}} e^{iwx}$$

$$\mathcal{F}\colon L^2([-\pi,\pi]) \to \mathcal{F}(\mathbb{R})$$

$$\mathcal{F}(\psi)(w) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-iwx} \psi(x) dx$$

$\mathrm{Ran}(\mathcal{F}) = PW_\pi$ $\quad$ Paley-Wiener space

S. Saitoh. *Integral transforms, reproducing kernels and their applications*. Longman, Essex, England, 1997.

R. Paley and N. Wiener. *Fourier transforms in the complex domain*, volume 19, AMS Colloq. Publ., AMS. New York, 1934.

Sampling formulas

$w_n \in \Omega\,, \quad e_n = K(w_n)\,, \quad n = 1, 2, \ldots$ orthonormal basis

$$f = \mathcal{F}(\psi) \in \mathcal{F}(\mathbb{H}) \qquad \psi = \sum_n \langle e_n, \psi \rangle e_n$$

$$f(w_n) = \langle K(w_n), \psi \rangle$$

$$f(w) = \mathcal{F}(\psi)(w) = \sum_n \langle K(w_n), \psi \rangle \mathcal{F}(e_n) = \sum_n f(w_n) S_n(w)$$

Example (contd.) $\qquad w_n = n \in \mathbb{Z}$

$$f(w) = \sum_{n \in \mathbb{Z}} f(n) S_n(w) \qquad S_n(w) = \frac{\sin \pi(w - n)}{\pi(w - n)}$$

If $\ker \mathcal{F} = 0$

$$\langle f, g \rangle_K = \langle \psi, \phi \rangle_{\mathbb{H}}\,, \quad f = \mathcal{F}(\psi), g = \mathcal{F}(\phi)$$

$$k(w, u) = \langle K(w), K(u) \rangle \qquad f(w) = \langle k(w, \cdot), f \rangle_K$$

$\left(\mathcal{F}(\mathbb{H}), \langle \cdot, \cdot \rangle_K\right)$ is a Reproducing Kernel Hilbert Space (RKHS)

4. Sampling theorems on groups

G Lie group

Equivalence classes of continuous irreducible unitary representations

$$\widehat{G} = \{[\mathcal{H}^a, U^a] \mid U^a \colon G \to \mathcal{U}(\mathcal{H}^a)\} = \{a\}$$

Examples: $G = U(1)\,, \quad \widehat{U(1)} = \mathbb{Z}$

$G = \mathbb{Z}\,, \quad \widehat{\mathbb{Z}} = U(1)$

$G = \mathbb{Z}_n\,, \quad \widehat{\mathbb{Z}}_n = \mathbb{Z}_n$

G compact Lie group

$\widehat{G}$ countable, $\{a\} = [n_a]$, $a = 1, 2, \ldots,$ $n_a = \dim \mathcal{H}^a$

In what follows $\qquad n_a = \dim \mathcal{H}^a < \infty\,, \quad \forall a \in \widehat{G}$

$$\chi_a(g) = \mathrm{Tr}(U^a(g)) \quad \text{character}$$

$$K \colon \widehat{G} \to L^2(G) \quad K(a) = n_a \chi_a \quad \text{Kernel group map}$$

4. Sampling theorems on groups (Cont.)

$$K\colon \widehat{G} \to L^2(G) \qquad K(a) = \chi_a \quad \text{Kernel group map}$$

$$\mathcal{F}\colon L^2(G) \to \mathcal{F}(\widehat{G})\,, \quad \mathcal{F}(\psi)(a) = \langle K(a), \Psi \rangle = \int_G \overline{\chi}_a(g)\psi(g)d\mu_G(g)$$

Example: G is a locally compact Abelian group (LCA)
$\widehat{G}$ is LCA too. Plancherel's theorem

$$\mathcal{F}\colon L^2(G) \to L^2(\widehat{G})\,, \quad \mathcal{F}(\psi)(a) = \int_G \overline{\chi}_a(g)\psi(g)d\mu_G(g)$$

G Abelian finite

$|G| = |\widehat{G}|\,, \quad G \cong \widehat{G}$

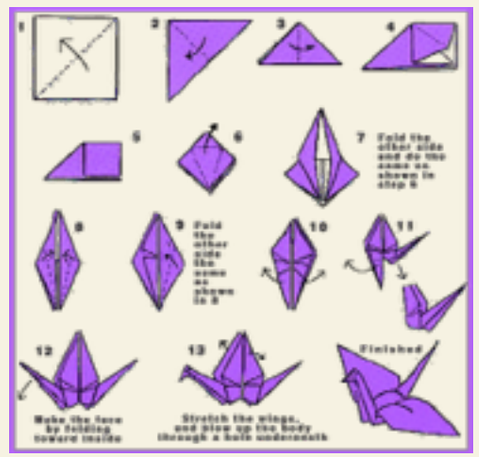
$$\mathcal{F}(|g\rangle) = \frac{1}{\sqrt{|G|}} \sum_{g'} \overline{\chi}_{g'}(g)|g'\rangle$$

Sampling with subgroups

G group $\qquad$ K normal subgroup

$H = G/K$ quotient group of right cosets

$$e \to K \to G \xrightarrow{\pi} H \to e$$



Sampling and subgroups

G group $\qquad$ K normal subgroup

$H = G/K$ quotient group of right cosets

$$e \to K \to G \xrightarrow{\pi} H \to e$$

$$\pi^* \colon \widehat{H} \to \widehat{G}\,, \quad \pi^*(U)(g) = U(\pi(g))$$

$$\begin{array}{ccc} K_H \colon \widehat{H} & \to & L^2(H) \\ \pi^* \downarrow & & \downarrow \sigma \\ K_G \colon \widehat{G} & \to & L^2(G) \end{array}$$

Suppose from now on that K is a discrete subgroup

$$\sigma \colon H \to G, \quad \pi \circ \sigma = id_H$$

cross-section

If H is Abelian, functions on $\mathcal{F}_G \circ \sigma$ can be sampled with respect to the system $S_a = \mathcal{F}_G(\chi_a)$, $a \in \widehat{H}$ and for them

$$f(\alpha) = \sum_{a \in \widehat{H}} f(a) S_a(\alpha)$$

Example: $\quad G = \mathbb{R}\,, \quad K = \mathbb{Z}\,, \quad H = G/K = U(1)$

Samplig formula = Shannon's sampling theorem

5. Sampling theorems and representations of compact groups

Sampling theorems on invariant subspaces

$U : G \to \mathcal{U}(\mathcal{H})$ Square integrable unitary representation

G compact group

$$|f(\Omega)\rangle = \int_G f(g) U(g) |\Omega\rangle d\mu(g)\,, \quad f \in L^2(G)$$

$\mathcal{H}_\Omega = \{|f(\Omega)\rangle\}$ Invariant subspace

$$|f(\Omega)\rangle = \sum_{a \in \widehat{H}} f(a) |S_n(\Omega)\rangle$$

A. García-García, M.A. Hernández-Media, A. Ibort,. Towards a quantum sampling theory: the case of finite groups (2016).

6. The Hidden subgroup problem.

G group $\qquad$ K subgroup

$H = G/K$ homogenous space of right cosets

$e \to K \to G \xrightarrow{\pi} H$

$F\colon G \to S\,, \quad F(gk) = F(g)\,, \quad \forall k \in K, g \in G$

Hidden subgroup problem: Given a function F as above, determine the subgroup K (i.e., find a set of generators)

Hidden Abelian subgroup problem: Given G an Abelian group, and a function F as above, determine the Abelian subgroup K (i.e., find a set of generators).

Example $\quad G = \mathbb{Z}_n\,, \quad n = pq \qquad K = \mathbb{Z}_p$ Schor's algorithm

A. Kitaev. *Quantum measurements and the abelian stabilizer problem,* ECCC, Report TR96-003, 1996.

Solution Hidden Abelian subgroup problem: G Finite

$|g\rangle \in L^2(G)$ standard orthonormal basis $\quad |\psi\rangle = \sum_{g\in G} \psi(g)|g\rangle$

Kitaev's algorithm:

1. Superposition: $\quad \frac{1}{\sqrt{|G|}} \sum_g |g\rangle$

2. Computing F: $\quad \frac{1}{\sqrt{|G|}} \sum_g |g\rangle \otimes |0\rangle$

$$U_F \left(\frac{1}{\sqrt{|G|}} \sum_g |g\rangle \otimes |0\rangle \right) = \frac{1}{\sqrt{|G|}} \sum_g |g\rangle \otimes |F(g)\rangle$$

3. Measure the second register

$$|\psi\rangle = \frac{1}{\sqrt{|K|}} \sum_{k\in K} |\tilde{g}k\rangle \otimes |F(\tilde{g})\rangle$$

Kitaev's algorithm (Cont.):

4. Quantum Fourier transform on the first factor

$$\mathcal{F}(|\psi\rangle) = \frac{1}{\sqrt{|G||K|}} \sum_{g\in G} \sum_{k\in K} \chi_g(k)\chi_g(\tilde{g})|g\rangle \otimes |F(\tilde{g})\rangle$$

$$\frac{1}{\sqrt{|H|}} \sum_{k\in K} \chi_g(k) = \delta(g, K^\perp) \qquad K^\perp = \{\chi_g \in \widehat{G} \mid \chi_g(k) = 1\,, \forall k \in K\}$$

$$\mathcal{F}(|\psi\rangle) = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \delta(g, K^\perp)\chi_g(\tilde{g})|g\rangle \otimes |F(\tilde{g})\rangle$$

5. Measure the first register. If non zero, you get a generator of $K^\perp$

An application: new public keys

Given G and and two Abelian (non commuting) subgroups K and H, define a function f by means of the corresponding sampling formula. Alice transmit the values $f(a)$ (S_a are public). Hence Bob (that nows K) reconstruct $F(g)$ for all g.

Scolium

"La meccanica quantistica è scritta in lingua geometrica senza la quale è impossibile capire nulla"

G. Marmo's dictum

Thanks again Beppe!

That's almost all...!