Fernando Falceto
Theoretical Physics Department.
University of Zaragoza

Complex geometry in the entanglement entropy of fermionic chains.

Geometria è Fisica, Policeta, July 12, 2016.

In collaboration with:
Filiberto Ares
José G. Esteve
Amilcar de Queiroz

Aim of the talk

- We review the scaling behavior of the **Rényi entanglement entropy** of the homogeneous, free fermionic chain and the rôle played by the geometry of **Riemann surfaces**.
- We discuss the properties of the entropy under **Möbius transformations** in critical and non critical theories.
- We compare the latter with **conformal transformations** in real space.

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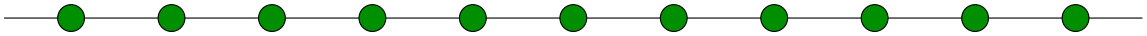
Based on:

F Ares, J G Esteve, F F, A R de Queiroz, JSTAT (2016), 043106; `arXiv:1511.02382`

F Ares, J G Esteve, F F, A R de Queiroz, *Möbius transformations in critical fermionic chains*. In preparation.

Fermionic chain



$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N}, \quad \{a_n, a_m^\dagger\} = \delta_{nm}, \quad \{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0.$$

- General quadratic, periodic, translational invariant Hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^{N} \sum_{l=-L}^{L} \left(2A_l a_n^\dagger a_{n+l} + B_l a_n^\dagger a_{n+l}^\dagger - \overline{B}_l a_n a_{n+l} \right),$$

with $A_{-l} = \overline{A}_l$ and $B_l = -B_{-l}$. Periodic B.C. $a_n = a_{n+N}$

- Note that:
  - If $B_l = 0$ for all l, fermionic number is preserved.
  - $\mathrm{Im}(A_l) \neq 0$ breaks reflection symmetry, $P : a_n \mapsto i\, a_{N-n}$.
  - $\mathrm{Im}(B_l) \neq 0$ breaks r. s. + charge conjugation, $PC : a_n \mapsto i\, a_{N-n}^\dagger$.

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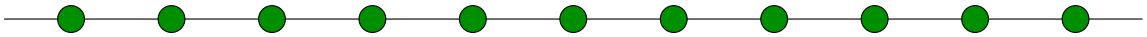

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Bogoliubov modes.

$$H = \frac{1}{2} \sum_{n=1}^{N} \sum_{l=-L}^{L} \left(2 A_l a_n^\dagger a_{n+l} + B_l a_n^\dagger a_{n+l}^\dagger - \overline{B}_l a_n a_{n+l} \right),$$

$$b_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta_k} a_n, \qquad \theta_k = \frac{2\pi k}{N}$$

$$F(\theta) = \sum_{l=-L}^{L} A_l e^{i\theta l}, \quad G(\theta) = \sum_{l=-L}^{L} B_l e^{i\theta l}.$$

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$$\begin{aligned} H &= \frac{1}{2}\sum_{k=0}^{N-1}(b_k^\dagger, b_{-k})\begin{pmatrix} F(\theta_k) & G(\theta_k) \\ \overline{G}(\theta_k) & -F(-\theta_k)\end{pmatrix}\begin{pmatrix} b_k \\ b_{-k}^\dagger\end{pmatrix} \\ &= \sum_{k=0}^{N-1}\Lambda(\theta_k)\, d_k^\dagger d_k. \qquad \text{Bogoliubov modes}\end{aligned}$$

$$\Lambda(\theta) = \sqrt{F^+(\theta)^2 + |G(\theta)|^2} + F^-(\theta), \quad F^\pm(\theta) \equiv \frac{F(\theta) \pm F(-\theta)}{2}$$

Stationary states

For any set of modes $\mathbf{K} \subset \{-N/2, \ldots, N/2-1\}$,

$$|\mathbf{K}\rangle = \prod_{k\in\mathbf{K}} d_k^\dagger |0\rangle, \qquad E_{\mathbf{K}} = \sum_{k\in\mathbf{K}} \Lambda(\theta_k),$$

$|0\rangle$: Fock space vacuum for Bogoliubov modes, $d_k |0\rangle = 0$.

Ground state: $|\mathrm{GS}\rangle = \prod_{\Lambda_k<0} d_k^\dagger |0\rangle$

Reflection. $P : a_n \mapsto i a_{N-n}, \qquad b_k \mapsto i b_{-k}$

The Hamiltonian is not reflection invariant, but:

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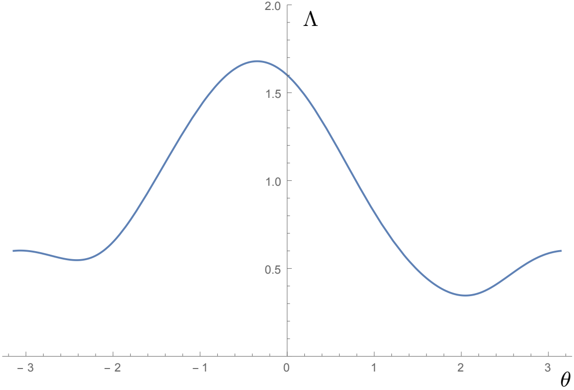
Non-critical Hamiltonian.



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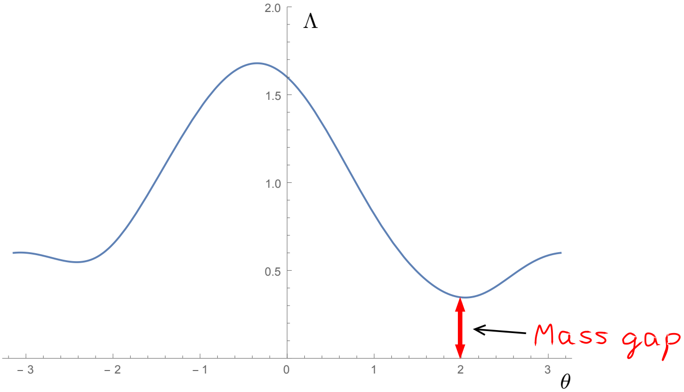

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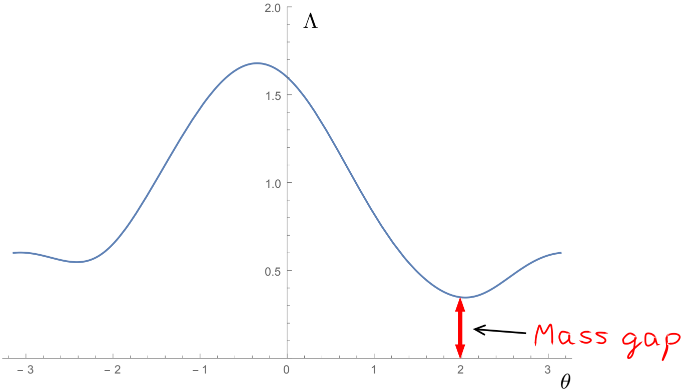

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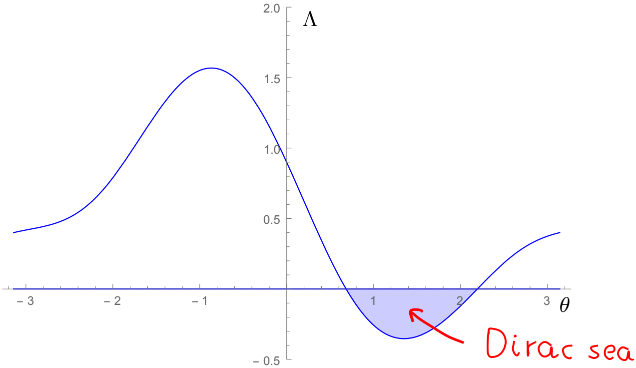

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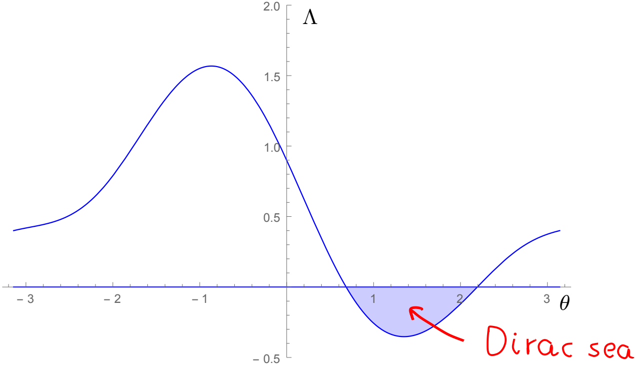

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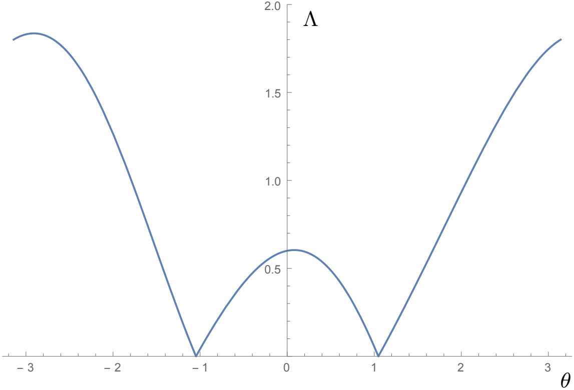

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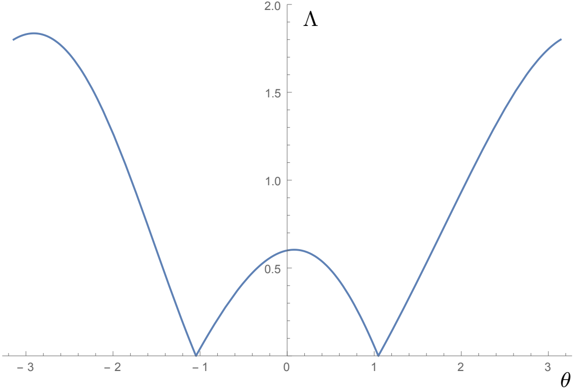

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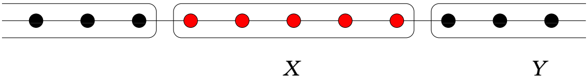
Bipartite entanglement entropy.



$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$$

- Introduce the *partition function*

$$Z_{\alpha,X} = \text{Tr}(\rho_X^\alpha)$$

Where ρ_X is the reduced density matrix

$$\rho_X = \text{Tr}_{\mathcal{H}_Y}(|\mathbf{K}\rangle\langle\mathbf{K}|).$$

- The Rényi entanglement entropy

$$S_{\alpha,X} = \frac{1}{1-\alpha}\log Z_{\alpha,X}$$

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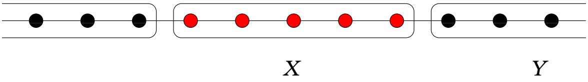

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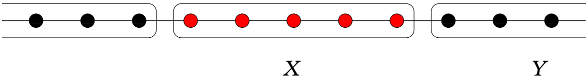

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Wick decomposition

$|\mathbf{K}\rangle = \prod_{k \in \mathbf{K}} d_k^\dagger |0\rangle$, Slater det. $\Rightarrow$ Wick decomposition holds

$$\begin{aligned} \mathrm{Tr}(\rho_X a_{n_1}^\dagger a_{n_2}^\dagger a_{m_1} a_{m_2}) &= \mathrm{Tr}(\rho_X a_{n_1}^\dagger a_{n_2}^\dagger)\, \mathrm{Tr}(\rho_X a_{m_1} a_{m_2}) \\ &- \mathrm{Tr}(\rho_X a_{n_1}^\dagger a_{m_1})\, \mathrm{Tr}(\rho_X a_{n_2}^\dagger a_{m_2}) \\ &+ \mathrm{Tr}(\rho_X a_{n_1}^\dagger a_{m_2})\, \mathrm{Tr}(\rho_X a_{n_2}^\dagger a_{m_1}) \end{aligned}$$

Therefore, one can show

$$\boxed{Z_{\alpha,X} = \det f_\alpha(V_X)}$$

with

$$f_\alpha(x) = \left[\left(\frac{1+x}{2} \right)^\alpha + \left(\frac{1-x}{2} \right)^\alpha \right],$$

and the correlation matrix

$$(V_X)_{nm} = \left\langle \mathbf{K} \middle| \left[\begin{pmatrix} a_n \\ a_n^\dagger \end{pmatrix}, (a_m^\dagger, a_m) \right] \middle| \mathbf{K} \right\rangle, \quad n, m \in X.$$

R. D. Sorkin, arXiv:1402.3589 (1983); L. Bombelli, R.K. Koul, J. Lee, R.D. Sorkin, PRD 34 (1986); G. Vidal, J.I. Latorre, E. Rico, A. Kitaev, PRL 90 (2003); I. Peschel, JPA 36 (2003).

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Due to translational invariance it is a block Toeplitz matrix with 2×2 symbol $\mathcal{G}$. In the thermodynamic limit

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$\mathcal{G}$ depends on the state $|\mathbf{K}\rangle$. For the ground state one has

$$\mathcal{G}(\theta) = \begin{cases} -I, & \text{if } \Lambda(\theta) < 0 \text{ and } \Lambda(-\theta) > 0, \\ M(\theta), & \text{if } \Lambda(\theta) > 0 \text{ and } \Lambda(-\theta) > 0, \\ I, & \text{if } \Lambda(\theta) > 0 \text{ and } \Lambda(-\theta) < 0, \end{cases}$$

Where

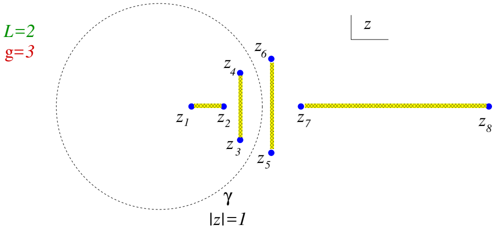
$$M(\theta) = \frac{\begin{pmatrix} F^+(\theta) & G(\theta) \\ \overline{G}(\theta) & -F^+(\theta) \end{pmatrix}}{\sqrt{(F^+(\theta))^2 + |G(\theta)|^2}}$$

Non-critical Hamiltonian ($\Lambda(\theta) > 0$)

$\mathcal{G}(\theta) = M(\theta)$, continous.

$\mathcal{M}(z)$ its analytic continuation from the unit circle.

It is meromorphic in a two-sheeted cover of the Riemann sphere.



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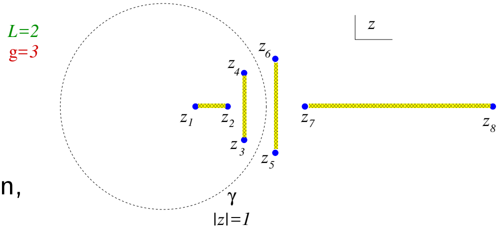

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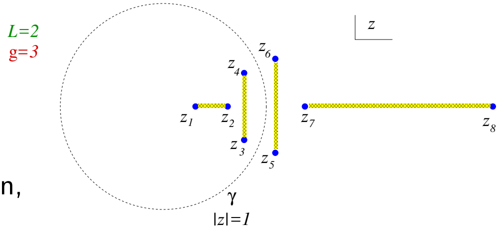

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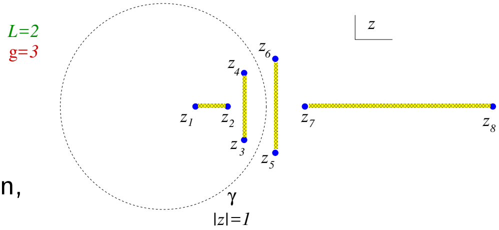

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$$K\mathbf{v}(z) = \mathbf{v}(z) + \frac{1}{2\pi i}\oint_\gamma \frac{I - \mathcal{M}(y)}{z - y}\mathbf{v}(y)\,\mathrm{d}y, \quad \mathbf{v} \in L^2(\gamma) \otimes \mathbb{C}^2$$

$$\boxed{Z_\alpha \equiv \det f_\alpha(V) = \det f_\alpha(K)}$$

Möbius transformations: $z' = \frac{az+b}{cz+d}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$

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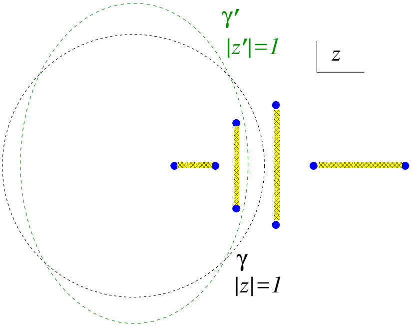

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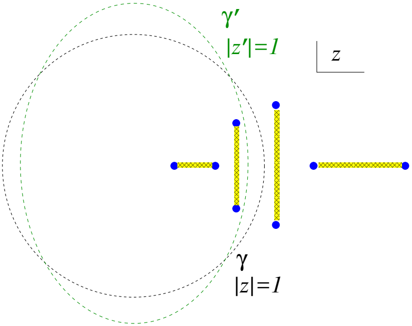

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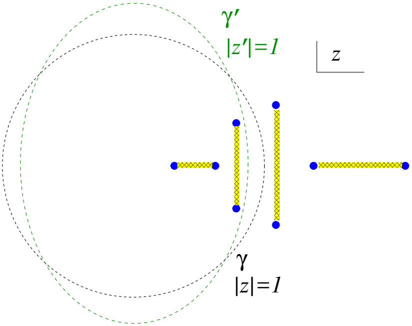
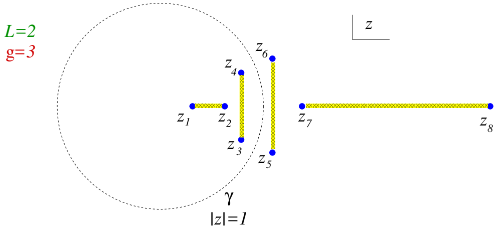
$$\det f_\alpha(K') = \det f_\alpha(K)$$

$$\boxed{Z'_\alpha = Z_\alpha, \quad S'_\alpha = S_\alpha}$$



Admissible Möbius transformations

But a physical Möbius transformation should preserve the relations between the branch points, e.g $z'_3 = \overline{z}'_4 = {z'_6}^{-1}$.



It implies that it should commute with inversion and conjugation, equivalently, it should preserve the real line and the unit circle.

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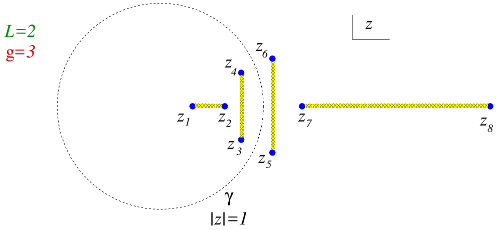

It implies that it should commute with inversion and conjugation, equivalently, it should preserve the real line and the unit circle.

Therefore we are left with transformations in $SO(1,1) \subset SL(2,\mathbb{C})$

$$z' = \frac{z\cosh\zeta + \sinh\zeta}{z\sinh\zeta + \cosh\zeta}$$

Admissible Möbius transformations: $SO(1,1) \subset SL(2,\mathbb{C})$

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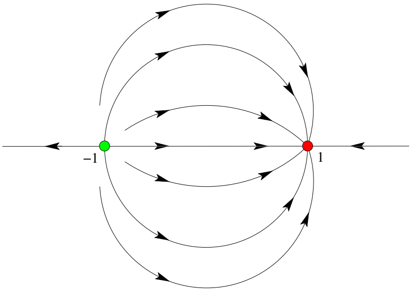
$z = 1$ stable, $z = -1$ unstable.



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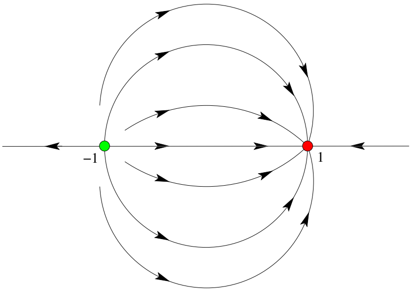

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Couplings A_l, B_l:
spin L repr. of $SL(2,\mathbb{C})$

$$\begin{pmatrix} A'_L \\ \vdots \\ A'_0 \\ \vdots \\ A'_{-L} \end{pmatrix} = \mathrm{e}^{\zeta \cdot (J_x)_L} \begin{pmatrix} A_L \\ \vdots \\ A_0 \\ \vdots \\ A_{-L} \end{pmatrix}, \quad \begin{pmatrix} B'_L \\ \vdots \\ B'_0 \\ \vdots \\ B'_{-L} \end{pmatrix} = \mathrm{e}^{\zeta \cdot (J_x)_L} \begin{pmatrix} B_L \\ \vdots \\ B_0 \\ \vdots \\ B_{-L} \end{pmatrix}$$

Recall: $H = \frac{1}{2}\sum_n \sum_{l=-L}^{L} \left[A_l a_n^\dagger a_{n+l} + B_l a_n^\dagger a_{n+l}^\dagger - \overline{B}_l a_n a_{n+l} \right]$

Example: XY model or Kitaev chain

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Nearest neighbours interaction $L = 1$.

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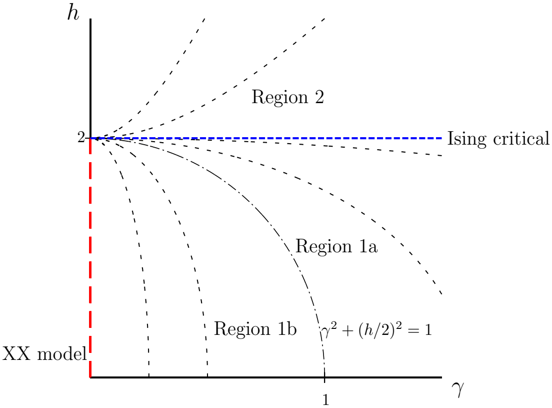

Ellipses of constant entropy: flow of the Möbius transformations...

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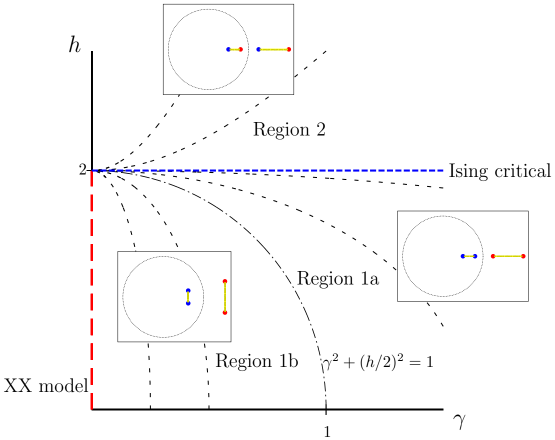

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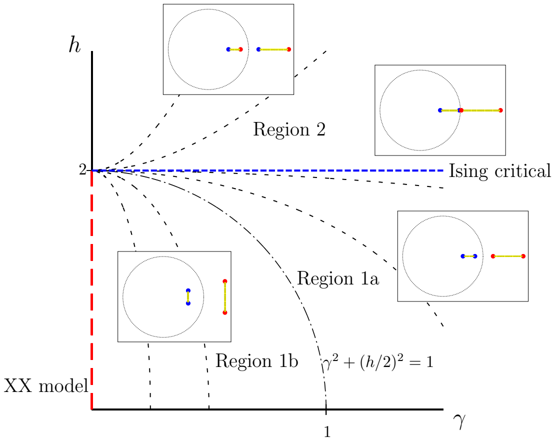

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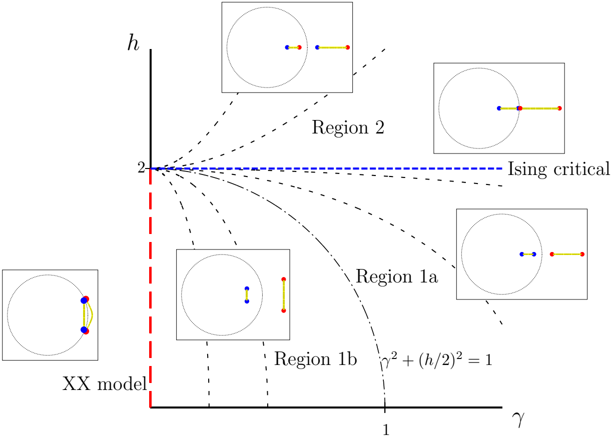

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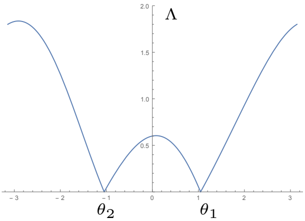
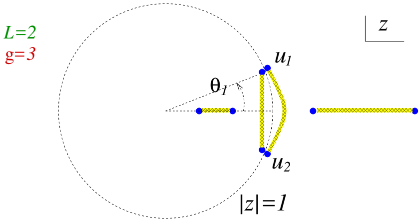
Möbius transformations and critical theories.





Critical theories → pinchings at $\underline{u} = (u_1, u_2, \ldots, u_R)$, $u_r = \mathrm{e}^{i\theta_r}$

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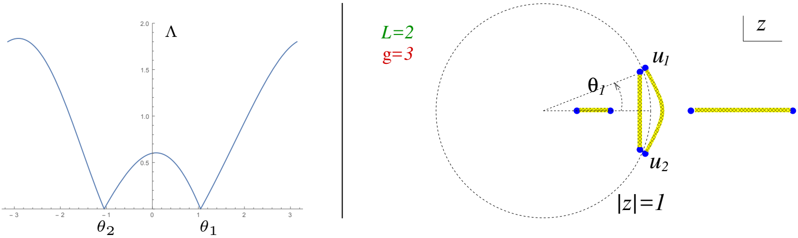

Critical theories → pinchings at $\underline{u} = (u_1, u_2, \ldots, u_R)$, $u_r = \mathrm{e}^{i\theta_r}$

Conjecture: Under admissible Möbius transf. and large $|X|$ limit.

$$\boxed{Z'_{\alpha,X}(\underline{u}') = \prod_{r=1}^{R} \left(\frac{\partial u'_r}{\partial u_r}\right)^{2\cdot\Delta_\alpha} Z_{\alpha,X}(\underline{u})} \qquad \Delta_\alpha = \frac{1-\alpha^2}{24\alpha}.$$

Möbius transformations and critical theories.

$$S'_{\alpha,X}(\underline{u}') = \frac{1+\alpha}{12\alpha} \sum_{r=1}^{R} \log \frac{\partial u'_r}{\partial u_r} + S_{\alpha,X}(\underline{u}).$$

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Can be proven assuming fermionic number conservation ($B_l = 0$)

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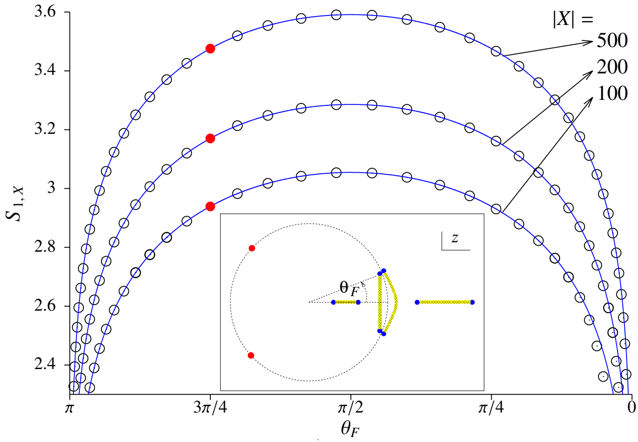
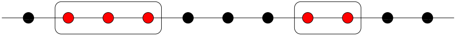
$$S'_{\alpha,X}(\underline{u}') = \frac{1+\alpha}{12\alpha}\sum_{r=1}^{R}\log\frac{\partial u'_r}{\partial u_r} + S_{\alpha,X}(\underline{u}).$$

Numerical test for $L = 2$ and $B_l \neq 0$



Möbius transformations and critical theories.

Consider the multi-interval $X = (x_1, x_2) \cup \cdots \cup (x_{2P-1}, x_{2P})$



$$Z_\alpha(\underline{u}, \underline{x}) = \mathrm{Tr}(\rho_X^\alpha)$$

- Under a Möbius transformation $(A_l, B_l, u_r) \mapsto (A'_l, B'_l, u'_r)$

$$Z'_\alpha(\underline{u}', \underline{x}) = \prod_{r=1}^{R} \left(\frac{\partial u'_r}{\partial u_r} \right)^{2P \cdot \Delta_\alpha} Z_\alpha(\underline{u}, \underline{x}).$$

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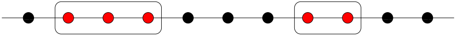

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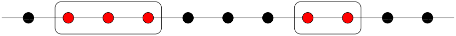

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$$Z_\alpha'(\underline{u}', \underline{x}) = \prod_{r=1}^{R} \left(\frac{\partial u_r'}{\partial u_r}\right)^{2P \cdot \Delta_\alpha} Z_\alpha(\underline{u}, \underline{x}).$$

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Möbius transformations and critical theories.

Combining Möbius & conformal $(A_l, B_l, u_r, x_p) \mapsto (A'_l, B'_l, u'_r, x'_p)$

$$Z'_\alpha(\underline{u}', \underline{x}') = \prod_{r,p} (J_{rp})^{\Delta_\alpha} Z_\alpha(\underline{u}, \underline{x}).$$

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$$J_{rp} = \frac{\partial u'_r}{\partial u_r} \cdot \frac{\partial x'_p}{\partial x_p}$$

of the $SO(1,1) \times SO(2,1)$ transformation

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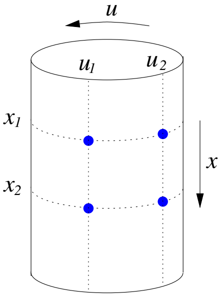

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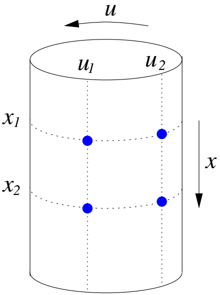
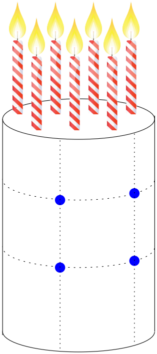


Can this be extended to a more general group?

HAPPY BIRTHDAY

BEPPE



Example: XY model or Kitaev chain

$$H_{\mathrm{XY}} = \sum_{n=1}^{N} \left[a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n + \gamma (a_n^\dagger a_{n+1}^\dagger - a_n a_{n+1}) + h a_n^\dagger a_n \right]$$

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  $\gamma = 0, h < 2$, XX universality class, $c = 1$

- Non critical regions:

  Nearest neighbors couplings $\Rightarrow L = 1$, hence $\mathrm{g} = 1$.

  The entropy involves elliptic integrals ....

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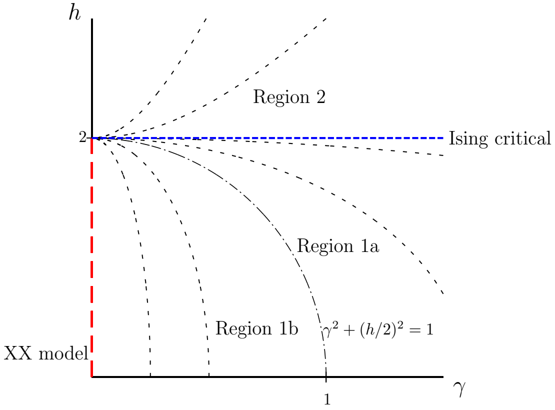

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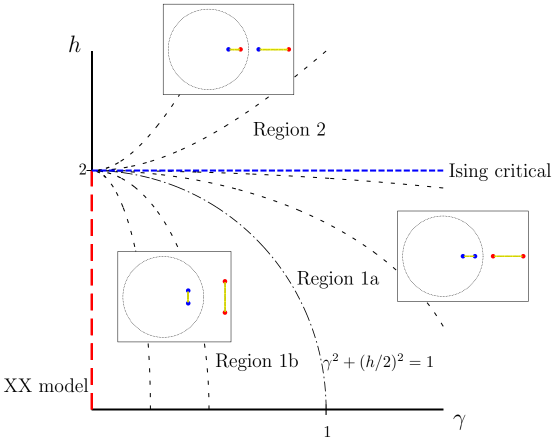

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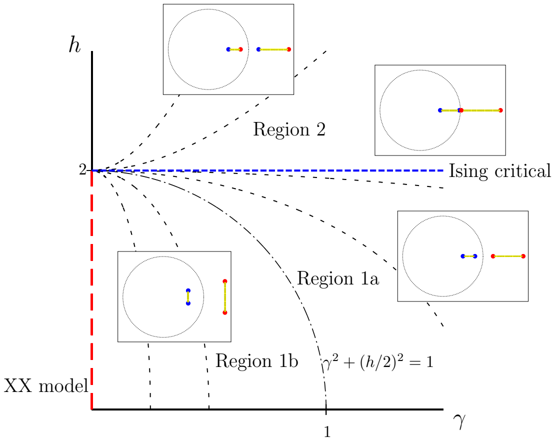

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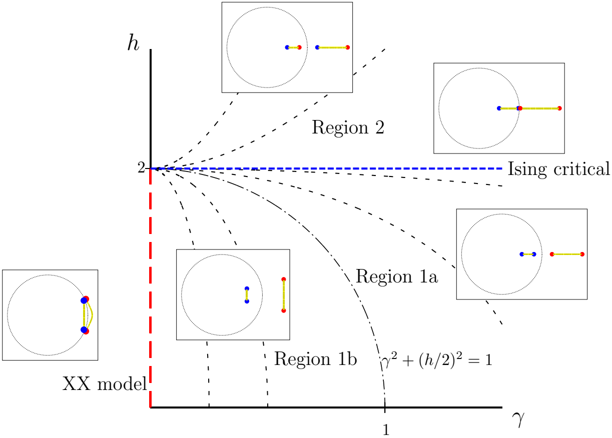

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